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Parabolic Kazhdan–Lusztig polynomials, plethysm and generalized Hall–Littlewood functions for classical types

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Abstract

We use power sums plethysm operators to introduce H functions which interpolate between the Weyl characters and the Hall–Littlewood functions Q' corresponding to classical Lie groups. The coefficients of these functions on the basis of Weyl characters are parabolic Kazhdan–Lusztig polynomials and thus, by works of Kashiwara and Tanisaki, are nonnegative. We prove that they can be regarded as quantizations of branching coefficients obtained by restriction to certain subgroups of Levi type. The H functions associated to linear groups coincide with the polynomials introduced by Lascoux, Leclerc and Thibon in [A. Lascoux, B. Leclerc, J.Y. Thibon, Ribbon tableaux, Hall Littelwood functions, quantum affine algebras, J. Math. Phys. 38 (1996) 1041–1068].

1. Introduction

Given μ a partition with at most n parts, the Hall Littlewood function Q'_μ can be defined by

$$Q'_\mu = \sum_\lambda K_{\lambda,\mu}(q) s_\lambda$$

where the sum runs over the partitions of length at most n, $K_{\lambda,\mu}(q)$ is the Lusztig q-analogue of weight multiplicity associated with (λ, μ) and s_λ the Schur function indexed by λ, that is the Weyl character of the irreducible finite dimensional GL_n-module $V(\lambda)$. Since $K_{\lambda,\mu}(1)$ is equal to the dimension of the weight space μ in $V(\lambda)$, Q'_μ can be regarded as a quantization

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of the homogeneous function h_μ. In [8], Lascoux, Leclerc and Thibon have introduced a new family of symmetric functions H_μ^ℓ depending on a fixed nonnegative integer ℓ which interpolate between the Schur functions s_μ and the Hall–Littlewood functions Q'_μ. The polynomials H_μ^ℓ can be combinatorially described in terms of the spin statistic on certain generalized Young tableaux called ℓ-ribbon tableaux. These ribbon tableaux naturally appear in the description of the action of the power sum plethysm ψ_ℓ on symmetric functions. Recall that for any symmetric function f, $\psi_\ell, (f)$ is obtained by replacing in f each variable x_i by x_i^ℓ. In particular, ψ_ℓ multiplies the degrees by ℓ. The space of symmetric functions is endowed with an inner product $\langle\cdot,\cdot\rangle$ which makes the basis of the Schur functions orthonormal. Then φ_ℓ, the adjoint operator of ψ_ℓ, divides the degrees by ℓ. It is well known that $\varphi_\ell(s_\mu)$ can be computed from the Jacobi–Trudi determinantal identity. Specifically, we have

$$\varphi_\ell(s_\mu) = 0 \quad \text{or} \quad \varphi_\ell(s_\mu) = \varepsilon(\sigma_0) s_{\mu^{(0)}} \cdots s_{\mu^{(\ell-1)}} \tag{1}$$

where $\varepsilon(\sigma_0) = \pm 1$ is the signature of a permutation $\sigma_0 \in S_n$ and $(\mu^{(0)}, \ldots, \mu^{(\ell-1)})$ a ℓ-tuple of partitions defined by ℓ and μ. By expanding $\varphi_\ell(s_\mu)$ on the basis of Schur functions, we obtain

$$\varphi_\ell(s_\mu) = \varepsilon(\sigma_0) \sum_\lambda c^\lambda_{\mu^{(0)},\ldots,\mu^{(\ell-1)}} s_\lambda \tag{2}$$

where $c^\lambda_{\mu^{(0)},\ldots,\mu^{(\ell-1)}}$ is the Littlewood–Richardson coefficient giving the multiplicity of $V(\lambda)$ in the tensor product $V(\mu^{(0)}) \otimes \cdots \otimes V(\mu^{(\ell-1)})$. When $\ell = 1$, one has $\varphi_\ell(s_{\ell\mu}) = s_\mu$ and when $\ell > n$, one can prove that $\varphi_\ell(s_{\ell\mu}) = h_\mu$. Thus the functions $h_\mu^{(\ell)} = \varepsilon(\sigma_0)\varphi_\ell(s_{\ell\mu})$ interpolate between the functions s_μ and h_μ and have nonnegative coefficients on the basis of the Schur functions.

In [8], the authors have interpreted the algebra of symmetric functions as the bosonic Fock space representation of the quantum affine Lie algebra $U_q(\widehat{sl_n})$. This permits us to introduce a natural quantization $\psi_{q,\ell}$ of the power sum plethysm ψ_ℓ. Let $\varphi_{q,\ell}$ be the adjoint operator of $\psi_{q,\ell}$ with respect to $\langle\cdot,\cdot\rangle$. The function H_μ^ℓ is then defined as a simple renormalization of $\varphi_{q,\ell}(s_{\ell\mu})$. This gives an identity of the form

$$H_\mu^\ell = \sum_\lambda c^\lambda_{\mu^{(0)},\ldots,\mu^{(\ell-1)}}(q) s_\lambda$$

where the polynomial $c^\lambda_{\mu^{(0)},\ldots,\mu^{(\ell-1)}}(q)$ is a q-analogue of $c^\lambda_{\mu^{(0)},\ldots,\mu^{(\ell-1)}}$.

Lusztig's q-analogues $K_{\lambda,\mu}(q)$ are particular affine Kazhdan–Lusztig polynomials. These polynomials arise in affine Hecke algebra theory as the entries of the transition matrix between the natural basis and a special basis defined by Lusztig. By replacing the affine Hecke algebra $\widehat{H}$ by one of its parabolic modules $\widehat{H}\nu$ (ν being a weight of the affine root system under consideration), Deodhar has introduced analogues of the Kazhdan–Lusztig polynomials. In [9], it is shown that the family constituted by these parabolic Kazhdan–Lusztig polynomials contains in particular the q-analogues $c^\lambda_{\mu^{(0)},\ldots,\mu^{(\ell-1)}}(q)$. By a result of Kashiwara and Tanisaki [7], this implies notably that the coefficients of the polynomial $c^\lambda_{\mu^{(0)},\ldots,\mu^{(\ell-1)}}(q)$ are nonnegative integers.

The aim of the paper is to introduce analogues of the polynomials H_μ^ℓ for the classical Lie groups $G = SO_{2n+1}, Sp_{2n}$ and SO_{2n} which interpolate between the Weyl characters and the Hall–Littlewood functions associated with G. Write also s_λ for the Weyl character of the irreducible G-module $V(\lambda)$ of highest weight λ. We define the plethysm operator φ_ℓ and its dual ψ_ℓ on the $\mathbb{Z}$-algebra generated by these Weyl characters. By a subgroup $L \subset G$ of Levi type, we mean a subgroup of G isomorphic to the Levi subgroup of one of its parabolic subgroups. Given γ a highest weight of L, we denote by $[V(\lambda) : V_L(\gamma)]$ the multiplicity of the irreducible L-module $V_L(\gamma)$ of highest weight γ in the restriction of $V(\lambda)$ to L. Then, provided that ℓ is odd when $G = Sp_{2n}$ or SO_{2n}, we establish for any Weyl character s_μ such that $\varphi_\ell(s_\mu) \neq 0$, a formula of the type

$$\varphi_\ell(s_\mu) = \varepsilon(w_0) \sum_\lambda \left[V(\lambda) : V_L \binom{\mu}{\ell} \right] s_\lambda \tag{3}$$

where $\varepsilon(w_0)$ is the signature of an element $w_0 \in W$ the Weyl group of G, L a subgroup of Levi type of G and $\binom{\mu}{\ell}$ a dominant weight associated with L. The procedure which yields w_0, L and $\binom{\mu}{\ell}$ from ℓ and μ can be regarded as an analogue of the algorithm computing the ℓ-quotient of a partition which implicitly appears in (1). The identity (2) can also be rewritten as in (3). Indeed, take $L = GL_{r_0} \times \cdots \times GL_{r_{\ell-1}}$ where for any $k = 1, \ldots, \ell - 1$, r_k is the length of $\mu^{(k)}$. Then $\binom{\mu}{\ell} = (\mu^{(0)}, \ldots, \mu^{(\ell-1)})$ can be interpreted as a dominant weight for the subgroup of Levi type L of GL_n, and we have the duality $c^\lambda_{\mu^{(0)},\ldots,\mu^{(\ell-1)}} = [V(\lambda) : V_L \binom{\mu}{\ell}]$.

The surprising constraint ℓ odd when $G = Sp_{2n}$ or SO_{2n} appearing in (3) follows from the fact that the procedure giving w_0, L and $\binom{\mu}{\ell}$ mentioned above depends not only on the Lie group under consideration, but also on the parity of the integer ℓ. For $G = SO_{2n+1}$ the coefficients of $\varepsilon(w_0)\varphi_\ell(s_\mu)$ on the basis of Weyl characters are always branching coefficients corresponding to restriction to L. For $G = Sp_{2n}$ or SO_{2n} this is only true when ℓ is odd. Note that this difficulty disappears for large ranks, that is for $n \geq \ell\,|\mu|$ (but see Section 6.4).

To define the functions H^ℓ_μ in type B, C or D, we prove the equalities

$$\left|\langle \psi_\ell(s_\lambda), s_\mu \rangle\right| = \left|\langle s_\lambda, \varphi_\ell(s_\mu)\rangle\right| = P^-_{\mu+\rho,\ell\lambda+\rho}(1)$$

which show that the coefficients of the expansion of $\psi_\ell(s_\lambda)$ on the basis of Weyl characters are, up to a sign, parabolic Kazhdan–Lusztig polynomials specialized at $q = 1$. By using (3) this gives, providing ℓ is odd for $G = Sp_{2n}$ or SO_{2n}

$$\left[V(\lambda) : V_L \binom{\mu}{\ell} \right] = P^-_{\mu+\rho,\ell\lambda+\rho}(1).$$

We then introduce the functions

$$G^\ell_\mu = \sum_\lambda \left[V(\lambda) : V_L \binom{\mu}{\ell} \right]_q s_\lambda$$

where $[V(\lambda) : V_L \binom{\mu}{\ell}]_q = P^-_{\mu+\rho,\ell\lambda+\rho}(q)$. This yields nonnegative q-analogues of the branching coefficients $[V(\lambda) : V_L \binom{\mu}{\ell}]$. The functions H^ℓ_μ are then defined by setting $H^\ell_\mu = G^\ell_{\ell\mu}$. We obtain the identities $H^1_\mu = s_\mu$ and $H^\ell_\mu = Q'_\mu$ when ℓ is sufficiently large. Thus the functions H^ℓ_μ interpolate between the Weyl characters and the Hall–Littlewood functions associated with G.

The paper is organized as follows. In Section 2 we recall the necessary background on classical root systems, Weyl characters, subgroups of Levi type and their corresponding branching coefficients. In Section 3, we define the plethysm operators ψ_ℓ and their dual operators φ_ℓ. By abuse of notation, we also denote by φ_ℓ the linear operator on the group algebra $\mathbb{Z}[\mathbb{Z}^n]$ with basis the formal exponentials (e^β) such that

$$\varphi_\ell(\mathrm{e}^\beta) = \begin{cases} \mathrm{e}^{\beta/\ell} & \text{if } \beta \in (\ell\mathbb{Z})^n \\ 0 & \text{otherwise} \end{cases} \qquad \text{for any } \beta \in \mathbb{Z}^n.$$

We then show how the determination of $\varphi_\ell(s_\mu)$ can be reduced to the computation of the polynomial

$$\varphi_\ell\left(\mathrm{e}^\mu \prod_{\alpha\in R_+}(1-\mathrm{e}^\alpha)\right)$$

where R_+ is the set of positive roots corresponding to the Lie group G. This permits us to establish formulas (3) providing ℓ is odd when $G = Sp_{2n}$ or SO_{2n}. For completeness, we have also included the case $G = GL_n$ and shown why (3) cannot hold when ℓ is even and $G = Sp_{2n}$ or SO_{2n}. To make the paper self-contained, we have summarized in Section 4 some necessary results on affine Hecke algebras and parabolic Kazhdan–Lusztig polynomials. Section 5 is devoted to the definition of the polynomials G_μ^ℓ and H_μ^ℓ and to their links with the Weyl characters and the Hall–Littlewood functions. Finally, we briefly discuss in Section 6 the problem of defining nonnegative q-analogues of tensor product multiplicities when $G \neq GL_n$. We add also a few remarks concerning the exceptional root systems.

2. Background

2.1. Classical root systems

In the sequel, G is one of the complex Lie groups GL_n, Sp_{2n}, SO_{2n+1} or SO_{2n} and $\mathfrak{g}$ its Lie algebra. We follow the convention of [7] to realize G as a subgroup of GL_N and $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}_N$, where

$$N = \begin{cases} n & \text{when } G = GL_n \\ 2n & \text{when } G = Sp_{2n} \text{ or } SO_{2n} \\ 2n+1 & \text{when } G = SO_{2n+1}. \end{cases}$$

With this convention, the maximal torus T of G and the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ coincide respectively with the subgroup and the subalgebra of diagonal matrices of G and $\mathfrak{g}$. Similarly, the Borel subgroup B of G and the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ coincide respectively with the subgroup and subalgebra of upper triangular matrices of G and $\mathfrak{g}$.

Let d_N be the linear subspace of $\mathfrak{gl}_N$ consisting of the diagonal matrices. For any $i \in I_n = \{1, \ldots, n\}$, write ε_i for the linear map $\varepsilon_i : d_N \to \mathbb{C}$ such that $\varepsilon_i(D) = \delta_{n-i+1}$ for any diagonal matrix D whose (i,i)-coefficient is δ_i. Then $(\varepsilon_1, \ldots, \varepsilon_n)$ is an orthonormal basis of the Euclidean space $\mathfrak{h}_{\mathbb{R}}^*$ (the real part of $\mathfrak{h}^*$). Let $(\cdot,\cdot)$ be the corresponding nondegenerate symmetric bilinear form defined on $\mathfrak{h}_{\mathbb{R}}^*$. Write R for the root system associated with G. For any $\alpha \in R$, we set $\alpha^\vee = \frac{\alpha}{(\alpha,\alpha)}$. The Lie algebra $\mathfrak{g}$ admits the diagonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \coprod_{\alpha\in R} \mathfrak{g}_\alpha$. We take for the set of positive roots:

$$\begin{cases} R^+ = \{\varepsilon_j - \varepsilon_i \text{ with } 1 \le i < j \le n\} \quad \text{for the root system } A_{n-1} \\ R^+ = \{\varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \text{ with } 1 \le i < j \le n\} \cup \{\varepsilon_i \text{ with } 1 \le i \le n\} \\ \quad \text{for the root system } B_n \\ R^+ = \{\varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \text{ with } 1 \le i < j \le n\} \cup \{2\varepsilon_i \text{ with } 1 \le i \le n\} \\ \quad \text{for the root system } C_n \\ R^+ = \{\varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \text{ with } 1 \le i < j \le n\} \quad \text{for the root system } D_n. \end{cases}$$

Let ρ be the half sum of positive roots. Set $J_n = \{\overline{n} < \cdots < \overline{1} < 1 < \cdots < n\}$ where, for each integer $i = 1, \ldots, n$, we have written $\overline{i}$ for the negative integer $-i$. For any $x \in J_n$ we have $\overline{\overline{x}} = x$ and we set $|x| = x$ if $x > 0$, $|x| = \overline{x}$ otherwise. Given a subset $U \subset J_n$, we define $|U| = \{|x| \mid x \in U\}$ and $\overline{U} = \{\overline{x} \mid x \in U\}$.

The Weyl group of GL_n is the symmetric group S_n and for $G = SO_{2n+1}$, Sp_{2n} or SO_{2n}, the Weyl group W of the Lie group G is the subgroup of the permutation group of J_n generated by the permutations

$$\begin{cases} s_i = (i, i+1)(\overline{i}, \overline{i+1}), \quad i = 1, \dots, n-1 \quad \text{and} \quad s_n = (n, \overline{n}) \\ \quad \text{for the root systems } B_n \text{ and } C_n \\ s_i = (i, i+1)(\overline{i}, \overline{i+1}), \quad i = 1, \dots, n-1 \quad \text{and} \quad s_n' = (n, \overline{n-1})(n-1, \overline{n}) \\ \quad \text{for the root system } D_n \end{cases}$$

where for $a \neq b$ (a, b) is the simple transposition which switches a and b. For types B_n and C_n, W is the group of signed permutations. It is the subgroup of the permutation group of J_n consisting of the permutations w such that $w(\overline{i}) = \overline{w(i)}$. For type D_n, the elements of W verify the additional constraint $\text{card}\{i \in I_n \mid w(i) < 0\} \in 2\mathbb{N}$. We identify the subgroup of W generated by $s_i = (i, i+1)(\overline{i}, \overline{i+1})$, $i = 1, \dots, n-1$ with the symmetric group S_n. The signature ε of $w \in W$ is defined by $\varepsilon(w) = (-1)^{l(w)}$, where l is the length function corresponding to the above sets of generators. Consider the increasing sequence $K = (\overline{i}_p, \dots, \overline{i}_1, i_1, \dots, i_p) \subset J_n$. For $X = B, D$ set

$$W_{X,K} = \{w \in W \text{ of type } X_n \mid w(x) = x \text{ for any } x \notin K\}.$$

Then, $W_{X,K}$ is isomorphic to the Weyl group of type X_p. Let $\varepsilon_{X,K}$ be the corresponding signature.

