1. INTRODUCTION

Let \( \Lambda \) be a finite-dimensional \( k \)-algebra where \( k \) is an algebraically closed field. Projective resolutions of \( \Lambda \) over its enveloping algebra \( \Lambda^e = \Lambda^{op} \otimes_k \Lambda \) make it possible to calculate the Hochschild cohomology groups \( H^n(\Lambda, \Lambda) = \text{Ext}^n(\Lambda, \Lambda) \), \( n \geq 0 \), of \( \Lambda \). One such resolution is the standard resolution (see [CE]). However, this resolution is too large to be of much use for many calculations. Smaller resolutions are desirable if one wants to pursue cohomology. In [Ci] Cibils gives a smaller resolution of \( \Lambda \) in the case where \( \Lambda/r \) is separable (here \( r \) is the Jacobson radical of \( \Lambda \)). Happel provides the projectives for a minimal projective resolution of a finite-dimensional \( k \)-algebra over its enveloping algebra (see [Ha]). However, the maps are not given. The need for maps corresponding to minimal resolutions is clear.

In this paper we present minimal projective resolutions for finite-dimensional monomial algebras which are quotients of path algebras. Monomial algebras are sometimes called zero relations algebras in the literature. It turns out that the syzygies for these resolutions exhibit an alternating behavior. This phenomenon is induced by certain properties of the associated sequence of paths which we will define in Section 3. Any even–odd behavior observed in \( H^n(\Lambda, \Lambda) \), for \( \Lambda \) monomial, follows naturally from
this resolution. Other more general modules may also have properties linked to the combinatorics of these monomial algebras. For example, suppose $M$ is a left $\Lambda$-module. Let the minimal $\Lambda$-projective resolution of $\Lambda$ be given by $\cdots \to P_n \xrightarrow{\phi_n} P_{n-1} \xrightarrow{\phi_{n-1}} \cdots \to P_0 \xrightarrow{\phi_0} \Lambda \to 0$ and let $\Omega_n$ denote the $n$th syzygy. Using the fact that $\Lambda$ is a right projective $\Lambda$-module, we see $\Omega_1$ must also be a right projective $\Lambda$-module since $0 \to \Omega_1 \to P_0 \to \Lambda \to 0$ splits. By continuing this process we have that $0 \to \Omega_{n+1} \to P_n \to \Omega_n \to 0$ splits as a sequence of right $\Lambda$-modules. So the following is an exact sequence of left $\Lambda$-modules:

$$0 \to \Omega_{n+1} \otimes_\Lambda M \to P_n \otimes_\Lambda M \to \Omega_n \otimes_\Lambda M \to 0.$$ 

Since $P_n$ is a summand of a free $\Lambda$-module and $\Lambda \otimes_\Lambda M$ is a projective left $\Lambda$-module, we have that $P_n \otimes_\Lambda M$ is a left projective $\Lambda$-module. So

$$\cdots \to P_n \otimes_\Lambda M \xrightarrow{\phi_n \otimes 1_M} P_{n-1} \otimes_\Lambda M \xrightarrow{\phi_{n-1} \otimes 1_M} \cdots \xrightarrow{\phi_1 \otimes 1_M} P_0 \otimes_\Lambda M \to M \to 0$$

is a $\Lambda$-projective resolution of $M$. Although this resolution need not be minimal, the minimal resolution is a summand of this resolution. Another example where this link may appear comes from the resolutions of the simple $A$-modulus given in [AG]. Here $A$ is a split basic $k$-algebra and there is a quiver $\Gamma$ such that $A = k\Gamma/I$. Although $I$ need not be a monomial ideal, a construction is given that produces what is called the associated monomial algebra $A_{mon}$ of $A$. The complexity of this construction determines how much the resolutions of the simple modules over $A_{mon}$ and $A$ differ. So it may be possible, in some cases, for the alternating syzygy behavior of monomial algebras to be inherited, at least in part, by more general modules. It is worth pointing out here that certain special properties of syzygies over monomial algebras have also been described in [ZH].

Throughout this paper $\Gamma$ will denote a finite quiver. We define the path algebra $k\Gamma$ over the field $k$ to be the vector space with basis $B$ consisting of all finite directed paths in $\Gamma$. Multiplication is defined by concatenation in the usual way. We denote the vertex set by $\Gamma_0 = \{v_1, v_2, \ldots, v_m\}$ and the arrow set by $\Gamma_1 = \{a_1, a_2, \ldots, a_n\}$. Each vertex will be regarded as a path of length zero. The algebras we are interested in have the form $\Lambda = k\Gamma/I$ where $J^N \subset I \subset J^2$ for some positive integer $N \geq 2$. Here $J$ is the two-sided ideal generated by the arrows. An ideal $I$ of this form is called admissible. Note that $\Lambda$ is finite dimensional over $k$. We call $\Lambda$ a monomial algebra if $I$ is generated by a finite number of paths in $\Gamma$. Since $\Lambda$ is a $\Lambda$-$\Lambda$ bimodule
we can view $\Lambda$ as a right $\Lambda'$-module, where $\Lambda' = \Lambda^{op} \otimes \Lambda$ is the enveloping algebra of $\Lambda$ (see [P]). From now on when we refer to a path we mean a directed path. If $p$ is a path we denote the origin of $p$ by $o(p)$, the terminus of $p$ by $t(p)$, and the length of $p$ by $l(p)$. Here the length of a path means the number of arrows in the path.

As a last prerequisite, we assume the reader is familiar with some of the terminology from noncommutative Grobner basis theory. Let us recall the basics of this subject. First, we need an admissible order $<$. Assume that $v_1 < \cdots < v_n < a_1 < \cdots < a_m$ and let $<$ be length-lexicographic order reading origin to terminus for each directed path in $B$. This is the admissible order that we will use for $B$. Let $x = \sum_{i=1}^r a_i p_i$, $p_i \in B$, where the $p_i$ are distinct and $a_i \in \mathbb{k} \setminus 0$. We define the support of $x$ to be $\{p_1, \ldots, p_r\}$ and $\text{tip}(x)$ to be the largest path under this order in the support of $x$. We say $x$ is uniform if there exists $u, v \in \Gamma_0$ such that $o(p_i) = u$ and $t(p_i) = v$ for $i = 1, \ldots, r$. $\text{Tip}(I)$ is the set of paths that are tips of elements of $I$. We define $\text{NonTip}(I) = B \setminus \text{Tip}(I)$. It is known that the path algebra $k\Gamma$ has a $k$-vector space decomposition as follows:

$$k\Gamma = I \oplus \text{Span}(\text{NonTip}(I)).$$

So every nonzero element $x \in k\Gamma$ can be written $x = I(x) + N(x)$ where $I(x) \in I$ and $N(x) \in \text{Span}(\text{NonTip}(I))$. $N(x)$ is called the normal form of $x$ modulo $I$. Since $\langle \text{Tip}(I) \rangle$ is a monomial ideal, there is a finite set of unique minimal generators $\xi_1, \ldots, \xi_w$ that generate $\langle \text{Tip}(I) \rangle$. So we can write $\xi_i = p_i + N(\xi_i)$ for each $i = 1, \ldots, w$. Now let us define $\text{Minsharp}_{<} (I) = \{p_1, \ldots, p_w\}$. Note that $\text{Minsharp}_{<} (I)$ is the noncommutative analogue of a reduced Grobner basis. For a monomial algebra, $\text{Minsharp}_{<} (I)$ is just a finite collection of paths where, if $p \in \text{Minsharp}_{<} (I)$, then no proper subdivisors of $p$ are in $\text{Minsharp}_{<} (I)$. Finally, if $x = \sum_{i=1}^r p_i \otimes q_i$, then reducing $x$ to its tensor normal form modulo $I$ will mean reducing each $p_i$ and $q_i$ to their respective normal forms modulo $I$. For a monomial algebra this will just mean that each $p_i$ and $q_i$ is not divisible by any path in $\text{Minsharp}_{<} (I)$. Further discussion of the terminology above can be found in [FFG, FG].

The main result of this paper is the construction of the minimal projective resolution for a monomial algebra (see Theorem 4.1). A different approach to minimal resolutions of finite-dimensional algebras is provided in [BK]. Section 2 begins with the projective presentation of $\Lambda = k\Gamma/I$ where $I$ is admissible but not necessarily monomial. The second projective and syzygy are also given in this general setting. Section 3 describes the associated sequence of paths which determine the higher projectives and syzygies for the monomial case. This sequence has also been used to describe the cohomology rings of monomial algebras (see
[GZ]) and to provide projective resolutions of vertex simple representations of trees (see [GHZ]). After stating the main theorem in Section 4 we extend the order < above to the projectives \( P_n \). The subsequent sections prove that the complex presented in 4.1 is exact. Following the evident alternating behavior of the maps, the proof is done inductively on pairs of maps. Since the second projective and map are given without the assumption that \( I \) is monomial, the induction begins with the third and fourth projectives and syzygies. The proof for the higher projectives and maps is then a generalization of this argument. The last section provides two small examples of Hochschild cohomology. Here we show how the alternating behavior of the associated sequence induces a period doubling in the complex of one of the examples.

2. THE PROJECTIVE PRESENTATION AND SECOND SYZYGY

Let \( \Lambda = K \Gamma / I \) where \( I = \langle r_1, r_2, \ldots, r_m \rangle \) and each relation \( r_i \) does not have to be a path. We do assume each \( r_i \) is uniform. To construct a projective resolution of \( \Lambda \) as a right \( \Lambda \)-module we start with a projective presentation of \( \Lambda \):

\[
\begin{array}{c}
P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\pi} \Lambda \to 0.
\end{array}
\]

Here \( \Lambda \equiv \text{coker } \phi_1 \), that is, we choose the above sequence to be exact so that \( \Lambda \equiv P_0 / \text{im } \phi_2 \). The projective modules we desire are

\[
P_0 = \coprod_{v \in \Gamma_0} \Lambda v \otimes v \Lambda \quad \text{and} \quad P_1 = \coprod_{a \in \Gamma_1} \Lambda o(a) \otimes t(a) \Lambda,
\]

where \( \pi \) is the multiplication map and \( \phi_2(o(a) \otimes t(a)) = a \otimes t(a) - o(a) \otimes a \).

Now, let

\[
P_2 = \coprod_{i=1}^{m} \Lambda o(r_i) \otimes t(r_i) \Lambda,
\]

where each \( r_i \in \text{Minsharp } \text{coker } \phi_2 \) can be written \( r_i = \sum_{j=1}^{k} \alpha_{ij} p_{ij} \). Let \( p \) be a path in the support of \( r_i \), say \( p = a_{p_1} a_{p_2} \cdots a_{p_{(p)}} \),

\[
\begin{array}{c}
o(r_i) \xrightarrow{a_{p_1}} a_{p_2} \cdots a_{p_{(p)-1}} a_{p_{(p)}} \xrightarrow{t(r_i)} \Lambda.
\end{array}
\]
Define
\[ x_p = \sum_{d=1}^{l(p)} a_{p_d} \cdots a_{p_{d-1}} \otimes a_{p_{d+1}} \cdots a_{p_{l(p)}}, \]
where, for notational convenience, we let \( a_{p_0} a_{p_0} = o(r_i) \) for \( d = 1 \) and \( a_{p_{l(p)+1}} a_{p_{l(p)}} = t(r_i) \) for \( d = l(p) \). Define \( \phi_2 : P_2 \rightarrow P_1 \) by
\[ \phi_2 : o(r_i) \otimes t(r_i) \rightarrow \sum_{j=1}^{k(i)} \alpha_{ij} x_{p_{ij}}. \]

**Proposition 2.1.** The sequence \( P_2 \xrightarrow{\phi_2} P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\pi} \Lambda \rightarrow 0 \) is exact at \( P_1 \).

**Proof.** If \( p \in \text{support}(r_i) \) is the path we used in defining \( x_p \), we have \( \phi_2(x_p) \)
\[ = \sum_{d=1}^{l(p)} a_{p_d} \cdots a_{p_{d-1}} [\phi_1(o(a_{p_d}) \otimes t(a_{p_d}))] a_{p_{d+1}} \cdots a_{p_{l(p)}} \]
\[ = \sum_{d=1}^{l(p)} a_{p_d} \cdots a_{p_{d-1}} [a_{p_d} \otimes t(a_{p_d}) - o(a_{p_d}) \otimes a_{p_d}] a_{p_{d+1}} \cdots a_{p_{l(p)}} \]
\[ = a_{p_1} \cdots a_{p_{l(p)}} \otimes t(p) - o(p) \otimes a_{p_1} \cdots a_{p_{l(p)}} \]
\[ = p \otimes t(p) - o(p) \otimes p. \]

Then \( \phi_1 \phi_2(o(r_i) \otimes t(r_i)) = \phi_1(\sum_{j=1}^{k(i)} \alpha_{ij} x_{p_{ij}}) \)
\[ = \sum_{j=1}^{k(i)} x_{p_{ij}} \]
\[ = \sum_{j=1}^{k(i)} \alpha_{ij} [p_{ij} \otimes t(r_i) - o(r_i) \otimes p_{ij}] \]
\[ = \left( \sum_{j=1}^{k(i)} \alpha_{ij} p_{ij} \right) \otimes t(r_i) - o(r_i) \otimes \left( \sum_{j=1}^{k(i)} \alpha_{ij} p_{ij} \right) \]
\[ = r_i \otimes t(r_i) - o(r_i) \otimes r_i \]
\[ = 0. \]
It follows that \( \text{im } \phi_2 \subseteq \ker \phi_1 \).