Lemma 2.1.1. *Consider $X = B, D$ and $w \in W_{X,K}$. Then we have $\varepsilon_{X,K}(w) = \varepsilon(w)$.*

Proof. Suppose $X = B$. The generators of the Weyl group $W_{X,K}$ are the $t_k = (i_k, i_{k+1})(\overline{i}_k, \overline{i}_{k+1})$, $k = 1, \dots, p-1$ and $s_n = (\overline{i}_p, i_p)$. One verifies easily that, considered as elements of W, they are of odd length. We proceed similarly when $X = D$. ■

The action of $w \in W$ on $\beta = (\beta_1, \dots, \beta_n) \in \mathfrak{h}_{\mathbb{R}}^*$ is defined by

$$w \cdot (\beta_1, \dots, \beta_n) = (\beta_1^{w^{-1}}, \dots, \beta_n^{w^{-1}}) \tag{4}$$

where $\beta_i^w = \beta_{w(i)}$ if $w(i) \in I_n$ and $\beta_i^w = -\beta_{w(\overline{i})}$ otherwise. The dot action of W on $\beta = (\beta_1, \dots, \beta_n) \in \mathfrak{h}_{\mathbb{R}}^*$ is defined by

$$w \circ \beta = w \cdot (\beta + \rho) - \rho. \tag{5}$$

The fundamental weights of $\mathfrak{g}$ belong to $\left(\frac{\mathbb{Z}}{2}\right)^n$. More precisely, we have $\omega_i = (0^i, 1^i) \in \mathbb{N}^n$ for $i < n-1$ and also $i = n-1$ for $\mathfrak{g} \neq \mathfrak{so}_{2n}$, $\omega_n^{C_n} = (1^n)$, $\omega_n^{B_n} = \omega_n^{D_n} = (\frac{1}{2}^n)$ and $\omega_{n-1}^{D_n} = (-\frac{1}{2}, \frac{1}{2}^{n-1})$. The weight lattice P of $\mathfrak{g}$ can be considered as the $\mathbb{Z}$-sublattice of $\left(\frac{\mathbb{Z}}{2}\right)^n$ generated by the ω_i, $i \in I$. For any $\beta = (\beta_1, \dots, \beta_n) \in P$, we set $|\beta| = \beta_1 + \dots + \beta_n$. Write P^+ for the cone of dominant weights of G. With our convention, a partition of length m is a weakly *increasing* sequence of m nonnegative integers. Denote by $\mathcal{P}_n$ the set of partitions with at most n parts. Each partition $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathcal{P}_n$ will be identified with the dominant weight $\sum_{i=1}^m \lambda_i \varepsilon_i$. Then the irreducible finite dimensional polynomial representations of G are parametrized by the partitions of $\mathcal{P}_n$. For any $\lambda \in \mathcal{P}_n$, denote by $V(\lambda)$ the irreducible finite dimensional representation of G of highest weight λ. We will also need the irreducible rational representations of GL_n. They are indexed by the n-tuples

$$(\gamma^-, \gamma^+) = (-\gamma_q^-, \ldots, -\gamma_1^-, \gamma_1^+, \gamma_2^+, \ldots, \gamma_p^+) \tag{6}$$

where $\gamma^+ = (\gamma_1^+, \gamma_2^+, \ldots, \gamma_p^+)$ and $\gamma^- = (\gamma_1^-, \ldots, \gamma_q^-)$ are partitions of length p and q such that $p + q = n$. Write $\widetilde{\mathcal{P}}_n$ for the set of such n-tuples, and denote also by $V(\gamma)$ the irreducible rational representation of GL_n of highest weight $\gamma = (\gamma^-, \gamma^+) \in \widetilde{\mathcal{P}}_n$.

In the sequel, our computations will also make appear root subsystems of the root systems R described above. Suppose that G is of type X_n with $X_n = A_{n-1}, B_n, C_n$ or D_n. Let $I = (i_1, \ldots, i_r)$ be an increasing sequence of integers belonging to I_n, that is $i_k \in I_n$ for any $k = 1, \ldots, r$ and $i_1 < \cdots < i_r$. Then

$$R_I = \{\alpha \in R \cap \oplus_{i \in I} \mathbb{Z}\varepsilon_i\}$$

is a root subsystem of R of type X_r. Write R_I^+ for the set of positive roots in R_I. Then we have $R_I^+ = R_I \cap R^+$. The dominant weights associated with R_I have the form $\lambda = (\lambda_1, \ldots, \lambda_n)$ where $\lambda_i \neq 0$ only if $i \in I$ and $\lambda^{(I)} = (\lambda_{i_1}, \ldots, \lambda_{i_r}) \in \mathcal{P}_r$. We slightly abuse the notation by identifying λ with $\lambda^{(I)}$.

Consider an increasing sequence $X = (x_1, \ldots, x_r)$ of integers belonging to J_n such that $|x_k| = |x_{k'}|$ if and only if $k = k'$. For any integer $i = 1, \ldots, n$, set $\varepsilon_{\bar{i}} = -\varepsilon_i$. Then

$$R_{A,X} = \{\pm(\varepsilon_{x_j} - \varepsilon_{x_i}) \mid 1 \leq i < j \leq r\}$$

is a root subsystem of R of type A_{r-1}. To see this, consider the linear map $\theta_X : \mathbb{Z}^r \to \mathbb{Z}^n$ such that $\theta_X(\varepsilon_i) = \varepsilon_{x_i}$. The map θ is injective and preserves the scalar product in $\mathbb{Z}^r$ and $\mathbb{Z}^n$. Moreover the root system $\{\pm(\varepsilon_j - \varepsilon_i) \mid 1 \leq i < j \leq r\} \subset \mathbb{Z}^r$ of type A_r is sent on $R_{A,X}$ by θ_X. The set of positive roots in $R_{A,X}$ is equal to $R_{A,X}^+ = R_{A,X} \cap R^+$. Denote by $s \in \{1, \ldots, r\}$ the maximal integer such that $x_s < 0$. We associate to X, the increasing sequence of indices $I \subset I_n$ defined by

$$I = (\overline{x}_s, \ldots, \overline{x}_1, x_{s+1}, \ldots, x_r). \tag{7}$$

It will be useful to consider the weights corresponding to $R_{A,X}$ as the r-tuples $\beta = (\beta_{x_1}, \ldots, \beta_{x_r})$ with coordinates indexed by X. The coordinates $(\beta'_1, \ldots, \beta'_n)$ of β on the initial basis $(\varepsilon_1, \ldots, \varepsilon_n)$ are such that $\beta'_i = \beta_{x_a}$ if $i = x_a \in X$, $\beta'_i = -\beta_{x_a}$ if $\bar{i} = x_a \in X$ and $\beta'_i = 0$ otherwise. With this convention the dominant weights for $R_{A,X}$ have the form

$$\lambda^{(X)} = (\lambda_{x_1}, \ldots, \lambda_{x_r}) \in \widetilde{\mathcal{P}}_r. \tag{8}$$

This simply means that we have chosen to expand the weights of $R_{A,X}$ on the basis $\{\varepsilon_x \mid x \in X\}$ rather than on the basis $\{\varepsilon_i \mid i \in I\}$ to preserve the identification of the dominant weights with the nondecreasing r-tuples of integers.

Example 2.1.2. Take $G = Sp_{10}$.

- For $I = (2, 4, 5)$ we have

  $$R_I^+ = \{\varepsilon_5 \pm \varepsilon_4, \varepsilon_5 \pm \varepsilon_2, \varepsilon_4 \pm \varepsilon_2, 2\varepsilon_2, 2\varepsilon_4, 2\varepsilon_5\}$$

  which is the set of positive roots of a root system of type C_3. The weight $\lambda = (1, 2, 2)$ is dominant for G_I. Considered as a weight of Sp_{10}, we have $\lambda = (0, 1, 0, 2, 2)$.
- For $X = (\overline{5}, \overline{2}, 1, 4)$ we have

$$R^+_{A,X} = \{\varepsilon_4 - \varepsilon_1, \varepsilon_5 - \varepsilon_2, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_5, \varepsilon_4 + \varepsilon_2, \varepsilon_4 + \varepsilon_5\}$$

which is the set of positive roots of a root system of type A_3. The weight $\gamma = (-3, -1, 4, 5)$ is dominant for G_X. Considered as a weight of Sp_{10}, we have $\gamma = (4, 1, 0, 5, 3)$.

2.2. Subgroups of Levi type

Suppose G is a classical Lie group and consider R the corresponding root system. We shall need Lie subgroups of G associated with particular sub-root systems of R. Each of these subgroups will be of Levy type, that is, will be isomorphic to the Levi subgroup of one of the parabolic subgroups of G.

Consider $p \geq 1$ an integer. Let $I^{(0)} = (i_1^{(0)}, \ldots, i_{r_0}^{(0)})$ be an increasing sequence of integers in I_n. For $k = 1, \ldots, p$, consider increasing sequences $X^{(k)} = (x_1^{(k)}, \ldots, x_{r_k}^{(k)}) \subset J_n$ such that $\mathrm{card}(X^{(k)}) = r_k$. Let s_k be maximal in $\{1, \ldots, r_k\}$ such that $x_{s_k}^{(k)} < 0$. Set

$$I^{(k)} = (\overline{x}_{s_k}^{(k)}, \ldots, \overline{x}_1^{(k)}, x_{s_k+1}^{(k)}, \ldots, x_{r_k}^{(k)}) \subset I_n. \tag{9}$$

We suppose that the sets $I^{(k)}, k = 0, \ldots, p$ are pairwise disjoint and verify $\cup_{k=0}^{p} I^{(k)} = I_n$. Set $\mathcal{I} = \{I^{(0)}, X^{(1)}, \ldots, X^{(p)}\}$ and

$$R_{\mathcal{I}} = R_{I^{(0)}} \cup \bigcup_{k=1}^{p} R_{A,X^{(k)}}.$$

Then $\mathfrak{g}_{\mathcal{I}} = \mathfrak{h} \oplus \coprod_{\alpha \in R_{\mathcal{I}}} \mathfrak{g}_\alpha$ is a Lie subalgebra of $\mathfrak{g}$. Its corresponding Lie group $G_{\mathcal{I}}$ is a subgroup of G of Levi type, and we have

$$G_{\mathcal{I}} \simeq \begin{cases} GL_{r_0} \times GL_{r_1} \times \cdots \times GL_{r_p} & \text{for } G = GL_n \\ SO_{2r_0+1} \times GL_{r_1} \times \cdots \times GL_{r_p} & \text{for } G = SO_{2n+1} \\ Sp_{2r_0} \times GL_{r_1} \times \cdots \times GL_{r_p} & \text{for } G = Sp_{2n} \\ SO_{2r_0} \times GL_{r_1} \times \cdots \times GL_{r_p} & \text{for } G = SO_{2n}. \end{cases}$$

The root system associated with $G_{\mathcal{I}}$ is $R_{\mathcal{I}}$. Denote by $P^+_{\mathcal{I}}$ its cone of dominant weights. The weight lattice of $G_{\mathcal{I}}$ coincides with that of G, since the Lie algebras $\mathfrak{g}_{\mathcal{I}}$ and $\mathfrak{g}$ have the same Cartan subalgebra. The elements of $P^+_{\mathcal{I}}$ are the $(p+1)$-tuples $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(p)})$ where $\lambda^{(0)} = (\lambda_i \mid i \in I^{(0)})$ is a dominant weight of $R_{G,I^{(0)}}$ and for any $k = 1, \ldots, p, \lambda^{(k)} = (\lambda_i \mid i \in X^{(k)})$ is a dominant weight of $R_{G,X^{(k)}}$. For any $\lambda \in P^+_{\mathcal{I}}$, we denote by $V_{\mathcal{I}}(\lambda)$ the irreducible finite dimensional $G_{\mathcal{I}}$-module of highest weight λ. Each weight $\beta = (\beta^{(0)}, \beta^{(1)}, \ldots, \beta^{(p)}) \in P_{\mathcal{I}}$ can be considered as a weight $\beta = (\beta'_1, \ldots, \beta'_n)$ of P. With the convention (8), we have then $\beta'_i = \beta_{i_a^{(0)}}$ if $i = i_a^{(0)} \in I^{(0)}$ and for any $k = 1, \ldots, p$, $\beta'_i = \beta_{i_a^{(k)}}$ if $i = i_a^{(k)} \in X^{(k)}$, $\beta'_i = -\beta_{i_a^{(k)}}$ if $\overline{i} = i_a^{(k)} \in X^{(k)}$. In the sequel we identify the two expressions

$$\beta = (\beta^{(0)}, \beta^{(1)}, \ldots, \beta^{(p)}) \quad \text{and} \quad \beta = (\beta'_1, \ldots, \beta'_n) \tag{10}$$

of the weights of $P_{\mathcal{I}}$.

2.3. Weyl characters and dual bases

We refer the reader to [13,15] for a detailed exposition of the results used in this paragraph. We use as a basis of the group algebra $\mathbb{Z}[\mathbb{Z}^n]$, the formal exponentials $(e^{\beta})_{\beta \in \mathbb{Z}^n}$ satisfying the relations $e^{\beta_1}e^{\beta_2} = e^{\beta_1+\beta_2}$. We furthermore introduce n independent indeterminates $x_1, \dots, x_n$ in order to identify $\mathbb{Z}[\mathbb{Z}^n]$ with the ring of polynomials $\mathbb{Z}[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$ by writing $e^{\beta} = x_1^{\beta_1} \cdots x_n^{\beta_n} = x^{\beta}$ for any $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$. Define the action of the Weyl group W on $\mathbb{Z}[\mathbb{Z}^n]$ by $w \cdot x^{\beta} = x^{w(\beta)}$. The Weyl character s_β is defined by

$$s_\beta = \frac{a_{\beta+\rho}}{a_\rho} \quad \text{where } a_\beta = \sum_{w \in W} \varepsilon(\sigma)(w \cdot x^{\beta}).$$

For any $\beta \in \mathbb{Z}^n$, we have

$$s_\beta = \begin{cases} \varepsilon(w)s_\lambda & \text{if there exists } w \in W \text{ and } \lambda \in \mathcal{P}_n \text{ such that } \lambda = w \circ \beta \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Let A be the $\mathbb{Z}$-algebra generated by the characters s_λ, $\lambda \in \mathcal{P}_n$. For any $\beta \in \mathbb{Z}^n$, denote by W_β the stabilizer of β under the action of the Weyl group W and by W^β a set of representatives in W/W_β with minimal length. Then the functions

$$m_\beta = \sum_{w \in W^\beta} w \cdot x^{\beta}$$

belong to A. Moreover $\{m_\lambda \mid \lambda \in \mathcal{P}_n\}$ is a basis of A. We have the decomposition

$$s_\lambda = \sum_{\mu \in \mathcal{P}_n} K_{\lambda,\mu} m_\mu \tag{12}$$

where $K_{\lambda,\mu}$ is equal to the dimension of the weight space μ in the irreducible representation $V(\lambda)$. There exists an inner product $\langle \cdot, \cdot \rangle$ on A which makes the characters s_λ orthonormal. We denote by $\{h_\mu \mid \mu \in \mathcal{P}_n\}$ the dual basis of $\{m_\lambda \mid \lambda \in \mathcal{P}_n\}$ with respect to $\langle \cdot, \cdot \rangle$. The homogeneous functions h_μ are given in terms of the Weyl characters by the decomposition

$$h_\mu = \sum_{\lambda \in \mathcal{P}_n} K_{\lambda,\mu} s_\lambda. \tag{13}$$

This decomposition is infinite in general when $G \neq GL_n$. Nevertheless, by embedding A in the ring $\widehat{A}$ of universal characters defined by Koike and Terada [7], it makes sense to consider formal series in the characters s_λ, $\lambda \in \mathcal{P}_n$. Note that the function h_μ is not the character of the representation $V(\mu_1\omega_1) \otimes \cdots \otimes V(\mu_n\omega_1)$ when $G \neq GL_n$,. For any $\beta \in \mathbb{Z}^n$, we define the function h_β by

$$h_\beta = h_\mu \tag{14}$$

where μ is the unique dominant weight contained in the orbit $W \cdot \beta$.