To show the reverse inclusion we first need to extend our order \( < \) to \( P_0 \) and \( P_1 \). So let \( a_i, a_j \in \Gamma \). Suppose \( p \otimes q \in \Lambda o(a_i) \otimes t(a_i) \Lambda \) and


\[ r \otimes s \in \Lambda o(a_j) \otimes t(a_j) \Lambda. \text{ Then } p \otimes q > r \otimes s \text{ iff } \]

1. \( l(q) < l(s) \) or

2. \( l(q) = l(s) \) and \( l(p) > l(r) \); or

3. \( l(q) = l(s), \) \( l(p) = l(r), \) and \( q < s \); or

4. \( q = s, \) \( l(p) = l(r), \) and \( p > r \); or

5. \( q = s, p = r, \) and \( a_i > a_j. \)

By replacing the arrows in the above order with two vertices and dropping the last requirement we obtain the desired order on \( P_0. \) Now let \( x \in \ker \phi_I \) and reduce each term of \( x \) to its tensor normal form modulo \( I. \) Suppose \( \text{tip}(x) = p \otimes q \in \Lambda o(a) \otimes t(a) \Lambda. \text{ Then } \phi_I(p \otimes q) = p(a \otimes t(a) - o(a) \otimes a)q = pa \otimes q - p \otimes aq \text{ has tip } pa \otimes q. \) Let \( r \otimes s \in \text{support}(x), \) where \( r \otimes s \in \Lambda o(\hat{a}) \otimes t(\hat{a}) \Lambda. \text{ Then } \phi_I(r \otimes s) = \hat{r} \otimes s - r \otimes \hat{a}s \text{ has tip } \hat{r} \otimes s. \text{ So under the order on } P_1, p \otimes q > r \otimes s \text{ implies } pa \otimes q > \hat{r} \otimes s \text{ in } P_0. \text{ This means } pa \otimes q \text{ cannot cancel with any other term in the image of } x. \text{ Since } q \text{ is in its normal form modulo } I \text{ it must be the case that } \text{tip}(r_i) = \hat{p} \text{ divides } pa \text{ for some } r_i \in \text{Minsharp}_I(I). \text{ Since } \hat{p} \text{ does not divide } p, \text{ the last arrow of } \hat{p} \text{ must be } a:\

\begin{align*}
  & p \\
  & \downarrow \hat{p} \\
  & a_{p_1} \quad a_{p_2} \\
  & \downarrow \hat{p} \\
  & a_{p^{(p)-1}} \quad a_{p^{(p)}} = a \\
  & q
\end{align*}

Without loss of generality suppose \( r_i = a_i \hat{p} + \sum_{j=2}^{r_i} \alpha_j a_j: \)

\begin{align*}
  & o(r_i), \\
  & \downarrow a_{p_1} \\
  & \downarrow a_{p_2} \\
  & \ldots \\
  & q_r \\
  & \downarrow \hat{p} \\
  & \downarrow a_{p^{(p)-1}} \quad a_{p^{(p)}} = a \\
  & t(r_i)
\end{align*}
It follows that \( \phi_2(o(r_j) \otimes t(r_j)) = \alpha_j x_{p_j} + \sum_{j=2}^{r} \alpha_j x_{d_j} \) has tip \( a_{p_1} a_{p_2} \cdots \).

3. THE ASSOCIATED SEQUENCE OF PATHS

For the first two definitions suppose \( \text{Minsharp}_\prec(I) = \{p_1, p_2, \ldots, p_d\} \) is a finite collection of paths that lie along some directed path \( T \). By definition this means \( p_i \) does not divide \( p_j \) if \( i \neq j \).

**Definition.** Let \( p_i \in \text{Minsharp}_\prec(I) \). We define the associated sequence of paths corresponding to \( p_i \) inductively as follows: Let \( r_2 \in \text{Minsharp}_\prec(I) \) be the path (if it exists) in \( \text{Minsharp}_\prec(I) \) such that \( o(p_i) < o(r_2) < t(p_i) \) and \( o(r_2) \) is minimal with respect to this double equality. Now assume \( r_1, r_2, \ldots, r_j \) have been constructed, where \( r_1 = p_i \). Let

\[
L_{j+1} = \{ r \in \text{Minsharp}_\prec(I) : t(r_{j-1}) \leq o(r) < t(r_j) \}.
\]

If \( L_{j+1} \neq \emptyset \), let \( r_{j+1} \) be such that \( o(r_{j+1}) \) is minimal with respect to \( r_{j+1} \in L_{j+1} \).

This sequence of paths was first described in [GHZ]. We shall refer to this construction as the left construction for the associated paths. There is also a dual construction which we shall refer to as the right construction:

**Definition.** Assuming the same hypotheses as in the previous definition let \( r_2 \in \text{Minsharp}_\prec(I) \) be the path (if it exists) in \( \text{Minsharp}_\prec(I) \) such that \( o(r_2) < o(p_i) < t(r_2) \) and \( t(r_2) \) is maximal with respect to this double inequality. Now assume \( r_1, r_2, \ldots, r_j \) have been constructed. Let

\[
R_{j+1} = \{ r \in \text{Minsharp}_\prec(I) : o(r) < t(r) \leq o(r_{j-1}) \}.
\]

If \( R_{j+1} \neq \emptyset \), let \( r_{j+1} \) be such that \( t(r_{j+1}) \) is maximal with respect to \( r_{j+1} \in R_{j+1} \).

Consider the left construction. Given an integer \( n \), we refer to the sequence of the first \( n \) associated paths corresponding to \( r_1 = p_i \) by \( (r_1, r_2, \ldots, r_n) \) (if it exists). So far we have been doing this construction along the directed path \( T \). However, if \( p_i \in \text{Minsharp}_\prec(I) \) then \( p_i \) may be the starting point of many directed paths. In this case we will want to form
the associated sequence of paths over all possible directed paths \( T_k \) beginning at \( p_i \):

**Definition.** Let \( r_1 = p_i \) and define

\[
AS_i(n) = \{(r_1, \ldots, r_{n-1}) : (r_1, \ldots, r_{n-1}) \text{ is an associated sequence of paths}\}.
\]

For each \((r_1, \ldots, r_{n-1}) \in AS_i(n)\) define \( p_i^n \) to be the path from \( o(p_i) \) to \( t(r_{n-1}) \) along the directed path \( T_k \) used in the construction of \((r_1, \ldots, r_{n-1})\).