2.4. Jacobi–Trudi identities

Denote by $\mathcal{L}_n = \mathbb{K}[[x^{\beta}]]$ the vector space of formal series in the monomials x^{β} with $\beta \in \mathbb{Z}$. We identify the ring of polynomials $\mathcal{F}_n = \mathbb{K}[x^{\beta}]$ with the subspace of $\mathcal{L}_n$ containing the finite formal series. The vector space $\mathcal{L}_n$ is not a ring since $\beta \in \mathbb{Z}$. More precisely, the product $F_1 \cdots F_r$ of the formal series $F_i = \sum_{\beta \in E_i} x^{\beta^{(i)}} i = 1, \dots, r$ is defined if and only if, for any $\gamma \in \mathbb{Z}^n$, the number N_γ of decompositions $\gamma = \beta^{(1)} + \cdots + \beta^{(r)}$ such that $\beta^{(i)} \in E_i$ is finite, and in this case we have

$$F_1 \cdots F_r = \sum_{\gamma \in \mathbb{Z}^n} N_\gamma x^\gamma.$$

In particular, the product $P \cdot F$ with $P \in \mathcal{P}_n$ and $F \in \mathcal{L}_n$ is well defined.

Set

$$\nabla = \prod_{\alpha \in R_+} \frac{1}{(1 - x^\alpha)} \quad \text{and} \quad \Delta = \prod_{\alpha \in R_+} (1 - x^\alpha).$$

Then $\Delta \in \mathcal{F}_n$ and $\nabla \in \mathcal{L}_n$. We define two linear maps

$$\mathrm{S} : \begin{cases} \mathcal{L}_n \to \widehat{A} \\ x^\beta \mapsto s_\beta \end{cases} \quad \text{and} \quad \mathrm{H} : \begin{cases} \mathcal{L}_n \to \widehat{A} \\ x^\beta \mapsto h_\beta. \end{cases}$$

From Theorem 2.14 of [15], we obtain

Proposition 2.4.1. *For any $\beta \in \mathbb{Z}^n$, $s_\beta = \sum_{w \in W} \varepsilon(w) h_{\beta + \rho - w \cdot \rho}$.*

By using the identity

$$\Delta = x^\rho \sum_{w \in W} \varepsilon(w) x^{-w \cdot \rho} \tag{15}$$

the previous proposition is equivalent to the following identity:

$$\mathrm{S}(x^\beta) = \mathrm{H}(\Delta \times x^\beta). \tag{16}$$

Proposition 2.4.2. *For any $\beta \in \mathbb{Z}^n$ we have* $\mathrm{H}(x^\beta) = \mathrm{S}(\nabla \times x^\beta)$.

Proof. Denote by χ_Δ and χ_∇ the linear maps defined on $\mathcal{L}_n$ by setting $\chi_\Delta(x^\beta) = \Delta \times x^\beta$ and $\chi_\nabla(x^\beta) = \nabla \times x^\beta$ respectively. By (16) we have $\mathrm{S} = \mathrm{H} \circ \chi_\Delta$. Moreover for any $\beta \in \mathbb{Z}^n$, $\chi_\Delta \circ \chi_\nabla(x^\beta) = x^\beta$. This gives $\mathrm{S}(\nabla \times x^\beta) = \mathrm{S} \circ \chi_\nabla(x^\beta) = \mathrm{H} \circ \chi_\Delta \circ \chi_\nabla(x^\beta) = \mathrm{H}(x^\beta)$. ■

2.5. *Branching coefficients for the restriction to subgroups of Levi type*

Consider $\mathcal{I} = \{I_0, X_1, \ldots, X_p\}$ as in Section 2.2. The set $\mathcal{I}$ characterizes a subgroup $G_\mathcal{I} \subset G$ of Levi type. Set

$$\Delta_\mathcal{I} = \prod_{\alpha \in R_\mathcal{I}^+} (1 - x^\alpha) \quad \text{and} \quad \nabla_\mathcal{I} = \prod_{\alpha \in R^+ - R_\mathcal{I}^+} \frac{1}{(1 - x^\alpha)}.$$

Then $\Delta_\mathcal{I} \in \mathcal{F}_n$ and $\nabla_\mathcal{I} \in \mathcal{L}_n$. Note that $\nabla_\mathcal{I} = \nabla \times \Delta_\mathcal{I}$.

As a formal series, $\nabla_\mathcal{I}$ can be expanded in the form

$$\nabla_\mathcal{I} = \sum_{\gamma \in \mathbb{Z}^n} \mathcal{P}_\mathcal{I}(\gamma) x^\gamma. \tag{17}$$

Consider $\lambda \in \mathcal{P}_n$ and $\mu = (\mu^{(0)}, \ldots, \mu^{(p)})$ a dominant weight associated with $G_\mathcal{I}$. We denote by $[V(\lambda) : V_\mathcal{I}(\mu)]$ the multiplicity of the irreducible representation $V_\mathcal{I}(\mu)$ in the restriction of $V(\lambda)$ from G to $G_\mathcal{I}$. The proposition below follows from Theorem 8.2.1 in [1]:

Proposition 2.5.1. *Consider* $\lambda \in \mathcal{P}_n$ *and* $\mu = (\mu^{(0)}, \ldots, \mu^{(p)})$ *a dominant weight of* $P_{\mathcal{I}}^+$. *Then*

$$[V(\lambda) : V_{\mathcal{I}}(\mu)] = \sum_{w \in W} \varepsilon(w) \mathcal{P}_{\mathcal{I}}(w \circ \lambda - \mu).$$

Define the linear map

$$\begin{cases} \mathrm{S}_{\mathcal{I}} : \mathcal{L}_n \to \widehat{A} \\ x^{\beta} \mapsto \mathrm{H}(\Delta_{\mathcal{I}} \times x^{\beta}). \end{cases}$$

For any dominant weight $\mu \in P_{\mathcal{I}}^+$, set

$$S_{\mu,\mathcal{I}} = \mathrm{H}(\Delta_{\mathcal{I}} \times x^{\mu}) = \mathrm{S}_{\mathcal{I}}(x^{\mu}). \tag{18}$$

Proposition 2.5.2. *With the above notations, we have*

$$S_{\mu,\mathcal{I}} = \sum_{\lambda \in \mathcal{P}_n} [V(\lambda) : V_{\mathcal{I}}(\mu)] s_{\lambda}. \tag{19}$$

Proof. For any $\beta \in \mathbb{Z}^n$, we have obtained in the proof of Proposition 2.4.2, the identity $\mathrm{H}(x^{\beta}) = \mathrm{S} \circ \chi_{\nabla}(x^{\beta})$. Denote by $\chi_{\Delta,\mathcal{I}}$ the linear map defined on $\mathcal{L}_n$ by setting $\chi_{\Delta,\mathcal{I}}(x^{\beta}) = \Delta_{\mathcal{I}} \times x^{\beta}$. We obtain $\mathrm{S}_{\mathcal{I}}(x^{\beta}) = \mathrm{H}(\Delta_{\mathcal{I}} \times x^{\beta}) = \mathrm{S} \circ \chi_{\nabla} \circ \chi_{\Delta,\mathcal{I}}(x^{\beta}) = \mathrm{S}(\nabla_{\mathcal{I}} \times x^{\beta})$since $\nabla_{\mathcal{I}} = \nabla \times \Delta_{\mathcal{I}}$. Thus by (17) this yields $\mathrm{S}_{\mathcal{I}}(x^{\beta}) = \sum_{\gamma \in \mathbb{Z}^n} \mathcal{P}_{\mathcal{I}}(\gamma) s_{\beta+\gamma}$. For any γ, we know by (11) that $s_{\beta+\gamma} = 0$ or there exists $\lambda \in \mathcal{P}_n$, $w \in W$ such that $\lambda = w \circ (\beta + \gamma)$ and $s_{\beta+\gamma} = \varepsilon(w) s_{\lambda}$. This permits us to write

$$\mathrm{S}_{\mathcal{I}}(x^{\beta}) = \sum_{\lambda \in \mathcal{P}_n} \sum_{w \in W} \varepsilon(w) \mathcal{P}_{\mathcal{I}}(w \circ \lambda - \beta) s_{\lambda}.$$

When $\beta = \mu$ is a dominant weight of $P_{\mathcal{I}}^+$, we obtain the desired identity by using Proposition 2.5.1. ■

Remarks. (i) When $G = G_{\mathcal{I}}$ that is, when $r_0 = n$ and $r_1 = \cdots = r_p = 0$, we have $\mu = \mu^{(0)}$, $\Delta_{\mathcal{I}} = \Delta$ and $\mathrm{H}_{\mathcal{I}} = \mathrm{H}$. Thus $S_{\mu,\mathcal{I}} = s_{\mu^{(0)}}$. This can be recovered by using (19) since in this case $[V(\lambda) : V_{\mathcal{I}}(\mu)] = 0$, except when $\lambda = \mu^{(0)}$.

(ii) When $G_{\mathcal{I}} = H$ the maximal torus of G, that is when $n = p + 1$ and $r_k = 1$ for any $k = 0, \ldots, p$, we have $\mu_i = \mu^{(i-1)}$ for any $i = 1, \ldots, n$, $\Delta_{\mathcal{I}} = 1$ and $\mathrm{H}_{\mathcal{I}}(x^{\beta}) = h_{\beta}$ for any $\beta \in \mathbb{Z}^n$. Hence $S_{\mu,\mathcal{I}} = h_{\mu}$. In this case $[V(\lambda) : V_{\mathcal{I}}(\mu)] = K_{\lambda,\mu}$ for any $\lambda \in \mathcal{P}_n$. Thus (19) reduces to (13).

(iii) By (i) and (ii), the functions $S_{\mu,\mathcal{I}}$ interpolate between the Weyl characters s_{μ} and the homogeneous functions h_{μ}.

(iv) When $G = GL_n$, we have the duality

$$[V(\lambda) : V_{\mathcal{I}}(\mu)] = c^{\lambda}_{\mu^{(0)},\ldots,\mu^{(p)}}$$

where $c^{\lambda}_{\mu^{(0)},\ldots,\mu^{(p)}}$ is the Littlewood–Richardson coefficient associated with the multiplicity of $V(\lambda)$ in the tensor product $V_{\mu} = V(\mu^{(0)}) \otimes \cdots \otimes V(\mu^{(p)})$. Thus we can write $S_{\mu,\mathcal{I}} = \sum_{\lambda \in \mathcal{P}_n} c^{\lambda}_{\mu^{(0)},\ldots,\mu^{(p)}} s_{\lambda}$. This means that $S_{\mu,\mathcal{I}}$ is the character of V_{μ}. Such a duality does not exist for $G = Sp_{2n}, SO_{2n+1}$ or SO_{2n}, (but see Section 6).

3. Plethysm on Weyl characters

3.1. The operators Ψ_ℓ and φ_ℓ

Consider ℓ a positive integer. The power sum plethysm operator Ψ_ℓ is defined on A by setting $\Psi_\ell(m_\beta) = m_{\ell\beta}$ for any $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$, where $\ell\beta = (\ell\beta_1, \ldots, \ell\beta_n)$. Since $\{m_\lambda \mid \lambda \in \mathcal{P}_n\}$ and $\{h_\lambda \mid \lambda \in \mathcal{P}_n\}$ are dual bases for the inner product $\langle \cdot, \cdot \rangle$, the adjoint operator φ_ℓ of Ψ_ℓ verifies

$$\varphi_\ell(h_\beta) = \begin{cases} h_{\beta/\ell} & \text{if } \beta \in (\ell\mathbb{Z})^n \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

where $\beta/\ell = (\beta_1/\ell, \ldots, \beta_n/\ell)$ when $\beta \in (\ell\mathbb{Z})^n$.

By abuse of notation, we also denote by Ψ_ℓ and φ_ℓ the linear operators respectively defined on $\mathcal{L}_n$ by setting

$$\Psi_\ell(x^\beta) = x^{\ell\beta} \quad \text{and} \quad \varphi_\ell(x^\beta) = \begin{cases} x^{\beta/\ell} & \text{if } \beta \in (\ell\mathbb{Z})^n \\ 0 & \text{otherwise} \end{cases} \quad \text{for any } \beta \in \mathbb{Z}^n. \tag{21}$$

Remark. Since $\Psi_\ell(x^\beta \times x^{\beta'}) = \Psi_\ell(x^\beta) \times \Psi_\ell(x^{\beta'})$ for any $\beta, \beta' \in \mathbb{Z}^n$ the map Ψ_ℓ is a morphism of algebra. This is not true for φ_ℓ. Nevertheless, if $\{i_1, \ldots, i_r\}$ and $\{j_1, \ldots, j_s\}$ are disjoint subsets of I_n, $\iota = (\iota_1, \ldots, \iota_r) \in \mathbb{Z}^r$ and $\gamma = (\gamma_1, \ldots, \gamma_s) \in \mathbb{Z}^s$, we have

$$\varphi_\ell(x_{i_1}^{\iota_1} \cdots x_{i_r}^{\iota_r} \times x_{j_1}^{\gamma_1} \cdots x_{j_r}^{\gamma_r}) = \varphi_\ell(x_{i_1}^{\iota_1} \cdots x_{i_r}^{\iota_r}) \times \varphi_\ell(x_{j_1}^{\gamma_1} \cdots x_{j_r}^{\gamma_r}). \tag{22}$$

For any $\lambda \in \mathcal{P}_n$, $\Psi_\ell(s_\lambda)$ belongs to A, and thus decomposes on the basis $\{s_\mu \mid \mu \in \mathcal{P}_n\}$. Let us write

$$\Psi_\ell(s_\lambda) = \sum_{\mu \in \mathcal{P}_n} n_{\lambda,\mu} s_\mu.$$

Since Ψ_ℓ and φ_ℓ are dual operators with respect to the scalar product $\langle \cdot, \cdot \rangle$, we can write $n_{\lambda,\mu} = \langle \Psi_\ell(s_\lambda), s_\mu \rangle = \langle s_\lambda, \varphi_\ell(s_\mu) \rangle$. So we have

$$\varphi_\ell(s_\mu) = \sum_{\lambda \in \mathcal{P}_n} n_{\lambda,\mu} s_\lambda.$$

By (16) and Proposition 2.4.1, we obtain the identity

$$s_\mu = \sum_{w \in W} \varepsilon(w) h_{\mu+\rho-w\cdot\rho} = \mathrm{H}(\Delta \times x^\mu).$$

Thus from (20) and (21), we derive $\varphi_\ell(s_\mu) = \mathrm{H}(\varphi_\ell(\Delta \times x^\mu))$. Set $P_\mu = \Delta \times x^\mu$. From the previous arguments, the coefficients $n_{\lambda,\mu}$ are determined by the computation of $\varphi_\ell(P_\mu)$.

3.2. Computation of $\varphi_\ell(P_\mu)$

For any $i \in \{\overline{n}, \ldots, \overline{1}\}$, we set $x_i = \frac{1}{x_{\overline{i}}}$. This permits us to consider also variables indexed by negative integers. Given $X = (i_1, \ldots, i_r)$, an increasing sequence contained in J_n and $\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{Z}^r$, we set $x_X^\beta = x_{i_1}^{\beta_1} \cdots x_{i_r}^{\beta_r}$. We also denote by S_X the group of permutations of the set X. Each $\sigma \in S_X$ determines a unique permutation σ^* of the set $\{1, \ldots, r\}$ defined by

$$\sigma(i_p) = i_{\sigma^*(p)} \quad \text{for any } p = 1, \dots r. \tag{23}$$

In the sequel, we identify for short σ and σ^*. Similarly, given $Z = (\overline{u}_r, \dots, \overline{u}_1, u_1, \dots, u_r)$ an increasing sequence such that $\{u_1, \dots, u_r\} \subset I_n$, each signed permutation w defined on Z will be identified with the signed permutation w^* defined on J_r by $w(u_p) = u_{w^*(p)}$ for any $p = 1, \dots, r$.