Let \( AP_i(n) \) be the set of all \( p_i^n \) constructed from \( AS_i(n) \). Suppose \( Minsharp_{<} (I) = \{p_1, \ldots, p_m\} \). Then the following definition is what we really need:

\[
AP(n) = \bigcup_{i=1}^{m} AP_i(n).
\]

Now, we can also dualize the above definitions for the right construction and obtain \( AP(n)^{op} \) which is the obvious analogue to \( AP(n) \). Note that \( AP(2) = AP(2)^{op} = Minsharp_{<} (I) \). The importance of using both constructions lies in the following result:

**Lemma 3.1.** \( AP(n) = AP(n)^{op} \) for \( n \geq 2 \).

The proof is left to the reader.

**Lemma 3.2.** Suppose \( m \) is even and \( p_1^m \neq p_2^m \in AP(m) \) overlap along a directed path \( T \) as follows:

\[ \alpha \quad p_1^m \quad p_2^m \quad \beta \]

If \( \alpha \) and \( \beta \) are not divisible by any path in \( Minsharp_{<} (I) \), then there exists some \( p_1^{m+1} \in AP(m+1) \) such that \( p_1^{m+1} \) divides the path from \( o(p_1^m) \) to \( t(p_2^m) \) along \( T \).

The proof of Lemma 3.2 is similar to the proof of Lemma 3.1 and is also omitted. The next result is crucial to the aforementioned alternating behavior. First we need one more definition:

**Definition.** Suppose \( p^n \in AP(n) \). Define \( Sub(p^n) = \{p^{n-1} \in AP(n-1) : p^{n-1} \text{ divides } p^n\} \).
Lemma 3.3. \( \text{Sub}(p^n) \) contains two paths \( p_o^{n-1} \) and \( p_t^{n-1} \), where \( o(p_o^{n-1}) = o(p^n) \) and \( t(p_t^{n-1}) = t(p^n) \). Furthermore, if \( n \) is odd then \( \text{Sub}(p^n) = \{p_o^{n-1}, p_t^{n-1}\} \).

Proof. The first part is immediate from the construction of \( AP(n) \) and \( AP(n)^p \) and the fact that these two constructions are the same. Note that \( p_o^{n-1} \) and \( p_t^{n-1} \) must properly divide \( p^n \). Now suppose \( n \) is odd:

![Diagram](image_url)

We know \( p_o^{n-1} \) and \( p_t^{n-1} \) overlap since, otherwise, \( p_t^{n-1} \) would have to properly divide the last path, call it \( p \), in the left construction of \( p^n \). For the same reason \( \alpha \) and \( \beta \) cannot be zero. Now suppose there exists some \( p^n-1 \in AP(n-1) \) such that \( p^n-1 \) divides \( p^n \) but is not \( p_o^{n-1} \) or \( p_t^{n-1} \), that is, \( o(p_o^{n-1}) < o(p^{n-1}) < o(p_t^{n-1}) \). We have the following situation:

![Diagram](image_url)

Since \( \alpha, \beta \neq 0 \), we know \( \hat{\alpha}, \hat{\beta} \neq 0 \). Now, \( n-1 \) is even so by Lemma 3.2 there exists some \( q^n \in AP(n) \) so that \( q^n \) divides the path from \( o(p_o^{n-1}) \) to \( t(p^{n-1}) \). But this means \( q^n \) properly divides \( p^n \), a contradiction.
Example. Suppose $\Gamma$ is the directed path $a_1a_2 \cdots a_{19}$ and, for notational convenience, we write each subpath of the form $a_i \cdots a_j$ as $a_{ij}$. Assume $\Gamma$ is generated by $AP(2) = \{a_1 \cdots a_4, a_2 \cdots a_5, a_3 \cdots a_6, a_4 \cdots a_8, a_6 \cdots a_9, a_7 \cdots a_{10}, a_8 \cdots a_{11}, a_9 \cdots a_{12}, a_{11} \cdots a_{13}, a_{12} \cdots a_{15}, a_{13} \cdots a_{16}, a_{15} \cdots a_{17}, a_{16} \cdots a_{18}, a_{17} \cdots a_{19}\}$. Then $AP(n) = \emptyset$ for $n \geq 10$. The nonempty $AP$ sets of paths are as follows:

$$AP(4) = \{a_2 \cdots a_9, a_3 \cdots a_{10}, a_4 \cdots a_{12}, a_7 \cdots a_8, a_8 \cdots a_{15}\},$$

$$AP(5) = \{a_2 \cdots a_{10}, a_3 \cdots a_{12}, a_4 \cdots a_{13}, a_7 \cdots a_8, a_8 \cdots a_{16}\},$$

$$AP(6) = \{a_3 \cdots a_{13}, a_4 \cdots a_{16}, a_7 \cdots a_8, a_8 \cdots a_{19}\},$$

$$AP(7) = \{a_3 \cdots a_{16}, a_4 \cdots a_{17}, a_7 \cdots a_8, a_8 \cdots a_{19}\},$$

$$AP(8) = \{a_3 \cdots a_{17}, a_4 \cdots a_{19}\},$$

$$AP(9) = \{a_3 \cdots a_{19}\}.$$  

It is clear that $Sub(a_1 \cdots a_{19}) = AP(8)$. However, the two $AP(8)$ paths each have a different number of divisors in $AP(7)$, i.e., $Sub(a_3 \cdots a_{17}) = \{a_3 \cdots a_{16}, a_4 \cdots a_{17}\}$ but $Sub(a_4 \cdots a_{19}) = \{a_4 \cdots a_{17}, a_7 \cdots a_{18}, a_8 \cdots a_{19}\}$. We see that each $AP(7)$ path has exactly two divisors in $AP(6)$. As things are more complicated for $AP(6)$. We have $Sub(a_3 \cdots a_{13}) = \{a_3 \cdots a_{12}, a_4 \cdots a_{13}\}$ but the rest of the $AP(6)$ paths each have three divisors in $AP(5)$. As expected, each $AP(5)$ path has two divisors in $AP(4)$. Most of the $AP(4)$ paths have three divisors in $AP(3)$. However, $Sub(a_4 \cdots a_{12}) = \{a_4 \cdots a_9, a_6 \cdots a_{10}, a_7 \cdots a_{11}, a_8 \cdots a_{12}\}$. Finally, it is clear that each $AP(3)$ path (which we did not list) has only two divisors in $AP(2)$.

To facilitate future computations we introduce some new notation: 

Definition. Suppose $m$ is even and $p_i^m, p_j^m \in AP(m)$ lie along the directed path $T$ with $o(p_i^m) < o(p_j^m)$ (relative to $T$). Define $L_i^j$ to be the subpath of $T$ from $o(p_i^m)$ to $o(p_j^m)$ and $R_i^j$ to be the subpath of $T$ from $t(p_i^m)$ to $t(p_j^m)$.