Set $\rho_n = (1, 2, \dots, n)$. For any $w \in W$ we have $w \cdot \rho_n = (w(1), \dots, w(n))$. This permits us to write

$$\sum_{w \in W} \varepsilon(w) x^{-w \cdot \rho_n} = \sum_{w \in W} \varepsilon(w) x_1^{-w(1)} \cdots x_n^{-w(n)}. \tag{24}$$

3.2.1. For $G = GL_n$

Set $\kappa_n = (1, \dots, 1) \in \mathbb{Z}^n$. Since $\sigma(\kappa_n) = \kappa_n$ for any $\sigma \in S_n$, one can replace ρ by $\rho_n = (1, 2, \dots, n)$ in (15). By using (24) we can write

$$P_\mu = x_1^{(\mu_1+1)} \cdots x_n^{(\mu_n+n)} \sum_{\sigma \in S_n} \varepsilon(\sigma) x_1^{-\sigma(1)} \cdots x_n^{-\sigma(n)}$$

where $\mu = (\mu_1, \dots, \mu_n)$. For any $k \in \{0, \dots, \ell - 1\}$ consider the ordering sequences

$$I^{(k)} = (i \in I_n \mid \mu_i + i \equiv k \bmod \ell) \quad \text{and} \quad J^{(k)} = (i \in I_n \mid i \equiv k \bmod \ell). \tag{25}$$

Set $r_k = \mathrm{card}(I^{(k)})$ and write $I^{(k)} = (i_1^{(k)}, \dots, i_{r_k}^{(k)})$. Then

$$\mu^{(k)} = \left(\frac{\mu_i + i + \ell - k}{\ell} \,\middle|\, i \in I^{(k)} \right) \in \mathbb{Z}^{r_k}.$$

We derive

$$P_\mu = x_{I^{(0)}}^{\ell\mu^{(0)}} x_{I^{(1)}}^{\ell\mu^{(1)}} \cdots x_{I^{(\ell-1)}}^{\ell\mu^{(\ell-1)}} \sum_{\sigma \in S_n} \varepsilon(\sigma) x_1^{-\sigma(1)} \cdots x_n^{-\sigma(n)} \times \prod_{k=0}^{\ell-1} \prod_{a=1}^{r_k} x_{i_a^{(k)}}^{-(\ell-k)}.$$

This gives

$$\varphi_\ell(P_\mu) = x_{I^{(0)}}^{\mu^{(0)}} x_{I^{(1)}}^{\mu^{(1)}} \cdots x_{I^{(\ell-1)}}^{\mu^{(\ell-1)}} \sum_{\sigma \in S_n} \varepsilon(\sigma) \varphi_\ell \left(\prod_{k=0}^{\ell-1} \prod_{a=1}^{r_k} x_{i_a^{(k)}}^{-\sigma(i_a^{(k)})-(\ell-k)} \right). \tag{26}$$

The contribution of a fixed permutation $\sigma \in S_n$ in the above sum is nonzero if and only if for any $k = 0, \dots, \ell - 1$

$$i \in I^{(k)} \Longrightarrow \sigma(i) \equiv k \bmod \ell.$$

Thus we must have $\sigma(I^{(k)}) \subset J^{(k)}$ for any $k = 0, \dots, \ell - 1$. Since σ is a bijection, $I^{(k)} \cap I^{(k')} = J^{(k)} \cap J^{(k')} = \emptyset$ if $k \neq k'$ and $\cup_{0 \le k \le \ell-1} I^{(k)} = \cup_{0 \le k \le \ell-1} J^{(k)} = I_n$, the restriction of σ on I_k is a bijection from $I^{(k)}$ to $J^{(k)}$. In particular, $\mathrm{card}(J^{(k)}) = \mathrm{card}(I^{(k)}) = r_k$. This means that we have the equivalences

$$\varphi_\ell \left(\prod_{k=0}^{\ell-1} \prod_{a=1}^{r_k} x_{i_a^{(k)}}^{-\sigma(i_a^{(k)})-(\ell-k)} \right) \neq 0 \Longleftrightarrow \sigma(I^{(k)}) = J^{(k)} \quad \text{for any } k = 0, \dots, \ell - 1. \tag{27}$$

Write $J^{(k)} = (k, k+\ell, \ldots, k+(r_k-1)\ell)$. Denote by $\sigma_0 \in S_n$ the permutation verifying

$$\sigma_0(i_a^{(k)}) = k + (a-1)\ell \tag{28}$$

for any $k = 0, \ldots, \ell-1$ and any $a = 1, \ldots, r_k$. Let $S_{I^{(k)}}$ be the permutation group of the set $I^{(k)}$. The permutations σ which verify the right-hand side of (27) can be written $\sigma = \sigma_0\tau$, where $\tau = (\tau^{(0)}, \ldots, \tau^{(\ell-1)})$ belongs to the direct product $S_{I^{(0)}} \times \cdots \times S_{I^{(\ell-1)}}$. We have then $\varepsilon(\sigma) = \varepsilon(\sigma_0)(-1)^{l(\tau^{(0)})} \times \cdots \times (-1)^{l(\tau^{(p)})}$.

For any $k \in \{0, \ldots, \ell-1\}$, set

$$P_k = \sum_{\tau^{(k)} \in S_{I^{(k)}}} (-1)^{l(\tau)} \varphi_\ell\left(\prod_{a=1}^{r_k} x_{i_a^{(k)}}^{-\sigma_0\tau^{(k)}(i_a^{(k)})-(\ell-k)}\right).$$

From (22) and (26), we derive

$$\varphi_\ell(P_\mu) = \varepsilon(\sigma_0)\prod_{k=0}^{\ell-1} x_{I^{(k)}}^{\mu^{(k)}} P_k.$$

Since $\sigma_0(i_a^{(k)}) = k + (a-1)\ell$, we can write by (23) $\sigma_0\tau^{(k)}(i_a^{(k)}) = k + (\tau^{(k)}(a)-1)\ell$. Thus we obtain

$$P_k = \sum_{\tau^{(k)} \in S_{I^{(k)}}} (-1)^{l(\tau)} x_{i_1^{(k)}}^{-\tau^{(k)}(1)} \cdots x_{i_{r_k}^{(k)}}^{-\tau^{(k)}(r_k)} = x_{I^{(k)}}^{-\rho_{r_k}} \Delta_{I^{(k)}}$$

where $\rho_{r_k} = (1, 2, \ldots, r_k)$ and $\Delta_{I^{(k)}} = \prod_{i<j\ i,j \in I^{(k)}} (1 - x_j/x_i)$. Finally, this gives

$$\varphi_\ell(P_\mu) = \varepsilon(\sigma_0)\prod_{k=0}^{\ell-1} x_{I^{(k)}}^{\mu^{(k)}-\rho_{r_k}} \Delta_{I^{(k)}} = \varepsilon(\sigma_0)\prod_{k=0}^{\ell-1} x_{I^{(k)}}^{\mu^{(k)}} \Delta_{I^{(k)}}$$

where for any $k = 0, \ldots, \ell-1$,

$$\mu^{(k)} = \left(\frac{\mu_i + i + \ell - k}{\ell} \,\middle|\, i \in I^{(k)}\right) - (1, 2, \ldots, r_k) \in \mathbb{Z}^{r_k}. \tag{29}$$

Theorem 3.2.1. *Consider a partition μ of length n and ℓ a positive integer. For any $k = 0, \ldots, \ell-1$, define the sets $I^{(k)}$ and $J^{(k)}$ as in (25).*

- *If there exists $k \in \{0, \ldots, \ell-1\}$ such that $\mathrm{card}(I^{(k)}) \neq \mathrm{card}(J^{(k)})$ then $\varphi_\ell(s_\mu) = 0$.*
- *Otherwise, for any $k = 0, \ldots, \ell-1$, set $r_k = \mathrm{card}(I^{(k)}) = \mathrm{card}(J^{(k)})$ and define σ_0 as in (28). Then each r_k-tuple defined by (29) is a Partition, and we have*

  $$\varphi_\ell(s_\mu) = \varepsilon(\sigma_0) S_{\left(\frac{\mu}{\ell}\right),\mathcal{I}} = \varepsilon(\sigma_0)\mathrm{char}(V_\mu)$$

  where $\mathcal{I} = \{I^{(0)}, \ldots, I^{(\ell-1)}\}$, $\left(\frac{\mu}{\ell}\right) = (\mu^{(0)}, \ldots, \mu^{(\ell-1)})$ and $\mathrm{char}(V_\mu)$ is the character of the GL_n-module $V_\mu = V(\mu^{(0)}) \otimes \cdots \otimes V(\mu^{(\ell-1)})$.

Proof. One verifies easily from (29) that each $\mu^{(k)}$ is a partition. By the previous computation, we obtain

$$\varphi_\ell(\Delta \times x^\mu) = \varepsilon(\sigma_0)\Delta_{\mathcal{I}} \times x^{\left(\frac{\mu}{\ell}\right)}$$

(with the notation of Section 2.5). By definition of φ_ℓ, we have also

$$\varphi_\ell(s_\mu) = \varepsilon(\sigma_0)\mathrm{H} \circ \varphi_\ell(\Delta \times x^\mu) = \varepsilon(\sigma_0)\mathrm{H}(\Delta_{\mathcal{I}} \times x^{(\frac{\mu}{\ell})}) = \varepsilon(\sigma_0) S_{(\frac{\mu}{\ell}),\mathcal{I}}$$

where the last equality follows from (18). ■

Remark. The subgroup $G_{\mathcal{I}}$ appearing in Theorem 3.2.1 is characterized by $\mathcal{I} = \{I^{(0)}, \dots, I^{(\ell-1)}\}$. This means that for type A, we have $X^{(k)} = I^{(k)}$ for any $k > 0$ with the notation of Section 2.2; that is the sets $X^{(k)}$ contain only positive indices.

Example 3.2.2. Consider $\mu = (1, 2, 3, 4, 4, 4, 6, 6)$ and take $\ell = 3$. We have $\mu + \rho_8 = (2, 4, 6, 8, 9, 10, 13, 14)$. Thus $I^{(0)} = \{3, 5\}$, $I^{(1)} = \{2, 6, 7\}$, $I^{(2)} = \{1, 4, 8\}$ and $J^{(0)} = \{3, 6\}$, $J^{(1)} = \{1, 4, 7\}$, $J^{(2)} = \{2, 5, 8\}$. Then $\mu^{(0)} = (1, 1)$, $\mu^{(1)} = (1, 2, 2)$ and $\mu^{(2)} = (0, 1, 2)$. We have $G_{\mathcal{I}} \simeq GL_2 \times GL_3 \times GL_3$.

3.2.2. For $G = Sp_{2n}$

We have $\rho = \rho_n = (1, 2, \dots, n)$. By using (24) we deduce the identity:

$$P_\mu = x_1^{(\mu_1+1)} \cdots x_n^{(\mu_n+n)} \sum_{w \in W} \varepsilon(w) x_1^{-w(1)} \cdots x_n^{-w(n)} \tag{30}$$

where W is the group of signed permutations defined on $J_n = \{\overline{n}, \dots, \overline{1}, 1, \dots, n\}$, that is the subgroup of permutations $w \in S_{J_n}$ verifying $w(\overline{x}) = \overline{w(x)}$ for any $x \in J_n$. Given $k \in \{0, \dots, \ell - 1\}$, consider the ordering sequences

$$I^{(k)} = (i \in I_n \mid \mu_i + i \equiv k \bmod \ell) \quad \text{and} \quad J^{(k)} = (x \in J_n \mid x \equiv k \bmod \ell). \tag{31}$$

Set $p = \frac{\ell}{2}$ if ℓ is even and $p = \frac{\ell-1}{2}$ otherwise.

3.2.2.1. The odd case $\ell = 2p - 1$. Set $r_0 = \mathrm{card}(I^{(0)})$ and for any $k = 1, \dots, p - 1$, $s_k = \mathrm{card}(I_k)$, $r_k = \mathrm{card}(I_k) + \mathrm{card}(I_{\ell-k})$. Write $X^{(k)}, k = 1, \dots, p$ for the increasing reordering of $\overline{I}_k \cup I_{\ell-k}$. Set $I^{(0)} = (i_1^{(0)}, \dots, i_{r_0}^{(0)})$ and for $k > 0$

$$X^{(k)} = (i_1^{(k)}, \dots, i_{r_k}^{(k)}). \tag{32}$$

This means that $I^{(k)} = (\bar{i}_{s_k}^{(k)}, \dots, \bar{i}_1^{(k)})$ and $I^{(\ell-k)} = (i_{s_{k+1}}^{(k)}, \dots, i_{r_k}^{(k)})$. To simplify the computation, we are going to use the indices and the variables $x_i, i \in X^{(k)}$ rather than the variables $x_i, i \in I^{(k)} \cup I^{(\ell-k)}$ when $k \in \{1, \dots, p - 1\}$.

Consider

$$\mu^{(0)} = \left(\frac{\mu_i + i}{\ell} \,\middle|\, i \in I^{(0)} \right) \in \mathbb{Z}^{r_0} \quad \text{and for } k > 0,$$

$$\mu^{(k)} = \left(\mathrm{sign}(i) \frac{\mu_{|i|} + |i| + \mathrm{sign}(i)k}{\ell} \,\middle|\, i \in X^{(k)} \right) \in \mathbb{Z}^{r_k}$$

where for any $i \in J_n$, $\mathrm{sign}(i) = 1$ if $i > 0$ and -1 otherwise. For any $i \in I$, we have $x_i^{-w(i)} = x_{\bar{i}}^{-w(\bar{i})}$. Thus

$$\prod_{i \in X^{(k)}} x_i^{-w(i)} = \prod_{i \in I^{(k)}} x_i^{-w(i)} \prod_{i \in I^{(\ell-k)}} x_i^{-w(i)} \quad \text{and}$$

$$x_1^{(\mu_1+1)}\cdots x_n^{(\mu_n+n)} = x_{I^{(0)}}^{\ell\mu^{(0)}}\prod_{k=1}^{p-1}x_{X^{(k)}}^{\ell\mu^{(k)}}\prod_{i\in X^{(k)}}x_i^{-k}$$

by definition of the $\mu^{(k)}$'s. Then (30) can be rewritten

$$P_\mu = x_{I^{(0)}}^{\ell\mu^{(0)}}\prod_{k=1}^{p-1}x_{X^{(k)}}^{\ell\mu^{(k)}} \times \sum_{w\in W}\varepsilon(w)\prod_{i\in I^{(0)}}x_i^{-w(i)}\prod_{k=1}^{p-1}\prod_{i\in X^{(k)}}x_i^{-w(i)} \times \prod_{k=1}^{p-1}\prod_{i\in X^{(k)}}x_i^{-k}.$$

This gives

$$\varphi_\ell(P_\mu) = x_{I^{(0)}}^{\mu^{(0)}}\prod_{k=1}^{p-1}x_{X^{(k)}}^{\mu^{(k)}} \times \sum_{w\in W}\varepsilon(w)\varphi_\ell\left(\prod_{i\in I^{(0)}}x_i^{-w(i)}\prod_{k=1}^{p-1}\prod_{i\in X^{(k)}}x_i^{-w(i)-k}\right).$$

The contribution of a fixed $w \in W$ in the above sum is nonzero if and only if

$$\begin{cases} i \in I^{(0)} \Longrightarrow w(i) \equiv 0 \bmod \ell \\ i \in X^{(k)} \Longrightarrow w(i) \equiv -k \bmod \ell \quad \text{for any } k = 1,\ldots,p-1. \end{cases} \tag{33}$$

Thus we must have $w(\overline{I}^{(0)} \cup I^{(0)}) \subset J^{(0)}$ and for any $k = 1,\ldots,p-1$, $w(X^{(k)}) \subset J^{(\ell-k)}$. Recall that $\overline{J}^{(0)} = J^{(0)}$ and $\overline{J}^{(\ell-k)} = J^{(k)}$ for $k = 1,\ldots,p-1$. Moreover

$$I^{(0)} \cup \overline{I}^{(0)} \bigcup_{k=1}^{p-1} X^{(k)} \cup \overline{X}^{(k)} = J_n \quad \text{and} \quad J^{(0)} \cup \bigcup_{k=1}^{p-1} J^{(k)} \cup J^{(\ell-k)} = J_n.$$

Since the sets appearing on the left hand sides of these two equalities are pairwise disjoint, we must have $w(\overline{I}^{(0)} \cup I^{(0)}) = J^{(0)}$, and for $k = 1,\ldots,p-1$, $w(X^{(k)}) = J^{(\ell-k)}$. In particular $\mathrm{card}(J^{(0)}) = 2\,\mathrm{card}(I^{(0)}) = 2r_0$ and $\mathrm{card}(J^{(\ell-k)}) = \mathrm{card}(X^{(k)}) = r_k$ for any $k = 1,\ldots,p-1$. We have the equivalences

$$\varphi_\ell\left(\prod_{i\in I^{(0)}}x_i^{-w(i)}\prod_{k=1}^{p-1}\prod_{i\in X^{(k)}}x_i^{-w(i)+k}\right) \neq 0 \Longleftrightarrow \begin{cases} \text{(i)}\, w(I^{(0)} \cup \overline{I}^{(0)}) = J^{(0)} \\ \text{(ii)}\, w(X^{(k)}) = J^{(\ell-k)} \\ \quad \text{for any } k = 1,\ldots,p-1. \end{cases} \tag{34}$$

Note that condition (ii) can be rewritten: $w(\overline{X}^{(k)}) = J^{(k)}$ for any $k = 1,\ldots,p-1$.