Note that it should be clear from context what $m$ is when using the above definition. Now suppose $n$ is odd and $p_n^m \in AP(n)$. Then by Lemma 3.3 we have $Sub(p_n^m) = \{p_n^{p_{n-1}}, p_n^{p_{n-2}}\}$. So we can write $p_n^m = L_1^2p_n^{p_{n-1}} = p_n^{p_{n-1}}R_1^2$. Similarly, if $p_n^{n+1} \in AP(n + 1)$ and $Sub(p_n^{n+1}) = \{p_1^n, p_2^n, \ldots, p_m^n\}$, then we can write $p_n^{n+1} = \theta_i p_i^n \mu_i$, for $i = 1, \ldots, m$. Here $\theta_i$ and $\mu_i$ are the
obvious complements of \( p^n \) in \( p^{n+1} \). This leads to one final result on the associated paths:

**Lemma 3.4.** \( L_1^2, R_1^1, \theta_i, \text{ and } \mu_i \) are not divisible by any path in \( \text{Minsharp}_< (I) \).

**Proof.** If \( p^n \) is constructed from the left using \( p_i^{n-1} \) and \( p_2 \), then \( R_1^1 \) must properly divide \( p_2 \). Similarly, if \( p^n \) is constructed from the right using \( p_2 \) and \( p_i^{n-1} \), then \( L_1^2 \) must properly divide \( p_2 \):

\[
\begin{array}{c}
p^n \\
\hline
p_1 \\
\hline
p_2 \\
\hline
L_1^2 \\
\hline
p_i^{n-1} \\
\hline
R_1^1 \\
\hline
p_i^{n-1} \\
\hline
\end{array}
\]

Thus, \( L_1^2, R_2^1 \neq 0 \).

We can use a similar argument for \( p^{n+1} \). Using the left construction with \( p_i^1 \) we see \( \mu_1 \) cannot be divisible by any \( \text{Minsharp}_< (I) \) path. Using the right construction with \( p_i^m \) we see \( \theta_m \) cannot be divisible by any \( \text{Minsharp}_< (I) \) path. The result follows since \( \mu_i \) divides \( \mu_1 \) for \( i = 1, \ldots, m \) and \( \theta_i \) divides \( \theta_m \) for \( i = 1, \ldots, m \).

We now have the machinery necessary for the construction of the minimal resolutions.

### 4. The Minimal Resolutions

Now we need to define the projective \( P_n, n \geq 2 \), for the case where \( \Lambda \) is a monomial algebra. This is the motivation behind the associated sequence of paths in Section 3.

**Definition.**

\[
P_n = \prod_{AP(n)} \Lambda o(p^n) \otimes t(p^n) \Lambda.
\]
There is a different and more general description of these projectives in [Ha]. To define the maps recall from Lemma 3.3 that if \( n \) is odd and \( p^n \in AP(n) \), then \( \text{Sub}(p^n) = \{ p_1^{n-1}, p_2^{n-1} \} \) and for \( p^{n+1} \in AP(n+1) \), \( \text{Sub}(p^{n+1}) = \{ p_1^n, p_2^n, \ldots, p_m^n \} \). So we can write \( p^n = L^2_1 p_2^{n-1} = p_3^{1-1} R^1_3 \) and \( p^{n+1} = \theta_i p^n \mu_i \) for \( i = 1, \ldots, m \). Let us now define the following maps:

\[
\begin{align*}
\phi_n &: o(p^n) \otimes t(p^n) = L^2_1 \otimes t(p^n) - o(p^n) \otimes R^1_2 \\
\phi_{n+1} &: o(p^{n+1}) \otimes t(p^{n+1}) = \sum_{i=1}^m \theta_i \otimes \mu_i.
\end{align*}
\]

These maps show now the alternating behavior of the syzygies is inherited from the construction of the associated sequences of paths. The reader has probably noticed that the second part of Lemma 3.3 appears to contradict this construction. However, if we let \( AP(0) = \Gamma_0 \) and \( AP(1) = \Gamma_1 \), then this alternating behavior is clear through \( AP(3) \). Given an arrow \( a \in AP(1) \) \((p^3 \in AP(3))\), there are obviously only two vertices in \( AP(0) \) (two paths in \( AP(2) \)) that divide \( a \) (divide \( p^3 \)). Conversely, given a path \( p \in AP(2) \), there are at least two arrows in \( AP(1) \) that divide \( p \). So even though the implications of Lemma 3.3 seem counterintuitive for \( AP(n), n > 3 \), the analogue of this lemma for \( AP(3), AP(2), \) and \( AP(1) \) is quite natural. We are now ready for the main theorem of this paper.

**Theorem 4.1.** Let \( \Gamma \) be a finite quiver and suppose \( \Lambda = k \Gamma / I \) is a monomial algebra. Furthermore, assume \( J^N \subset I \subset J^2 \) for some integer \( N \geq 2 \). Then the sequence

\[
\cdots \to P_{n+1} \xrightarrow{\phi_{n+1}} P_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_2} P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\pi} \Lambda \to 0
\]

is a minimal projective resolution of \( \Lambda \) as a right \( \Lambda \)-module.

Recall that the above resolution is minimal if \( \text{im} \phi_n \subset P_{n-1} \mathfrak{r}^e \), where \( \mathfrak{r}^e \) is the Jacobson radical of \( \Lambda \). By construction we know \( L^2_1 \) and \( R^1_2 \) are not vertices. Similarly, \( \theta_i \) and \( \mu_i \) are not simultaneously vertices for any given \( i = 1, \ldots, m \). Since \( \mathfrak{r}^e \) is generated by elements of the form \( a_i^{n} \otimes w \otimes v \) and \( a w \otimes a_j \), \( \theta_i, \mu_i \in \Gamma_1 \), \( v, w \in \Gamma_0 \), the minimality follows.

Before we proceed with the proof of Theorem 4.1 we first need to extend the order \( < \) to \( P_n \). Suppose \( p^n, p^n_j \in AP(n) \) with \( p \otimes q \in \Lambda o(p^n) \otimes t(p^n) \Lambda \) and \( r \otimes s \in \Lambda o(p^n) \otimes t(p^n) \Lambda \). Then we say \( p \otimes q > r \otimes s \) iff

1. \( l(q) < l(s) \) or
2. \( l(q) = l(s) \) and \( l(p^n) > l(p^n_j) \); or
3. \( l(q) = l(s) \), \( l(p^n) = l(p^n_j) \), and \( q < s \); or
4. \( q = s \), \( l(p^n) = l(p^n_j) \), and \( p^n > p^n_j \).
Note that this order is just a generalization of the order we put on $P_0$ and $P_1$. The importance of this order is the following lemma. We leave the proof to the reader.

**Lemma 4.2.** Suppose $p \otimes q > r \otimes s$ in $P_n$. Then $\text{tip } \phi_n(p \otimes q) > \text{tip } \phi_n(r \otimes s)$ in $P_{n-1}$.

5. EXACTNESS AT $P_2$

We need to show that $\text{im } \phi_2 = \ker \phi_2$. So let $p^3 \in AP(3)$ and suppose $\text{Sub}(p^3) = \{p_i, p_j\}$. Composing the necessary maps gives $\phi_2 \phi_3(o(p^3) \otimes \tau(p^3))$

$$= \phi_2(L_i^j \otimes \tau(p^3) - o(p^3) \otimes R_j^i)$$

$$= L_i^j \phi_2(o(p_j) \otimes \tau(p_j)) - \phi_2(o(p_i) \otimes \tau(p_i)) R_j^i$$

$$= L_i^j x_{p_j} - x_{p_i} R_j^i.$$

Now, suppose $p_i = a_1 a_2 \cdots a_{i-1} a_i \cdots a_{i+k}$ and $p_j = a_j a_{i+1} \cdots a_{i+k}$.

Then $L_i^j x_{p_j}$

$$= L_i^j \sum_{d=l}^{l+k+n} a_1 \cdots a_{d-1} \otimes a_{d+1} \cdots a_{l+k+n}$$

$$= L_i^j \sum_{d=l}^{l+k} a_1 \cdots a_{d-1} \otimes a_{d+1} \cdots a_{l+k+n}$$

$$= \sum_{d=l}^{l+k} a_1 \cdots a_{d-1} \otimes a_{d+1} \cdots a_{l+k+n}.$$

Similarly, $x_{p_i} R_j^i = \sum_{d=l}^{l+k+n} a_1 \cdots a_{d-1} \otimes a_{d+1} \cdots a_{l+k+n}$. It follows that $\phi_2 \phi_3 = 0$ and $\text{im } \phi_3 \subset \ker \phi_2$. 
To show the reverse inclusion let \( x \in \ker \phi_2 \) and reduce \( x \) to its tensor normal form modulo \( I \). Suppose \( \text{tip}(x) = p \otimes q \in \Lambda \circ \omega(p_i) \otimes t(p_i) \Lambda \), where \( p_i = a_1a_2 \cdots a_n \in \text{Minsharp}_< (I) \). Then \( \phi_2(p \otimes q) = px \cdot q \) has tip \( pa_1a_2 \cdots a_{n-1} \otimes q \). Now suppose \( r \otimes s \) is in the support of \( x \) where \( r \otimes s \in \Lambda \circ \omega(p_i) \otimes t(p_i) \Lambda \), \( p_j = b_1b_2 \cdots b_m \in \text{Minsharp}_< (I) \). Then \( \phi_2(r \otimes s) = rx \cdot p \) has tip \( rb_1b_2 \cdots b_{m-1} \otimes s \). So under the order on \( \mathbb{P} \), \( p \otimes q > r \otimes s \) implies \( pa_1a_2 \cdots a_{n-1} \otimes q > rb_1b_2 \cdots b_{m-1} \otimes s \) in \( \mathbb{P} \). Thus, \( \text{tip}(\phi_2(x)) = pa_1a_2 \cdots a_{n-1} \otimes q \) and \( pa_1a_2 \cdots a_{n-1} \otimes q \notin \text{support}(\phi_2(y)) \) for any \( y \) in the support of \( x \) (except \( p \otimes q \)). Since \( q \) is in its normal form modulo \( I \), it must be the case that \( p_k \) divides \( pa_1a_2 \cdots a_{n-1} \) for some \( p_k \in \text{Minsharp}_< (I) \). But \( p_k \) does not divide \( p \) and \( p_k \) does not divide \( a_1a_2 \cdots a_{n-1} \), so we must have the following situation:

\[
\begin{array}{c}
p \\
\theta \\
\hline
L_k^i \\
\hline
p_k \\
\hline
R_k^i \\
\end{array}
\]

Without loss of generality we can choose \( p_k \) such that \( p_k \) is minimal with respect to \( o(p_k) < o(p_i) < t(p_k) \). Then \( \Lambda \circ \omega(p_k) \otimes t(p_k) \Lambda \) is a summand of \( P_3 \) and \( \phi_3(o(p_k) \otimes t(p_k)) = L_k^i \otimes t(p_k) - o(p_k) \otimes R_k^i \) has tip \( L_k^i \otimes t(p_k) \).

We see \( L_k^i \otimes o(p_k) \) divides \( p \otimes q \) since \( (L_k^i \otimes t(p_k))(\theta \otimes q) = \theta L_k^i \otimes q = p \otimes q \). So \( x \) reduces over \( \text{im} \phi_3 \) and it follows that \( \text{im} \phi_3 = \ker \phi_2 \).

6. EXACTNESS AT \( P_3 \)

To show that \( \text{im} \phi_4 = \ker \phi_3 \) let \( p^4 \in AP(4) \), \( \text{Sub}(p^4) = \{ p_1^3, \ldots, p_n^3 \} \), and consider

\[
\phi_3 \phi_2(o(p^4) \otimes t(p^4)) = \sum_{i=1}^{n} \theta_i \phi_3(o(p_i^3) \otimes t(p_i^3)) \mu_i.
\]
To calculate this suppose $p_1^3$ is constructed from $p_1$ and $p_2$, $p_2^3$ is constructed from $p_2$ and $p_3$, ..., and $p_n^3$ is constructed from $p_n$ and $p_{n+1}$.

\begin{align*}
\theta_1 &= L_1^2 \\
\theta_3 &= L_1^2 L_2^3 = L_1^3 \\
\vdots \\
\theta_n &= L_1^2 \ldots L_{n-1}^3 = L_1^n \\
\mu_{n-1} &= R_{n+1}^n \\
\mu_{n-2} &= R_{n+1}^{n-1} R_{n+1}^n = R_{n+1}^{n-1} \\
\vdots \\
\mu_1 &= R_2^2 R_4^3 \ldots R_{n+1}^n = R_{n+1}^2
\end{align*}
Since \( \theta_1 = o(p_4) = L_1^1 \) and \( \mu_n = t(p_4) = R_{n+1}^{n+1} \), we conclude that \( \theta_i = L_i^1 \) and \( \mu_i = R_{i+1}^{i+1} \) for \( i = 1, \ldots, n \). Hence, \( \phi_3 \phi_4(o(p^4) \otimes t(p^4)) \)

\[
= \sum_{i=1}^n \theta_i \phi_3(o(p_i^3) \otimes t(p_i^3)) \mu_i \\
= \sum_{i=1}^n L_i^1(L_i^{i+1} \otimes t(p_i^3) - o(p_i^3) \otimes R_{i+1}^{i+1}) \mu_i \\
= \sum_{i=1}^n (L_i^{i+1} \otimes R_{i+1}^{i+1} - L_i^1 \otimes R_i^{i+1}) \\
= L_i^{i+1} \otimes t(p^4) - o(p^4) \otimes R_{i+1}^{i+1}.
\]

By the left construction of \( p^4 \) there must be some \( p \in \text{Minsharp}_<(I) \) such that \( t(p_1) \leq o(p) < t(p_2) \) and \( t(p) = t(p_3) \). But this means that \( p \) divides \( R_{n+1}^{n+1} \). Similarly, by the right construction of \( p^4 \), there must be some \( q \in \text{Minsharp}_<(I) \) such that \( t(q) \leq o(p_{n+1}) \) and \( o(q) = o(p^4) \). Thus, \( q \) divides \( L_{n+1}^{n+1} \). It follows that \( \phi_3 \phi_4 = 0 \) and \( \text{im } \phi_4 \subset \ker \phi_3 \).