We can set $J^{(0)} = (-r_0\ell,\ldots,r_0\ell)$ and for $k = 1,\ldots,p-1$,

$$J^{(\ell-k)} = (-k-\alpha_k\ell,\ldots,-k+\beta_k\ell), \qquad J^{(k)} = (k-\beta_k\ell,\ldots,k+\alpha_k\ell)$$

with $\alpha_k + \beta_k + 1 = r_k$.

Consider $w_0 \in W$ defined by

$$\begin{aligned} &w_0(i_a^{(0)}) = a\ell \text{ for } a \in \{1,\ldots,r_0\} \\ &w_0(i_a^{(k)}) = -k-\alpha_k\ell+(a-1)\ell \quad \text{for any } k = 1,\ldots,p-1. \end{aligned} \tag{35}$$

Denote by $\mathcal{W}$ the set of signed permutations w which verify (i) and (ii) in (34). We have $w_0 \in \mathcal{W}$. Each $w \in \mathcal{W}$ can be written $w = w_0 v$ where $v = (v^{(0)}, \tau^{(1)},\ldots,\tau^{(p-1)})$ belongs to the direct product $W_{I^{(0)}} \times S_{X^{(1)}} \times \cdots \times S_{X^{(p-1)}}$. Here $W_{I^{(0)}}$ is the group of signed permutations defined on $\overline{I}^{(0)} \cup I^{(0)}$ and for $k = 1,\ldots,p-1$, $S_{X^{(k)}}$ is the group of signed permutations $\tau^{(k)}$ defined on $\overline{X}^{(k)} \cup X^{(k)}$ and verifying $\tau^{(k)}(X^{(k)}) = X^{(k)}$. Indeed if $\tau^{(k)}(x) \in \overline{X}^{(k)}$ and $x \in X^{(k)}$, we would have $w(x) \in J^{(k)}$ and $x \in X^{(k)}$ which contradicts (ii). This means that $S_{X^{(k)}}$ is, in fact, isomorphic to the symmetric group S_{r_k}. Since the sets $I^{(0)}$ and $X^{(k)}, k = 1, \ldots, p-1$ are increasing subsequences of J_n, we have by Lemma 2.1.1 $\varepsilon(w) = \varepsilon(w_0)(-1)^{l(v^{(0)})}(-1)^{l(\tau^{(1)})} \times \cdots \times (-1)^{l(\tau^{(p-1)})}$.

Set

$$P_0 = \sum_{v^{(0)} \in W_{I^{(0)}}} (-1)^{l(v^{(0)})} \varphi_\ell \left(\prod_{i \in I^{(0)}} x_i^{-w_0 v^{(0)}(i)} \right) \quad \text{and}$$

$$P_k = \sum_{\tau^{(k)} \in S_{X^{(k)}}} (-1)^{l(\tau^{(k)})} \varphi_\ell \left(\prod_{i \in X^{(k)}} x_i^{-w_0 \tau^{(k)}(i) - k} \right), \quad k \in \{1, \ldots, p-1\}.$$

We obtain

$$\varphi_\ell(P_\mu) = \varepsilon(w_0) x_{I^{(0)}}^{\mu^{(0)}} P_0 \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)}} P_k.$$

From (23) and (35), we have $w_0 v^{(0)}(i_a^{(0)}) = v^{(0)}(a)\ell$ for any $a = 1, \ldots, r_0$ and

$$w_0 \tau^{(k)}(i_a^{(k)}) = -k - \alpha_k \ell + (\tau^{(k)}(a) - 1)\ell \quad \text{for any } a = 1, \ldots, r_k.$$

This yields

$$P_0 = \sum_{v^{(0)} \in W_{I^{(0)}}} (-1)^{l(v^{(0)})} \prod_{a=1}^{r_0} x_{i_a^{(0)}}^{-v^{(0)}(a)} = x_{I^{(0)}}^{-\rho_{r_0}} \Delta_{I^{(0)}} \quad \text{and}$$

$$P_k = \sum_{\tau^{(k)} \in S_{X^{(k)}}} (-1)^{l(\tau^{(k)})} \prod_{a=1}^{r_k} x_{i_a^{(k)}}^{-\tau^{(k)}(a) + (\alpha_k + 1)} = x_{X^{(k)}}^{\eta_{r_k}} \Delta_{X^{(k)}}$$

where for any $k = 1, \ldots, p-1$, $\eta_{r_k} = -\rho_{r_k} + (\alpha_k + 1, \ldots, \alpha_k + 1) \in \mathbb{Z}^{r_k}$,

$$\Delta_{I^{(0)}} = \prod_{i<j \ i,j \in I^{(0)}} \left(1 - \frac{x_j}{x_i}\right) \prod_{r \le s \ r,s \in I^{(0)}} (1 - x_r x_s)$$

$$\text{and} \quad \Delta_{X^{(k)}} = \prod_{i<j \ i,j \in X^{(k)}} \left(1 - \frac{x_j}{x_i}\right) \quad \text{for any } k = 1, \ldots, p-1.$$

Finally, this gives

$$\varphi_\ell(P_\mu) = \varepsilon(w_0) x_{I^{(0)}}^{\mu^{(0)} - \rho_{r_0}} \Delta_{I^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)} - \eta_{r_k}} \Delta_{X^{(k)}} = \varepsilon(w_0) x_{I^{(0)}}^{\mu^{(0)}} \Delta_{I^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)}} \Delta_{X^{(k)}}$$

where

$$\mu^{(0)} = \left(\frac{\mu_i + i}{\ell} \,\middle|\, i \in I^{(0)} \right) - (1, \ldots, r_0) \in \mathbb{Z}^{r_0} \tag{36}$$

and for any $k = 1, \ldots, p-1$

$$\mu^{(k)} = \left(\mathrm{sign}(i)\frac{\mu_{|i|} + |i| + \mathrm{sign}(i)k}{\ell}\,\middle|\, i \in X^{(k)}\right) - (1, \ldots, r_k) \\ + (\alpha_k + 1, \ldots, \alpha_k + 1) \in \mathbb{Z}^{r_k}. \tag{37}$$

Recall that the weights corresponding to the subgroup of Levi type $G_{\mathcal{I}}$ are written following the convention (8).

Theorem 3.2.3. *Consider a partition μ of length n and $\ell = 2p-1$ a positive integer. Let $I^{(0)}$ and $J^{(0)}$ be as in (31). For any $k = 1, \ldots, p-1$, define the sets $X^{(k)}$ and $J^{(k)}$ by (31) and (32).*

- *If $\mathrm{card}(I^{(0)}) \neq \frac{1}{2}\mathrm{card}(J^{(0)})$ or if there exists $k \in \{1, \ldots, p-1\}$ such that $\mathrm{card}(X^{(k)}) \neq \mathrm{card}(J^{(k)})$ then $\varphi_\ell(s_\mu) = 0$.*
- *Otherwise, set $r_0 = \mathrm{card}(I^{(0)})$ and for any $k = 1, \ldots, p-1$, $r_k = \mathrm{card}(X^{(k)})$. Let $w_0 \in W$ be as in (35). Consider $\binom{\mu}{\ell} = (\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(p-1)})$ where the $\mu^{(k)}$'s are defined by (36) and (37). Then $\binom{\mu}{\ell}$ is a dominant weight of $P_{\mathcal{I}}^{+}$ with $\mathcal{I} = \{I^{(0)}, X^{(1)}, \ldots, X^{(p-1)}\}$, and we have*

$$\varphi_\ell(s_\mu) = \varepsilon(w_0) S_{\binom{\mu}{\ell}, \mathcal{I}}.$$

Proof. The proof is essentially the same as in Theorem 3.2.1. We obtain

$$\varphi_\ell(\Delta \times x^\mu) = \varepsilon(w_0)\Delta_{\mathcal{I}} \times x^{\binom{\mu}{\ell}}$$

where on the right-hand side of the preceding equality $\binom{\mu}{\ell}$ is expressed on the basis $\{\varepsilon_i \mid i \in I_n\}$ (see (10)). This permits us to write, as in the case $G = GL_n$,

$$\varphi_\ell(s_\mu) = \varepsilon(w_0)\mathrm{H} \circ \varphi_\ell(\Delta \times x^\mu) = \varepsilon(w_0)\mathrm{H}(\Delta_{\mathcal{I}} \times x^{\binom{\mu}{\ell}}) = \varepsilon(w_0) S_{\binom{\mu}{\ell}, \mathcal{I}}. \quad \blacksquare$$

Example 3.2.4. Consider $\mu = (1, 2, 3, 4, 4, 4, 6, 6)$ and take $\ell = 3$. We have $\mu + \rho_8 = (2, 4, 6, 8, 9, 10, 13, 14)$. Thus $I^{(0)} = \{3, 5\}$, $X^{(1)} = \{\overline{7}, \overline{6}, \overline{2}, 1, 4, 8\}$ and $J^{(0)} = \{\overline{6}, \overline{3}, 3, 6\}$, $J^{(1)} = \{\overline{8}, \overline{5}, \overline{2}, 1, 4, 7\}$, $J^{(2)} = \{\overline{7}, \overline{4}, \overline{1}, 2, 5, 8\}$. In particular $\alpha_1 = 2$. Then $\mu^{(0)} = (1, 1)$ and $\mu^{(1)} =$

$$\left(-\frac{13-1}{3} - 1 + 3, -\frac{10-1}{3} - 2 + 3, -\frac{4-1}{3} - 3 + 3, \frac{2+1}{3} - 4 + 3, \right.\\ \left. \frac{8+1}{3} - 5 + 3, \frac{14+1}{3} - 6 + 3\right) \\ = (-2, -2, -1, 0, 1, 2)$$

with the convention (8). We have $G_{\mathcal{I}} \simeq Sp_4 \times GL_6$.

3.2.2.2. The even case $\ell = 2p$. With the same notation as in the odd case, (30) can be rewritten

$$P_\mu = x_{I^{(0)}}^{\ell\mu^{(0)}} x_{I^{(p)}}^{\ell\mu^{(p)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\ell\mu^{(k)}} \times \sum_{w \in W} \varepsilon(w) \prod_{i \in I^{(0)}} x_i^{-w(i)} \prod_{i \in I^{(p)}} x_i^{-w(i)-p} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_i^{-w(i)-k}$$

where

$$\mu^{(p)} = \left(\frac{\mu_i + i + p}{\ell}\,\middle|\, i \in I^{(p)}\right).$$

This gives

$$\varphi_\ell(P_\mu) = x_{I^{(0)}}^{\mu^{(0)}} x_{I^{(p)}}^{\mu^{(p)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(p)}} \times \sum_{w\in W} \varepsilon(w)\varphi_\ell$$
$$\times \left(\prod_{i\in I^{(0)}} x_i^{-w(i)} \prod_{i\in I^{(p)}} x_i^{-w(i)-p} \prod_{k=1}^{p-1} \prod_{i\in X^{(k)}} x_i^{-w(i)-k} \right).$$

The contribution of a fixed $w \in W$ in the above sum is nonzero if conditions (34) are verified and

$$i \in I^{(p)} \Longrightarrow w(i) \in J^{(p)}.$$

Since $p \equiv -p \bmod \ell$ we have $J^{(p)} = \overline{J}^{(p)} = \{-p-\alpha_p\ell, \ldots, -p, p, \ldots, p+\alpha_p\ell\}$. This implies that $w(I^{(p)} \cup \overline{I}^{(p)}) = J^{(p)}$ and thus $\mathrm{card}(I^{(p)}) = \frac{1}{2}\mathrm{card}(J^{(p)})$. We then define w_0 by requiring (35) and $w_0(i_a^{(p)}) = p+(a-1)\ell$ for $a \in \{1, \ldots, r_p\}$. By using similar arguments as in the odd case, we obtain that w can be written $w = w_0\nu$ where $\nu = (v^{(0)}, \tau^{(1)}, \ldots, \tau^{(p-1)}, v^{(p)})$ belongs to the direct product $W_{I^{(0)}} \times S_{X^{(1)}} \times \cdots \times S_{X^{(\ell-1)}} \times W_{I^{(p)}}$ with $W_{I^{(p)}}$ the group of signed permutations defined on $\overline{I^{(p)}} \cup I^{(p)}$. Note that $W_{I^{(p)}}$ is a Weyl group of type B_{r_p}. By Lemma 2.1.1, we have also $\varepsilon(w) = \varepsilon(w_0)(-1)^{l(v^{(0)})}(-1)^{l(\tau^{(1)})} \times \cdots \times (-1)^{l(\tau^{(p)})} \times (-1)^{l(v^{(p)})}$. We obtain

$$\varphi_\ell(P_\mu) = \varepsilon(w_0) x_{I^{(0)}}^{\mu^{(0)}} P_0 \times x_{I^{(p)}}^{\mu^{(p)}} P_p \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)}} P_k \quad \text{where}$$

$$P_p = \sum_{v^{(p)}\in W_{I^{(p)}}} (-1)^{l(v^{(p)})} \varphi_\ell \left(\prod_{i\in I^{(p)}} x_i^{-w_0 v^{(p)}(i)-p} \right).$$

The functions $P_k, k = 0, \ldots, p-1$ can be computed as in the odd case. For P_p, observe that each $v^{(p)} \in W_{I^{(p)}}$ can be written $v^{(p)} = \zeta\sigma$ according to the decomposition of $W_{I^{(p)}}$ as the semidirect product $(\mathbb{Z}/2\mathbb{Z})^{r_p} \propto S_{I^{(p)}}$. We have then for any $a = 1, \ldots, r_p$, $w_0 v^{(p)}(i_a^{(p)}) = \xi(a)(p + (\sigma(a)-1)\ell)$. This yields

$$P_p = \sum_{v^{(p)}\in W_{I^{(p)}}} (-1)^{l(v^{(p)})} \varphi_\ell \left(\prod_{a=1}^{r_p} x_i^{-\xi(a)(p+(\sigma(a)-1)\ell)-p} \right)$$
$$= \sum_{v^{(p)}\in W_{I^{(p)}}} (-1)^{l(v^{(p)})} \prod_{a=1}^{r_p} x_{i_a^{(p)}}^{-\frac{1-\xi(a)}{2}-\xi\sigma(a)}.$$

Thus

$$P_p = \prod_{i\in I^{(p)}} x_i^{-1/2} \sum_{v^{(p)}\in W_{I^{(p)}}} (-1)^{l(v^{(p)})} \left(v^{(p)} \cdot \prod_{a=1}^{r_p} x_{i_a^{(p)}}^{-(a-\frac{1}{2})} \right) = x_{I^{(p)}}^{-\rho_{r_p}} \Delta_{I^{(p)},B_{r_p}}$$

where

$$\Delta_{I^{(p)},B_{r_p}} = \prod_{i<j\ i,j\in I^{(p)}} \left(1 - \frac{x_j}{x_i}\right) \prod_{r<s\ r,s\in I^{(p)}} (1 - x_r x_s) \prod_{i\in I^{(p)}} (1 - x_i).$$

Indeed the half sum of positive roots is equal to $(\frac{1}{2},\ldots,r_p-\frac{1}{2})$ in type B_{r_p}. This means that when ℓ is even

$$\varphi_\ell(P_\mu)=\varepsilon(w_0)x_{I^{(0)}}^{\mu^{(0)}}\Delta_{I^{(0)}}\prod_{k=1}^{p-1}x_{X^{(k)}}^{\mu^{(k)}}\Delta_{X^{(k)}}\times x_{I^{(p)}}^{\mu^{(p)}}\Delta_{I^{(p)},B_{r_p}}$$

where $\mu^{(p)}=\mu^{(p)}-(1,\ldots,r_p)$. In particular the computation of $\varphi_\ell(P_\mu)$ makes positive roots appear corresponding to a root system of type B_{r_p}. These roots do not belong to the root lattice associated with Sp_{2n}. Hence, there cannot exist an analogue of Theorem 3.2.3 when ℓ is even. With the previous notation, we only obtain:

Proposition 3.2.5. *Suppose $G=SP_{2n}$ and $\ell=2p$.*

- *If* $\mathrm{card}(I^{(0)})\neq\frac{1}{2}\mathrm{card}(J^{(0)})$, $\mathrm{card}(I^{(p)})\neq\frac{1}{2}\mathrm{card}(J^{(p)})$ *or there exists* $k\in\{1,\ldots,p-1\}$ *such that* $\mathrm{card}(X^{(k)})\neq\mathrm{card}(J^{(k)})$ *then* $\varphi_\ell(s_\mu)=0$.
- *Otherwise, the coefficients appearing in the decomposition of $\varphi_\ell(s_\mu)$ on the basis of Weyl characters cannot be interpreted as branching coefficients and have signs alternatively positive and negative.*

3.2.3. For $G=SO_{2n}$

As for $G=Sp_{2n}$, the coefficients appearing in the decomposition of $\varphi_\ell(s_\mu)$ with $\ell=2p$ on the basis of Weyl characters cannot be interpreted as branching coefficients. Note that there is an additional difficulty in this case. Indeed, $\varphi_\ell(P_\mu)$ cannot be factorized as a product of polynomials $(1-x^\beta)$ where $\beta\in\mathbb{Z}^n$. For example, we have for SO_4

$$\varphi_2(P_{(0,0)})=\varphi_2\left(\left(1-\frac{x_2}{x_1}\right)(1-x_1x_2)\right)=1+x_2.$$

This is due to the incompatibility between the signatures defined on the Weyl groups of types B and D when they are realized as subgroups of the permutation group S_{J_n}.

So we will suppose $\ell=2p-1$ in this paragraph. Recall that the elements of W are the signed permutations w defined on $J_n=\{\overline{n},\ldots,\overline{1},1,\ldots,n\}$ such that $\mathrm{card}(\{i\in I_n\mid w(i)<0\})$ is even. Set $K_n=\{\overline{n-1},\ldots,\overline{1},0,1,\ldots,n-1\}$. Each $w\in W$ can be written $w=\zeta\sigma$ according to the decomposition of W as the semidirect product $(\mathbb{Z}/2\mathbb{Z})^{n-1}\propto S_n$. For any $x\in J_n$, we have then $\xi(x)=1$ if $w(x)>0$ and $\xi(x)=-1$ otherwise. Given $w\in W$, we define $\widehat{w}:J_n\to K_n$ such that $\widehat{w}(x)=w(x)-\xi(x)$ for any $x\in J_n$. Then $\widehat{w}(\overline{x})=\overline{\widehat{w}(x)}$.

For type D_n, we have $\rho=\rho'_n=(0,1,\ldots,n-1)=\rho_n-(1,\ldots,1)$. Hence

$$w\cdot\rho'_n=w\cdot\rho_n-(\xi(1),\ldots,\xi)(n)=(\widehat{w}(1),\ldots,\widehat{w}(n))=\widehat{w}\cdot\rho_n.$$

Then we obtain

$$P_\mu=x_1^{(\mu_1+0)}\cdots x_n^{(\mu_n+n-1)}\sum_{w\in W}\varepsilon(w)x_1^{-\widehat{w}(1)}\cdots x_n^{-\widehat{w}(n)}.$$

For any $k=0,\ldots,\ell-1$, set

$$I^{(k)}=(i\in I_n\mid \mu_i+i-1\equiv k \bmod \ell)\quad\text{and}\quad J^{(k)}=(x\in K_n\mid x\equiv k\bmod \ell).\tag{38}$$

We then proceed essentially as in Section 3.2.2 by using $\widehat{w}$ instead of w and $\rho'_n=(0,1,\ldots,n-1)$ instead of $\rho_n=(1,\ldots,n)$. We only sketch below the main steps of the computation.

Set $r_0 = \mathrm{card}(I^{(0)})$ and for any $k = 1, \ldots, p-1$, $s_k = \mathrm{card}(I_k)$, $r_k = \mathrm{card}(I_k) + \mathrm{card}(I_{\ell-k})$. For $k = 1, \ldots, p-1$, $X^{(k)}$ is defined as the increasing reordering of $\overline{I}^{(k)} \cup I^{(\ell-k)}$. Consider

$$\mu^{(0)} = \left(\frac{\mu_i + i - 1}{\ell} \middle| i \in I^{(0)} \right) \in \mathbb{Z}^{r_0} \quad \text{and for } k > 0,$$
$$\mu^{(k)} = \left(\mathrm{sign}(i) \frac{\mu_{|i|} + |i| - 1 + \mathrm{sign}(i)k}{\ell} \middle| i \in X^{(k)} \right) \in \mathbb{Z}^{r_k}.$$

We obtain

$$\varphi_\ell(P_\mu) = x_{I^{(0)}}^{\mu^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(p)}} \times \sum_{w \in W} \varepsilon(w) \varphi_\ell \left(\prod_{i \in I^{(0)}} x_i^{-\widehat{w}(i)} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_i^{-\widehat{w}(i)-k} \right).$$

We also have the equivalences

$$\varphi_\ell \left(\prod_{i \in I^{(0)}} x_i^{-w(i)} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}}^{s_k} x_i^{-\widehat{w}(i)+k} \right) \neq 0$$
$$\iff \begin{cases} \text{(i)} \widehat{w}(I^{(0)} \cup \overline{I}^{(0)}) = J^{(0)} \\ \text{(ii)} \widehat{w}(X^{(k)}) = J^{(\ell-k)} \quad \text{for any } k = 1, \ldots, p-1 \end{cases}. \tag{39}$$

We can write $J^{(0)} = (-(r_0 - 1)\ell, \ldots, 0, \ldots, (r_0 - 1)\ell)$ and for $k = 1, \ldots, p$,

$$J^{(\ell-k)} = (-k - \alpha_k \ell, \ldots, -k + \beta_k \ell), \qquad J^{(k)} = (k - \beta_k \ell, \ldots, k + \alpha_k \ell)$$

with $\alpha_k + \beta_k + 1 = r_k$. Consider $w_0 \in W$ defined by

$$\begin{aligned} &\widehat{w}_0(i_a^{(0)}) = (a-1)\ell \quad \text{for } a \in \{1, \ldots, r_0\} \\ &\widehat{w}_0(i_a^{(k)}) = -k - \alpha_k \ell + (a-1)\ell \quad \text{for any } k = 1, \ldots, p-1. \end{aligned} \tag{40}$$

Denote by $\mathcal{W}$ the set of signed permutations $w \in W$ which verify (i) and (ii) in (39). We have $w_0 \in \mathcal{W}$. Each $w \in \mathcal{W}$ can be written $w = w_0 v$ where $v = (v^{(0)}, \tau^{(1)}, \ldots, \tau^{(p-1)})$ belongs to the direct product $W_{I^{(0)}} \times S_{X^{(1)}} \times \cdots \times S_{X^{(p-1)}}$ with $W_{I^{(0)}}$ the Weyl group of type D_{r_0} defined on $\overline{I}^{(0)} \cup I^{(0)}$. We have by Lemma 2.1.1 $\varepsilon(w) = \varepsilon(w_0)(-1)^{l(v^{(0)})}(-1)^{l(\tau^{(1)})} \times \cdots \times (-1)^{l(\tau^{(p-1)})}$.

Set

$$P_0 = \sum_{v^{(0)} \in W_{I^{(0)}}} (-1)^{l(v^{(0)})} \varphi_\ell \left(\prod_{i \in X^{(0)}} x_i^{-\widehat{w}_0 v^{(0)}(i)} \right)$$
$$P_k = \sum_{\tau^{(k)} \in S_{X^{(k)}}} (-1)^{l(\tau^{(k)})} \varphi_\ell \left(\prod_{i \in X^{(k)}} x_i^{-\widehat{w}_0 v^{(k)}(i)-k} \right) \quad \text{for any } k \in \{1, \ldots, p-1\}.$$

We obtain

$$\varphi_\ell(P_\mu) = \varepsilon(w_0) x_{I^{(0)}}^{\mu^{(0)}} P_0 \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)}} P_k.$$

Given $v^{(0)} \in W_{I^{(0)}}$, we define $\widehat{v}^{(0)} = v^{(0)} - \xi_v$ where $\xi_v(i_a) = 1$ if $v(i_a) > 0$ and -1 otherwise. By (40), we have for any $a = 1, \ldots, r_0$, $\widehat{w}_0 v^{(0)}(i_a^{(0)}) = \widehat{v}^{(0)}(a)\ell$. Moreover, since

$\tau^{(k)} \in S_{X^{(k)}}$,

$$\widehat{w}_0\tau^{(k)}(i_a^{(k)}) = -k - \alpha_k\ell + (\tau^{(k)}(a) - 1)\ell \quad \text{for any } a = 1, \dots, r_k.$$

This yields

$$P_0 = \sum_{v^{(0)} \in W_{I^{(0)}}} (-1)^{l(v^{(0)})} \prod_{a=1}^{r_0} x_{i_a^{(0)}}^{-\widehat{v}^{(0)}(a)} = x_{I^{(0)}}^{-\rho'_{r_0}} \Delta_{I^{(0)}} \quad \text{and}$$

$$P_k = \sum_{\tau^{(k)} \in S_{X^{(k)}}} (-1)^{l(\tau^{(k)})} \prod_{a=1}^{r_k} x_{i_a^{(k)}}^{-\tau^{(k)}(a)+(\alpha_k+1)} = x_{X^{(k)}}^{\eta_{r_k}} \Delta_{X^{(k)}}$$

where for any $k = 1, \dots, p-1$, $\eta_{r_k} = -\rho'_{r_k} + (\alpha_k, \dots, \alpha_k) = -\rho_{r_k} + (\alpha_k + 1, \dots, \alpha_k + 1) \in \mathbb{Z}^{r_k}$,

$$\Delta_{I^{(0)}} = \prod_{i<j, i,j \in I^{(0)}} \left(1 - \frac{x_j}{x_i}\right) \prod_{r<s, r,s \in I^{(0)}} (1 - x_r x_s) \quad \text{and}$$

$$\Delta_{X^{(k)}} = \prod_{i<j, i,j \in X^{(k)}} \left(1 - \frac{x_j}{x_i}\right) \quad \text{for any } k = 1, \dots, p-1.$$

This gives

$$\varphi_\ell(P_\mu) = \varepsilon(w_0) x_{I^{(0)}}^{\mu^{(0)} - \rho'_{r_0}} \Delta_{I^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)} - \eta_{r_k}} \Delta_{X^{(k)}} = \varepsilon(w_0) x_{I^{(0)}}^{\mu^{(0)}} \Delta_{I^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)}} \Delta_{X^{(k)}}$$

where

$$\mu^{(0)} = \left(\frac{\mu_i + i - 1}{\ell} \,\middle|\, i \in I^{(0)}\right) - (0, \dots, r_0 - 1) \in \mathbb{Z}^{r_0}, \tag{41}$$

and for any $k = 1, \dots, p-1$,

$$\mu^{(k)} = |\, i \in X^{(k)} \left(\operatorname{sign}(i) \frac{\mu_{|i|} + |i| - 1 + \operatorname{sign}(i)k}{\ell} \,|\, i \in X^{(k)}\right) \\ - (0, \dots, r_k - 1) + (\alpha_k, \dots., \alpha_k) \in \mathbb{Z}^{r_k}. \tag{42}$$

Note that these formulas are essentially the same as for $G = Sp_{2n}$, except that we use $\rho'_n = (0, \dots, n-1)$ instead of $\rho_n = (1, \dots, n)$ for the half sum of positive roots. This gives the following theorem, whose proof is identical to that of Theorem 3.2.3:

Theorem 3.2.6. *Consider a partition μ of length n and $\ell = 2p - 1$ a positive integer. Let $I^{(0)}$ and $J^{(0)}$ be as in (38). For any $k = 0, \dots, p-1$, define the sets $X^{(k)}$ and $J^{(k)}$ by (31) and (38).*

- *If* $\operatorname{card}(I^{(0)}) \neq \frac{1}{2}(\operatorname{card}(J^{(0)}) + 1)$ *or if there exists $k \in \{1, \dots, p-1\}$ such that* $\operatorname{card}(X^{(k)}) \neq \operatorname{card}(J^{(k)})$*, then $\varphi_\ell(s_\mu) = 0$.*
- *Otherwise, set $r_0 = \operatorname{card}(I^{(0)})$ and for any $k = 0, \dots, p-1$, $r_k = \operatorname{card}(X^{(k)})$. Let $w_0 \in W$ be as in (40). Consider $\binom{\mu}{\ell} = (\mu^{(0)}, \dots, \mu^{(\ell-1)})$ where the $\mu^{(k)}$'s are defined by (41) and (42). Then $\binom{\mu}{\ell}$ is a dominant weight of $P_{\mathcal{I}}^+$ with $\mathcal{I} = \{I^{(0)}, X^{(1)}, \dots, X^{(p-1)}\}$, and we have*

$$\varphi_\ell(s_\mu) = \varepsilon(w_0) S_{\binom{\mu}{\ell}, \mathcal{I}}.$$

Example 3.2.7. Consider $\mu = (1,2,3,4,4,4,6,6)$ and take $\ell = 3$. We have $\mu + \rho'_8 = (1,3,5,7,8,9,12,13)$. Thus $I^{(0)} = \{2,6,7\}$, $X^{(1)} = \{\overline{8},\overline{4},\overline{1},3,5\}$ and $J^{(0)} = \{\overline{6},\overline{3},0,3,6\}$, $J^{(1)} = \{\overline{5},\overline{2},1,4,7\}$ and $J^{(2)} = \{\overline{7},\overline{4},\overline{1},2,5\}$. In particular, $\alpha_1 = 2$. Then $\mu^{(0)} = (1,2,2)$ and

$$\mu^{(1)} = \left(-\frac{13-1}{3} - 1 + 3, -\frac{7-1}{3} - 2 + 3, -\frac{1-1}{3} - 3 + 3, \right. \\ \left. \times\ \frac{5+1}{3} - 4 + 3, \frac{8+1}{3} - 5 + 3\right) \\ = (-2,-1,0,1,1).$$

We have $G_{\mathcal{I}} \simeq SO_6 \times GL_5$.

3.2.4. For $G = SO_{2n+1}$

Set $L_n = \{\overline{n-1},\dots,\overline{1},0,1,\dots,n\}$. Each $w \in W$ can be written $w = \zeta\sigma$ according to the decomposition of W as the semidirect product $(\mathbb{Z}/2\mathbb{Z})^n \propto S_n$. Given $w \in W$ we define $\widetilde{w} : J_n \to L_n$ such that $\widetilde{w}(x) = w(x) + \frac{1}{2}(1 - \xi(x))$ for any $x \in J_n$. For any $y \in L_n$, set $y^* = \overline{y} + 1$. We have then $\widetilde{w}(\overline{x}) = (w(x))^* = \overline{w(x)} + 1$.

Observe that $\rho = \rho''_n = (\frac{1}{2}, \frac{3}{2}, \dots, n - \frac{1}{2}) = \rho_n - (\frac{1}{2}, \dots, \frac{1}{2})$. Thus

$$w \cdot \rho''_n = w \cdot \rho_n - \frac{1}{2}(\xi(1),\dots,\xi(n)) = (\widetilde{w}(1),\dots,\widetilde{w}(n)) - \frac{1}{2}(1,\dots,1).$$

This permits us to write

$$P_\mu = x_1^{(\mu_1+1)} \cdots x_n^{(\mu_n+n)} \sum_{w\in W} \varepsilon(w) x_1^{-\widetilde{w}(1)} \cdots x_n^{-\widetilde{w}(n)}. \tag{43}$$

For any $k = 1,\dots,\ell$ (observe that k does not run over $\{0,\dots,\ell-1\}$ as for $G = Sp_{2n}$ or SO_{2n}), set

$$I^{(k)} = (i \in I_n \mid \mu_i + i \equiv k \bmod \ell) \quad \text{and} \quad J^{(k)} = (x \in L_n \mid x \equiv k \bmod \ell). \tag{44}$$

Note that $(J^{(k)})^* = J^{(l-k+1)}$. We then proceed essentially as in Section 3.2.2 by using $\widetilde{w}$ instead of w. We are going to see that for $G = SO_{2n+1}$, there exists an analogue of Theorem 3.2.3 whatever the parity of ℓ.