To show the reverse inclusion let \( x \in \ker \phi_3 \) and reduce \( x \) to its tensor normal form modulo \( I \). Suppose \( \text{tip}(x) = p \otimes q \in \Lambda o(p_3^3) \otimes t(p_3^3) \Lambda \) where we assume \( p_3^3 \) is constructed from the \( \text{Minsharp}_<(I) \) paths \( p_1 \) and \( p_2 \):

\[
\begin{array}{c}
p \quad o(p_3^3) \\
\downarrow \quad \downarrow \quad \downarrow \\
L_1^2 \quad p_1 \quad R_2^1 \quad q \\
\downarrow \quad \downarrow \quad \downarrow \\
p_2 \quad t(p_3^3) \\
\end{array}
\]

Then \( \phi_3(p \otimes q) \)

\[
= p \phi_3(o(p_3^3) \otimes t(p_3^3))q \\
= p(L_1^2 \otimes t(p_3^3) - o(p_3^3) \otimes R_2^1)q \\
= pL_1^2 \otimes q - p \otimes R_2^1 q
\]

has tip \( pL_1^2 \otimes q \). By Lemma 4.2 \( \text{tip}(\phi_3(x)) = pL_1^2 \otimes q \) and \( pL_1^2 \otimes q \) is not an element of \( \text{support}(\phi_3(y)) \) for any \( y \in \text{support}(x) \) (except \( p \otimes q \)).
Now, $q$ is in its normal form modulo $I$ so it must be the case that $p_k$ divides $pL_2^1$ for some $p_k \in \operatorname{Minsharp}_< (I)$. Since $p_k$ does not divide $p$ and $p_k$ does not divide $L_2^1$, we must have the following situation:

\[
\begin{array}{c}
    p \rightarrow o(p_1^3) \rightarrow p_1 \rightarrow L_1^2 \rightarrow p_2 \rightarrow R_2^1 \rightarrow q \\
    p_k
\end{array}
\]

So $o(p) \leq o(p_k) < t(p)$ and $o(p_1) < t(p_k) \leq o(p_2)$. Without loss of generality we can assume $p_k$ is maximal with respect to $o(p_1) < t(p_k) \leq o(p_2)$. Then $\Lambda o(p_k) \otimes t(p_2) \Lambda$ is a summand of $P_2$, say $p_4^3$ is the path from $o(p_k)$ to $t(p_2)$ by the right construction. Suppose $\operatorname{Sub}(p_4^3) = (p_3^1, \ldots, p_2^3)$:

\[
\begin{array}{c}
    p_1^3 \rightarrow \theta_1 \rightarrow p_2^3 \rightarrow \theta_2 \rightarrow \cdots \rightarrow p_{n-1}^3 \rightarrow \theta_{n-1} \\
    o(p_4) \rightarrow \theta_n \rightarrow p_n^3 \rightarrow t(p_4)
\end{array}
\]

By construction $p_n^3 = p_2^3$. Then $\phi_4(o(p_4) \otimes t(p_4^3)) = \sum_{j=1}^n \theta_j \otimes \mu_j$ has tip $\theta_n \otimes t(p_4^3)$. We have the following:

\[
\begin{array}{c}
    p \rightarrow o(p_1^3) \rightarrow p_1 \rightarrow L_1^2 \rightarrow p_2 \rightarrow R_2^1 \rightarrow q \\
    \theta_n
\end{array}
\]
We see $\theta_n$ divides $p$ and $(\theta_n \otimes \tau(p)) \hat{\theta} \otimes q = \hat{\theta} \otimes q = p \otimes q$, that is, $\text{tip}(\theta_n \otimes \tau(p)) \text{tip}(x)$ divides $\text{tip}(x)$. Hence, $x$ reduces over $im \phi_4$, and it follows that $im \phi_4 = ker \phi_3$.

Now let us assume that Theorem 4.1 is true for $P_{n+1} \xrightarrow{\phi_{n+1}} P_n \xrightarrow{\phi_n} P_{n-1} \xrightarrow{\phi_{n-1}} \cdots \to \Lambda \to 0$. Exactness at $P_{n}$ follows from the same argument used to show $im \phi_4 = ker \phi_3$. So we just need exactness at $P_{n-1}$. The details are tedious so we will just outline the proof. To see $im \phi_{n-1} \phi_n = 0$, let $p^n \in AP(n)$ and suppose $\text{Sub}(p^n) = (p_{\alpha}^{\beta} - 1, p_{\beta}^{\alpha} - 1)$. Then $\phi_{n-1} \phi_n (o(p^n) \otimes \tau(p^n)) = L^2 \phi_{n-1} \phi_n (o(p_{\alpha}^{\beta} - 1) \otimes \tau(p_{\beta}^{\alpha} - 1)) - \phi_{n-1} \phi_n (o(p_{\alpha}^{\beta} - 1) \otimes \tau(p_{\beta}^{\alpha} - 1))R^1_{q^n}$. To calculate this we need to know the divisors of $p_{\alpha}^{\beta} - 1$ and $p_{\beta}^{\alpha} - 1$. Using the lemmas from Section 3 one can show that $\text{Sub}(p_{\alpha}^{\beta} - 1) \cap \text{Sub}(p_{\beta}^{\alpha} - 1) \neq \emptyset$. So suppose $\text{Sub}(p_{\alpha}^{\beta} - 1) = (p_{\alpha}^{\beta - 2}, \ldots, p_{\alpha}^{\beta - 2n}, \ldots, p_{\alpha}^{\beta - 2m})$ and $\text{Sub}(p_{\beta}^{\alpha} - 1) = (p_{\beta}^{\alpha - 2}, \ldots, p_{\beta}^{\alpha - 2n}, \ldots, p_{\beta}^{\alpha - 2m})$. When one evaluates $\phi_{n-1} \phi_n (o(p^n) \otimes \tau(p^n))$, a long sum is obtained where the middle $2m$ terms cancel. This cancellation results from the $AP(n-2)$ paths in the intersection of the two Sub sets. Using the lemmas from Section 3 one can then show that the remaining terms are zero. So $\phi_{n-1} \phi_n = 0$. The other inclusion follows from the reduction technique used for the lower syzygies.

7. TWO EXAMPLES

Example 1. First let us calculate the cohomology algebra for $\Lambda = k \Gamma/I$ where $I = (a_1a_2, a_2a_3, a_3a_4)$ and $\Gamma$ is the following quiver:

\[
\begin{array}{c}
V_2 \\
\downarrow a_1 \\
V_1 & \xrightarrow{a_2} & V_3
\end{array}
\]

By constructing the associated sequences of paths for each relation in $I$ we have the following:

\[
\begin{align*}
P_0 &= \Lambda v_1 \otimes v_1 \Lambda + \Lambda v_2 \otimes v_2 \Lambda + \Lambda v_3 \otimes v_3 \Lambda = P_3 = P_6 = \cdots \\
P_1 &= \Lambda v_1 \otimes v_2 \Lambda + \Lambda v_2 \otimes v_3 \Lambda + \Lambda v_3 \otimes v_1 \Lambda = P_4 = P_7 = \cdots \\
P_2 &= \Lambda v_1 \otimes v_3 \Lambda + \Lambda v_2 \otimes v_1 \Lambda + \Lambda v_3 \otimes v_2 \Lambda = P_5 = P_8 = \cdots
\end{align*}
\]

Applying $\text{Hom}_k(\cdot, \Lambda)$ to the projective resolution of $\Lambda$ yields the sequence

\[
0 \to \hat{P}_0 \xrightarrow{\hat{\phi}_1} \hat{P}_1 \xrightarrow{\hat{\phi}_2} 0 \xrightarrow{0} \hat{P}_0 \xrightarrow{\hat{\phi}_4} \hat{P}_1 \xrightarrow{0} 0 \to \cdots.
\]
where
\[
\hat{P}_0 = v_1\Lambda v_1 + v_2\Lambda v_2 + v_3\Lambda v_3 \\
\hat{P}_1 = v_1\Lambda v_2 + v_2\Lambda v_3 + v_3\Lambda v_1 \\
\hat{P}_2 = v_1\Lambda v_3 + v_2\Lambda v_1 + v_3\Lambda v_2 = 0
\]

and \(\hat{\phi}_t\) is the map induced by \(\phi_t\).

Since the new complex has a periodicity of six, the cohomology algebra of \(\Lambda\) is determined by \(H^k(\Lambda, \Lambda), \ldots, H^5(\Lambda, \Lambda)\). Obviously \(H^5(\Lambda, \Lambda) = H^5(\Lambda, \Lambda) = 0\). Let us first calculate \(\hat{P}_0 = (k, v_1 + k_2v_2 + k_3v_3; k_i \in k)\) and
\[
\begin{align*}
\hat{\phi}_1 &\to a_3 - a_1 \\
\hat{\phi}_2 &\to a_1 - a_2 \\
\hat{\phi}_3 &\to a_2 - a_3.
\end{align*}
\]

So if \(\alpha = k_1v_1 + k_2v_2 + k_3v_3\), then \(\hat{\phi}_t(\alpha) = (k_2 - k_1)a_1 + (k_3 - k_2)a_2 + (k_1 - k_3)a_3\). But then \(\alpha \in \ker \phi_t\) iff \(k_1 = k_2 = k_3\). Thus, \(H^3(\Lambda, \Lambda) = (k, v_1 + v_2 + v_3; k_i \in k) \cong k\), as expected. Since \(H^4(\Lambda, \Lambda) = \hat{P}_1 / \text{Im} \hat{\phi}_t\) and \(\phi_t\) is not surjective, we conclude that \(H^4(\Lambda, \Lambda) \neq 0\). To calculate \(H^5(\Lambda, \Lambda) = \ker \hat{\phi}_4\) we note that
\[
\begin{align*}
\hat{\phi}_1 &\to a_1 + a_3 \\
\hat{\phi}_2 &\to a_1 + a_2 \\
\hat{\phi}_3 &\to a_2 + a_3.
\end{align*}
\]

If \(\alpha = k_1v_1 + k_2v_2 + k_3v_3 \in \ker \hat{\phi}_4\), we have \(\hat{\phi}_4(\alpha) = (k_1 + k_2)a_1 + (k_2 + k_3)a_2 + (k_1 + k_3)a_3 = 0\). But this is possible iff \(k_1 = -k_2 = k_3 = -k_3\) = 0. So \(H^5(\Lambda, \Lambda) = 0\). Finally, let us calculate \(H^4(\Lambda, \Lambda) = \hat{P}_1 / \text{Im} \hat{\phi}_4\). To see that \(\phi_4\) is surjective, let \(l_1a_1 + l_2a_2 + l_3a_3 \in \hat{P}_1\). If we assume that the characteristic of \(k\) does not equal 2 and let \(k_1 = 1/2(l_2 - l_2 + l_3)\), \(k_2 = 1/2(l_2 + l_2 - l_3)\), and \(k_3 = 1/2(l_2 - l_1 + l_3)\), then \(\phi_4(k_1v_1 + k_2v_2 + k_3v_3) = l_1a_1 + l_2a_2 + l_3a_3\). It follows that \(H^4(\Lambda, \Lambda) = 0\). After reindexing we obtain the cohomology algebra of \(\Lambda\):
\[
H^*(\Lambda) = \prod_{1}^{\infty} H^0(\Lambda, \Lambda) \oplus H^1(\Lambda, \Lambda).
\]

**Example 2.** In the previous example we looked at an oriented cycle with three arrows and let \(I\) be the ideal generated by the three paths of length two. Now consider an oriented cycle \(I\) with four arrows and
suppose the ideal $I$ is generated by the four paths of length two. Then it is not hard to show that $H^*(\Lambda)$ is determined by the following complex:

$$0 \to \hat{P}_0 \xrightarrow{\phi_1} \hat{P}_1 \to 0 \to 0 \to \hat{P}_0 \xrightarrow{\phi_1} \hat{P}_1 \to 0 \to \cdots.$$ 

Note that this complex has a periodicity of four. As before, $H^*(\Lambda)$ is determined by $H^0(\Lambda, \Lambda)$ and $H^1(\Lambda, \Lambda)$. However, in this example the periodicity of our complex equals the number of relations whereas in the last example the periodicity equals twice the number of relations. This apparent discrepancy is a result of the fact that a periodic resolution must have an even period. Even though the projectives can have an odd period, the maps cannot. Otherwise the signs of the maps would not alternate in the manner previously described. So we see how the construction of the associated sequence of paths in the first example has led to a period doubling. This contrasts with the second example where the periodicity of the projectives equals the periodicity of the maps.

More general results on Hochschild cohomology will be presented in a future paper.

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