3.2.4.1. The even case $\ell = 2p$. For any $k = 1,\dots,p$, set $s_k = \operatorname{card}(I^{(k)})$, $r_k = \operatorname{card}(I^{(k)}) + \operatorname{card}(I^{(\ell-k+1)})$ and define $X^{(k)}$ as the increasing reordering of $\overline{I}^{(k)} \cup I^{(\ell-k+1)}$. Set

$$X^{(k)} = (i_1^{(k)},\dots,i_{r_k}^{(k)}). \tag{45}$$

For $k = 1,\dots,p$ consider the r_k-tuple $\mu^{(k)}$ such that

$$\mu^{(k)} = \left(\operatorname{sign}(i)\frac{\mu_{|i|} + |i| + \operatorname{sign}(i)k - \frac{1+\operatorname{sign}(i)}{2}}{\ell} \,\middle|\, i \in X^{(k)}\right) \in \mathbb{Z}^{r_k}.$$

For any $i \in I^{(k)}$ with $k = 1,\dots,p$, we have $x_i^{-\widetilde{w}(i)-1} = x_{\overline{i}}^{-\widetilde{w}(\overline{i})}$. Thus

$$\prod_{i\in X^{(k)}} x_i^{-\widetilde{w}(i)} = \prod_{i\in\overline{I}^{(k)}} x_i^{-\widetilde{w}(i)} \prod_{i\in I^{(\ell-k+1)}} x_i^{-\widetilde{w}(i)} = \prod_{i\in I^{(k)}} x_i^{-\widetilde{w}(i)-1} \prod_{i\in I^{(\ell-k+1)}} x_i^{-\widetilde{w}(i)}$$

and by definition of the $\mu^{(k)}$'s, (43) can be rewritten

$$P_\mu = \prod_{k=1}^{p} x_{X^{(k)}}^{\ell\mu^{(k)}} \times \sum_{w\in W} \varepsilon(w) \prod_{k=1}^{p} \prod_{i\in X^{(k)}} x_i^{-\widetilde{w}(i)} \times \prod_{k=1}^{p} \prod_{i\in X^{(k)}} x_i^{-k+1}.$$

We obtain

$$\varphi_\ell(P_\mu) = \prod_{k=1}^{p} x_{X^{(k)}}^{\mu^{(k)}} \times \sum_{w\in W} \varepsilon(w)\varphi_\ell\left(\prod_{k=1}^{p} \prod_{i\in X^{(k)}} x_i^{-\widetilde{w}(i)-k+1}\right).$$

We deduce the equivalences

$$\varphi_\ell\left(\prod_{k=0}^{p} \prod_{i\in X^{(k)}}^{s_k} x_i^{-w(i)+k-1}\right) \neq 0 \Longleftrightarrow \widetilde{w}(X^{(k)}) = J^{(\ell-k+1)} \quad \text{for any } k = 1,\dots,p. \tag{46}$$

In particular, we must have $\mathrm{card}(J^{(\ell-k+1)}) = \mathrm{card}(J^{(k)}) = r_k$. We can write

$$J^{(\ell-k+1)} = (-k+1-\alpha_k\ell,\dots,-k+1+\beta_k\ell) \quad \text{and}$$
$$J^{(k)} = (k-\beta_k\ell,\dots,k+\alpha_k\ell)$$

with $\alpha_k+\beta_k+1 = r_k$. Consider $w_0 \in W$ defined by

$$\widetilde{w}_0(i_a^{(k)}) = -k+1-\alpha_k\ell+(a-1)\ell \quad \text{for any } k = 1,\dots,p. \tag{47}$$

Denote by $\mathcal{W}$ the set of signed permutations $w \in W$ which verify the right-hand side of (46). We have $w_0 \in \mathcal{W}$. Each $w \in \mathcal{W}$ can be written $w = w_0 v$, where $\tau = (\tau^{(1)},\dots,\tau^{(p)})$ belongs to the direct product $S_{X^{(1)}} \times \cdots \times S_{X^{(p)}}$. We have also by Lemma 2.1.1 $\varepsilon(w) = \varepsilon(w_0)(-1)^{l(\tau^{(1)})} \times \cdots \times (-1)^{l(\tau^{(p)})}$.

For any $k = 1,\dots,p$, set

$$P_k = \sum_{\tau^{(k)}\in S_{X^{(k)}}} (-1)^{l(\tau^{(k)})}\varphi_\ell\left(\prod_{i\in X^{(k)}} x_i^{-\widetilde{w}_0\tau^{(k)}(i)-k+1}\right).$$

We obtain

$$\varphi_\ell(P_\mu) = \varepsilon(w_0)\prod_{k=1}^{p} x_{X^{(k)}}^{\mu^{(k)}} P_k.$$

By (47), we have

$$w_0\tau^{(k)}(i_a^{(k)}) = -k+1-\alpha_k\ell+(\tau^{(k)}(a)-1)\ell \quad \text{for any } a = 1,\dots,r_k.$$

This yields

$$P_k = \sum_{\tau^{(k)}\in S_{X^{(k)}}} (-1)^{l(\tau^{(k)})} \prod_{a=1}^{r_k} x_{i_a^{(k)}}^{-\tau^{(k)}(a)+(\alpha_k+1)} = x_{X^{(k)}}^{\eta_{r_k}} \Delta_{X^{(k)}}$$

where for any $k = 1,\dots,p$, $\eta_{r_k} = -\rho_{r_k} + (\alpha_k+1,\dots,\alpha_k+1) \in \mathbb{Z}^{r_k}$ and

$$\Delta_{X^{(k)}} = \prod_{i<j\ i,j\in X^{(k)}} \left(1-\frac{x_j}{x_i}\right).$$

Note that the computation only makes root systems of type A appear in this case. This gives

$$\varphi_\ell(P_\mu) = \varepsilon(w_0) \prod_{k=1}^{p} x_{X^{(k)}}^{\mu^{(k)} - \eta_{r_k}} \Delta_{X^{(k)}} = \varepsilon(w_0) \prod_{k=1}^{p} x_{X^{(k)}}^{\mu^{(k)}} \Delta_{X^{(k)}}$$

where for any $k = 1, \ldots, p$,

$$\mu^{(k)} = \left(\operatorname{sign}(i) \frac{\mu_{|i|} + |i| + \operatorname{sign}(i)k - \frac{1+\operatorname{sign}(i)}{2}}{\ell} \middle| i \in X^{(k)} \right)$$
$$- (1, \ldots, r_k) + (\alpha_{k+1}, \ldots, \alpha_{k+1}) \in \mathbb{Z}^{r_k}. \tag{48}$$

Similarly to Theorem 3.2.3 we obtain:

Theorem 3.2.8. *Consider a partition μ of length n and $\ell = 2p$ a positive integer. For any $k = 1, \ldots, p$ define the sets $X^{(k)}$, $J^{(k)}$ by (44) and (45).*

- *If there exists $k \in \{1, \ldots, p\}$ such that $\operatorname{card}(X^{(k)}) \neq \operatorname{card}(J^{(k)})$, then $\varphi_\ell(s_\mu) = 0$.*
- *Otherwise, for any $k = 1, \ldots, p$, set $r_k = \operatorname{card}(X^{(k)})$. Let $w_0 \in W$ be as in (47). Consider $\binom{\mu}{\ell} = (\mu^{(1)}, \ldots, \mu^{(p)})$ where the $\mu^{(k)}$'s are defined by (48). Then $\binom{\mu}{\ell}$ is a dominant weight of $P_{\mathcal{I}}^+$ with $\mathcal{I} = \{X^{(1)}, \ldots, X^{(p)}\}$ and we have*

$$\varphi_\ell(s_\mu) = \varepsilon(w_0) S_{\binom{\mu}{\ell}, \mathcal{I}}.$$

Example 3.2.9. Consider $\mu = (2, 5, 5, 6, 7, 9)$ and $\ell = 2$. Then $\mu + \rho_6 = (3, 7, 8, 10, 12, 15)$. Hence $I_1 = \{1, 2, 6\}$ and $I_2 = \{3, 4, 5\}$. Moreover, $J_2 = \{\overline{4}, \overline{2}, 0, 2, 4, 6\}$ and $J_1 = \{\overline{5}, \overline{3}, \overline{1}, 1, 3, 5\}$. Then $\widetilde{w}_0$ sends $X_1 = \{\overline{6}, \overline{2}, \overline{1}, 3, 4, 5\}$ on J_2. This gives

$$\widetilde{w}_0 = \begin{pmatrix} \overline{6} & \overline{5} & \overline{4} & \overline{3} & \overline{2} & \overline{1} & 1 & 2 & 3 & 4 & 5 & 6 \\ \overline{4} & 5 & \overline{3} & \overline{1} & \overline{2} & 0 & 1 & 3 & 2 & 4 & 6 & 5 \end{pmatrix}$$

by using (47). Hence

$$w_0 = \begin{pmatrix} \overline{6} & \overline{5} & \overline{4} & \overline{3} & \overline{2} & \overline{1} & 1 & 2 & 3 & 4 & 5 & 6 \\ \overline{5} & \overline{6} & \overline{4} & \overline{2} & \overline{3} & \overline{1} & 1 & 3 & 2 & 4 & 6 & 5 \end{pmatrix}.$$

We have $\varepsilon(\mu) = 1$, $\alpha_1 = 2$ and

$$\mu^{(1)} = (-7, -3, -1, 4, 5, 6) - (1, 2, 3, 4, 5, 6) + (3, 3, 3, 3, 3, 3)$$
$$= (-5, -2, -1, 3, 3, 3).$$

We have then $G_{\mathcal{I}} \simeq GL_6$.

3.2.4.2. The case $\ell = 2p + 1$. In addition to the sets $X^{(k)}, k = 1, \ldots, p$ defined in (45), we have also to consider $I^{(p+1)} = \{i_1^{(p+1)}, \ldots, i_{r_{p+1}}^{(p+1)}\}$. This yields

$$\mu^{(p+1)} = \left(\frac{\mu_i + i + p}{\ell} \mid i \in I^{(p+1)} \right)$$

We have

$$\varphi_\ell(P_\mu) = x_{I^{(p+1)}}^{\mu^{(p+1)}} \prod_{k=1}^{p} x_{X^{(k)}}^{\mu^{(k)}} \times \sum_{w \in W} \varepsilon(w) \varphi_\ell \left(\prod_{i \in I^{(p+1)}} x_i^{-\widetilde{w}(i) - p} \prod_{k=1}^{p} \prod_{i \in X^{(k)}} x_i^{-\widetilde{w}(i) - k - 1} \right)$$

and the equivalence

$$\varphi_\ell\left(\prod_{i\in I^{(p+1)}} x_i^{-\widetilde{w}(i)-p}\prod_{k=1}^{p}\prod_{i\in X^{(k)}} x_i^{-w(i)+k-1}\right)\neq 0$$
$$\Longleftrightarrow \begin{cases}\widetilde{w}(X^{(k)})=J^{(\ell-k+1)} & \text{for any } k=1,\dots,p\\ \widetilde{w}(I^{(p+1)}\cup\overline{I}^{(p+1)})=J^{(\ell-p)}=J^{(p+1)}.\end{cases} \tag{49}$$

Indeed, $(J^{(p+1)})^*=J^{(p+1)}$. In particular, we must have $\mathrm{card}(J^{(p+1)})=2\mathrm{card}(I^{(p+1)})=2r_{p+1}$. Thus we can set $J^{(p+1)}=(-p-(r_{p+1}-1)\ell,\dots,-p+r_{p+1}\ell)$. Consider $w_0\in W$ defined by (47) and

$$\widetilde{w}_0(i_a^{(p+1)})=-p+a\ell \quad \text{for any } a=1,\dots,r_{p+1}. \tag{50}$$

Denote by $\mathcal{W}$ the set of signed permutations $w\in W$ which verify the right-hand side of (49). We have $w_0\in\mathcal{W}$. Each $w\in\mathcal{W}$ can be written $w=w_0v$, where $v=(\tau^{(1)},\dots,\tau^{(p)},v^{(p+1)})$ belongs to the direct product $S_{X^{(1)}}\times\cdots\times S_{X^{(p)}}\times W_{I^{(p+1)}}$. We have also $\varepsilon(w)=\varepsilon(w_0)(-1)^{l(\tau^{(1)})}\times\cdots\times(-1)^{l(\tau^{(p)})}(-1)^{l(v^{(p+1)})}$. This permits us to write

$$\varphi_\ell(P_\mu)=\varepsilon(w_0)x_{I^{(p+1)}}^{\mu^{(p+1)}}P_{p+1}\prod_{k=1}^{p}x_{X^{(k)}}^{\mu^{(k)}}P_k \quad \text{where}$$
$$P_{p+1}=\sum_{v^{(p+1)}\in W_{I^{(p+1)}}}(-1)^{l(v^{(p+1)})}\varphi_\ell\left(\prod_{i\in I^{(p+1)}}x_i^{-\widetilde{w}_0v^{(p+1)}(i)-p}\right).$$

The functions P_k, $k=1,\dots,p$ can be computed as in the even case. For P_{p+1}, observe that each $v^{(p+1)}\in W_{I^{(p+1)}}$ can be written $v^{(p+1)}=\zeta\sigma$ with $\sigma\in S_{I^{(p+1)}}$. According to this decomposition, we have for any $a=1,\dots,r_{p+1}$, $\widetilde{w}_0v^{(p+1)}(i_a^{(p+1)})=\xi(a)(-p+\sigma(a)\ell)$.

$$\begin{aligned}P_{p+1}&=\sum_{v^{(p+1)}\in W_{I^{(p+1)}}}(-1)^{l(v^{(p+1)})}\varphi_\ell\left(\prod_{a=1}^{r_{p+1}}x_{i_a^{(p+1)}}^{-\xi(a)(-p+\sigma(a)\ell)-p}\right)\\ &=\sum_{v^{(p+1)}\in W_{I^{(p+1)}}}(-1)^{l(v^{(p+1)})}\prod_{a=1}^{r_{p+1}}x_{i_a^{(p+1)}}^{-\frac{1-\xi(a)}{2}-\xi\sigma(a)}.\end{aligned}$$

Thus

$$\begin{aligned}P_{p+1}&=\prod_{i\in I^{(p+1)}}x_i^{-1/2}\sum_{v^{(p+1)}\in W_{I^{(p+1)}}}(-1)^{l(v^{(p+1)})}\left(v^{(p+1)}\cdot\prod_{a=1}^{r_{p+1}}x_{i_a^{(p+1)}}^{-(a-\frac{1}{2})}\right)\\ &=x_{I^{(p+1)}}^{-\rho_{r_{p+1}}}\Delta_{I^{(p+1)}}\\ \Delta_{I^{(p+1)}}&=\prod_{i<j\ i,j\in I^{(p+1)}}\left(1-\frac{x_j}{x_i}\right)\prod_{r<s\ r,s\in I^{(p+1)}}(1-x_rx_s)\prod_{i\in I^{(p+1)}}(1-x_i).\end{aligned}$$

This means that when ℓ is odd

$$\varphi_\ell(P_\mu)=\varepsilon(w_0)\prod_{k=1}^{p}x_{X^{(k)}}^{\mu^{(k)}}\Delta_{X^{(k)}}\times x_{I^{(p+1)}}^{\mu^{(p+1)}}\Delta_{I^{(p+1)}}$$

where

$$\mu^{(p+1)} = \left(\frac{\mu_i + i + p}{\ell} \middle| i \in I^{(p+1)} \right) - (1, \ldots, r_{p+1}) \in \mathbb{Z}^{r_{p+1}}. \tag{51}$$

This gives the following theorem:

Theorem 3.2.10. *Consider a partition μ of length n and $\ell = 2p + 1$ a positive integer. Define $X^{(k)}$, $J^{(k)} k = 1, \ldots, p$ and $I^{(p+1)}$, $J^{(p+1)}$ by (44) and (45).*

- *If* $\mathrm{card}(I^{(p+1)}) \neq \frac{1}{2}\mathrm{card}(J^{(p+1)})$ *or if there exists $k \in \{1, \ldots, p\}$ such that* $\mathrm{card}(X^{(k)}) \neq \mathrm{card}(J^{(k)})$*, then $\varphi_\ell(s_\mu) = 0$.*
- *Otherwise, set $r_{p+1} = \mathrm{card}(I^{(p+1)})$ and for any $k = 1, \ldots, p$, $r_k = \mathrm{card}(X^{(k)})$. Let $w_0 \in W$ verifying (47) and (50). Consider $\binom{\mu}{\ell} = (\mu^{(p+1)}, \mu^{(1)}, \ldots, \mu^{(p)})$ where the $\mu^{(k)}$'s are defined by (48) and (51). Then $\binom{\mu}{\ell}$ is a dominant weight of $P_{\mathcal{I}}^{+}$ with $\mathcal{I} = \{I^{(p+1)}, X^{(1)}, \ldots, X^{(p)}\}$, and we have*

$$\varphi_\ell(s_\mu) = \varepsilon(w_0) S_{\binom{\mu}{\ell}, \mathcal{I}}.$$

Example 3.2.11. Consider $\mu = (1, 5, 5, 6, 7, 9)$ and take $\ell = 3$. We have $\mu + \rho_6 = (2, 7, 8, 10, 12, 15)$. Thus $X^{(1)} = \{\bar{4}, \bar{2}, 5, 6\}$, $I^{(2)} = \{1, 3\}$ and $J^{(1)} = \{\bar{5}, \bar{2}, 1, 4\}$, $J^{(2)} = \{\bar{4}, \bar{1}, 2, 5\}$. In particular $\alpha_2 = 1$. Then

$$\begin{aligned} \mu^{(1)} &= \left(-\frac{10-1}{3} - 1 + 1, -\frac{7-1}{3} - 2 + 1, \frac{12}{3} - 3 + 1, \frac{15}{3} - 4 + 1 \right) \\ &= (-2, -2, 3, 3) \end{aligned}$$

and $\mu^{(2)} = (\frac{2+1}{3} - 1, \frac{8+1}{3} - 2) = (0, 1)$. Moreover, one has by using (47)

$$\widetilde{w}_0 = \begin{pmatrix} \bar{6} & \bar{5} & \bar{4} & \bar{3} & \bar{2} & \bar{1} & 1 & 2 & 3 & 4 & 5 & 6 \\ \bar{3} & 0 & \bar{5} & \bar{4} & \bar{2} & \bar{1} & 2 & 3 & 5 & 6 & 1 & 4 \end{pmatrix}.$$

Hence

$$w_0 = \begin{pmatrix} \bar{6} & \bar{5} & \bar{4} & \bar{3} & \bar{2} & \bar{1} & 1 & 2 & 3 & 4 & 5 & 6 \\ \bar{4} & \bar{1} & \bar{6} & \bar{5} & \bar{3} & \bar{2} & 2 & 3 & 5 & 6 & 1 & 4 \end{pmatrix}$$

and $\varepsilon(\mu) = 1$. We have, moreover, $G_{\mathcal{I}} \simeq SO_5 \times GL_4$.

4. Parabolic Kazhdan–Lusztig polynomials

We recall briefly in this section some basics on Affine Hecke algebras and parabolic Kazhdan–Lusztig polynomials associated with classical root systems. The reader is referred to [14,16] for detailed expositions. Note that the definition of the Hecke algebra used in [14] coincides with that used in [9,16] (with generators H_w) up to the change $q \to q^{-1}$.

4.1. Extended affine Weyl group

Consider a root system of type A_{n-1}, B_n, C_n or D_n. For any $\beta \in P$, we denote by t_β the translation defined in $\mathfrak{h}_{\mathbb{R}}^*$ by $\gamma \longmapsto \gamma + \beta$. The extended affine Weyl group $\widehat{W}$ is the group

$$\widehat{W} = \{w t_\beta \mid w \in W, \beta \in P\}$$

with multiplication determined by the relations $t_\beta t_\gamma = t_{\beta+\gamma}$ and $wt_\beta = t_{w\cdot\beta}w$. The group $\widehat{W}$ is not a Coxeter group but contains the affine Weyl group $\widetilde{W}$ generated by reflections through the affine hyperplanes $H_{\alpha,k} = \{\beta \in \mathfrak{h}_{\mathbb{R}}^* \mid (\beta, \alpha^\vee) = k\}$. It makes sense to define a length function on $\widehat{W}$ verifying

$$l(wt_\beta) = \sum_{\alpha \in R_+} \left|(\beta, \alpha^\vee) + 1_{R_-}(w \cdot \alpha)\right| \tag{52}$$

where for any $w \in W$, $1_{R_-}(w \cdot \alpha) = 0$ if $w \cdot \alpha \in R_+$ and $1_{R_-}(w \cdot \alpha) = 1$ if $w \cdot \alpha \in -R_+ = R_-$. Write n_β for the element of maximal length in $Wt_\beta W$. It follows from (52) that for any $\lambda \in P_+$, we have $l(wt_\lambda) = l(w) + l(t_\lambda)$. This gives

$$n_\lambda = w_0 t_\lambda \tag{53}$$

where w_0 denotes the longest element of W. There exists a unique element $\eta \in R_+$ such that the fundamental alcove

$$\mathcal{A} = \{\beta \in \mathfrak{h}_{\mathbb{R}}^* \mid (\beta, \alpha^\vee) \geq 0 \;\forall \alpha \in R_+ \text{ and } (\beta, \eta^\vee) < 1\}$$

is a fundamental region for the action of $\widetilde{W}$ on $\mathfrak{h}_{\mathbb{R}}^*$. This means that, for any $\beta \in \mathfrak{h}_{\mathbb{R}}^*$, the orbit $\widetilde{W} \cdot \beta$ intersects $\mathcal{A}$ in a unique point. Each $w \in \widehat{W}$ can be written in the form $w = w_{\mathcal{A}} w_{\mathrm{aff}}$, where $w_{\mathrm{aff}} \in \widetilde{W}$ and $w_{\mathcal{A}}$ belongs to the stabilizer of $\mathcal{A}$ under the action of $\widehat{W}$. This implies that $\mathcal{A}$ is also a fundamental domain for the action of $\widehat{W}$ on $\mathfrak{h}_{\mathbb{R}}^*$. The Bruhat ordering on $\widehat{W}$ is defined by taking the transitive closure of the relations

$$w < sw \quad \text{whenever } l(w) < l(sw)$$

for all $w \in \widehat{W}$ and all (affine) reflections $s \in \widetilde{W}$.

In fact, the natural action of $\widehat{W}$ on the weight lattice P obtained by considering P as a sublattice of $\mathfrak{h}_{\mathbb{R}}^*$ is not one which is relevant for our purpose. For any integer $m \in \mathbb{Z}^*$, we obtain a faithful representation π_m of $\widehat{W}$ on P by setting for any $\beta, \gamma \in P$, $w \in W$

$$\pi_m(w) \cdot \gamma = w \cdot \gamma \quad \text{and} \quad \pi_m(t_\beta) \cdot \gamma = \gamma + m\beta.$$

Warning: In the sequel, the extended affine Weyl group $\widehat{W}$ acts on the weight lattice P via $\pi_{-\ell}$ where ℓ is a fixed nonnegative integer.

We write for simplicity $wt_\beta \cdot \gamma$ rather than $\pi_{-\ell}(wt_\beta) \cdot \gamma$. Hence for any $w \in W$ and any $\beta \in P$, we have $wt_\beta \cdot \gamma = w \cdot \gamma - \ell w \cdot \beta$. The fundamental region for this new action of $\widehat{W}$ on P is the alcove $\mathcal{A}_\ell$ obtained by expanding $\mathcal{A}$ with the factor $-\ell$. This gives

$$\mathcal{A}_\ell = \begin{cases} \{\nu = (\nu_1, \ldots, \nu_n) \mid 0 \geq \nu_1 \geq \cdots \geq \nu_n > -\ell\} & \text{for types } A, B, C \\ \{\nu = (\nu_1, \ldots, \nu_n) \mid 0 \geq -|\nu_1| \geq \nu_2 \geq \cdots \geq \nu_n > -\ell\} & \text{for type } D. \end{cases}$$

Consider a weight $\beta \in P$. Then its orbit intersects $\mathcal{A}_\ell$ in a unique weight ν. Then there is a unique $w(\beta) \in \widehat{W}$ of minimal length such that $w(\beta) \cdot \nu = \beta$. We denote by W_ν the stabilizer of $\nu \in \mathcal{A}_\ell$ in $\widehat{W}$. Since $\nu \in \mathcal{A}_\ell$, W_ν is in fact a subgroup of W.

Lemma 4.1.1. *Consider $\lambda \in P^+$ and suppose $\ell > n$. Then*

1. $w(\ell\lambda + \rho) = n_{\lambda^*}\tau^{-n+1}$ *with* $\lambda^* = -w_0(\lambda)$ *and* $\tau = s_1 s_2 \cdots s_{r-1} t_{\varepsilon_1}$ *for type* A.
2. $w(\ell\lambda + \rho) = n_\lambda$ *for types* B, C *and* D.

Proof. 1. See Lemma 2.3 in [9].

2. Observe first that $w_0 \cdot \rho = -\rho$ belongs to $\mathcal{A}_\ell$ for types B, C, D since $\ell > n$. We have

$$\ell\lambda + \rho = t_{-\lambda} \cdot \rho = t_{-\lambda} w_0 \cdot (w_0 \cdot \rho) = t_{-\lambda} w_0 \cdot (-\rho).$$

Moreover $W_{-\rho} = \{1\}$. Since $-\rho \in \mathcal{A}_\ell$, this means that $w(\ell\lambda + \rho) = t_{-\lambda} w_0 = w_0 t_{w_0 \cdot (-\lambda)} = w_0 t_\lambda = n_\lambda$, where the last equality follows from (53). ■

4.2. Affine Hecke algebra and K–L polynomials

The Hecke algebra associated with the root system R of type A_n, B_n, C_n or D_n is the $\mathbb{Z}[q, q^{-1}]$-algebra defined by the generators T_w, $w \in \widehat{W}$ and relations

$$T_{w_1} T_{w_2} = T_{w_1} T_{w_2} \quad \text{if } l(w_1 w_2) = l(w_1) + l(w_2),$$
$$T_{s_i} T_w = (q^{-1} - q) T_w + T_{s_i w} \quad \text{if } l(s_i w) < l(w) \text{ and } i \in I_n.$$

In particular, we have $T_i^2 = (q^{-1} - q) T_i + 1$ for any $i \in I_n$. The bar involution on $\widehat{H}$ is the $\mathbb{Z}$-linear automorphism defined by

$$\overline{q} = q^{-1} \quad \text{and} \quad \overline{T}_w = T_{w^{-1}}^{-1} \quad \text{for any } w \in \widehat{W}.$$

Kazhdan and Lusztig have proved that there exists a unique basis $\{C'_w \mid w \in \widehat{W}\}$ of $\widehat{H}$ such that

$$\overline{C}'_w = C'_w \quad \text{and} \quad C'_w = \sum_{y \le w} p_{y,w} T_y$$

where $p_{w,w} = 1$ and $p_{y,w} \in q\mathbb{Z}[q]$ for any $y < w$. We will refer to the polynomials $p_{y,w}(q)$ as Kazhdan–Lusztig polynomials. They are renormalizations of the polynomials $P_{y,w}$ originally introduced by Kazhdan and Lusztig in [6]. Specifically, we have $p_{y,w} = q^{l(w)-l(y)} P_{y,w}$.

Let us define the q-partition function $\mathcal{P}_q$ by

$$\prod_{\alpha \in R_+} \frac{1}{1 - q x^\alpha} = \sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q(\beta) x^\beta.$$

Given λ and μ in P, the Lusztig q-analogue $K_{\lambda,\mu}(q)$ is defined by

$$K_{\lambda,\mu}(q) = \sum_{w \in W} \varepsilon(w) \mathcal{P}_q(w \circ \lambda - \mu).$$

Then one has the following theorem due to Lusztig:

Theorem 4.2.1. *Suppose λ, μ are dominant weights. Then $K_{\lambda,\mu}(q) = p_{n_\mu, n_\mu}(q)$.*

One defines the action of the bar involution on the parabolic module $P_\nu = \widehat{H}\nu$, $\nu \in \mathcal{A}_\ell$, by setting $\overline{q} = q^{-1}$ and $\overline{h \cdot \nu} = \overline{h} \cdot \nu$ for any $h \in \widehat{H}$. Deodhar has proved that there exist two bases $\{C_\lambda^+ \mid \lambda \in \widehat{W} \cdot \nu\}$ and $\{C_\lambda^- \mid \lambda \in \widehat{W} \cdot \nu\}$ of P_ν belonging respectively to

$$L_\nu^+ = \coprod_{\lambda \in \widehat{W} \cdot \nu} \mathbb{Z}[q]\lambda \quad \text{and} \quad L_\nu^- = \coprod_{\lambda \in \widehat{W} \cdot \nu} \mathbb{Z}[q^{-1}]\lambda$$

characterized by

$$\begin{cases} \overline{C}_\lambda^+ = C_\lambda^+ \\ C_\lambda^+ \equiv \lambda \bmod q L_\nu^+ \end{cases} \quad \text{and} \quad \begin{cases} \overline{C}_\lambda^- = C_\lambda^- \\ C_\lambda^- \equiv \lambda \bmod q^{-1} L_\nu^-. \end{cases}$$

We will only need the basis $\{C_\lambda^- \mid \lambda \in \widehat{W} \cdot \nu\}$ in the sequel. The parabolic Kazhdan–Lusztig polynomials $P_{\lambda,\mu}^-$ are then defined by the expansion

$$C_\lambda^- = \sum_{\mu \in \widehat{W}\cdot\lambda} (-1)^{l(w(\lambda))+l(w(\mu))} P_{\mu,\lambda}^-(q^{-1})\mu$$

(see [16] Theorem 3.5). In particular, they belong to $\mathbb{Z}[q]$. Their expansion in terms of the ordinary Kazhdan–Lusztig polynomials is given by the following theorem due to Deodhar:

Theorem 4.2.2. *Consider $\nu \in \mathcal{A}_\ell$ and $\lambda \in \widehat{W} \cdot \nu$. Then for any $\mu \in \widehat{W} \cdot \lambda$, we have*

$$P_{\lambda,\mu}^-(q) = \sum_{z \in W_\nu} (-q)^{l(z)} p_{w(\mu)z,w(\lambda)}(q) \tag{54}$$

with the notation of Section 4.1.

Remark. When ν is regular, that is $W_\nu = \{1\}$, we have $P_{\lambda,\mu}^-(q) = p_{w(\mu),w(\lambda)}(q)$.

5. Generalized Hall–Littlewood functions

5.1. Plethysm and parabolic K–L polynomials

Consider ℓ a nonnegative integer and $\zeta \in \mathbb{C}$ such that ζ^2 is a primitive ℓ-th root of 1. We briefly recall in this paragraph the arguments of [9] which establish that the coefficients of the plethysm $\psi_\ell(s_\lambda)$ on the basis of Weyl characters are, up to a sign, parabolic Kazhdan–Lusztig polynomials specialized at $q = 1$.

For any $\lambda \in P_+$, denote by $V_q(\lambda)$ the finite dimensional $U_q(\mathfrak{g})$-module of highest weight λ. Its character is also the Weyl character s_λ. Let $U_{q,\mathbb{Z}}(\mathfrak{g})$ be the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by the elements

$$E_i^{(k)} = \frac{E_i^{(k)}}{[k]_i!}, \qquad F_i^{(k)} = \frac{F_i^{(k)}}{[k]_i!} \quad \text{and} \quad K_i^{\pm 1}$$

where $E_i, F_i, K_i^{\pm 1}, i \in I_n$ are the generators of $U_q(\mathfrak{g})$. The indeterminate q can be specialized at ζ in $U_{q,\mathbb{Z}}(\mathfrak{g})$. Thus it makes sense to define $U_\zeta(\mathfrak{g}) = U_{q,\mathbb{Z}}(\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$, where $\mathbb{Z}[q, q^{-1}]$ acts on $\mathbb{C}$ by $q \mapsto \zeta$. Fix a highest weight vector v_λ in $V_q(\lambda)$. We have $V_q(\lambda) = U_q(\mathfrak{g}) \cdot v_\lambda$. Similarly, $V_\zeta(\lambda) = U_\zeta(\mathfrak{g}) \cdot v_\lambda$ is a $U_\zeta(\mathfrak{g})$ module called a Weyl module, and one has $\mathrm{char}(V_\zeta(\lambda)) = s_\lambda$. The module $V_\zeta(\lambda)$ is not simple but admits a unique simple quotient denoted by $L(\lambda)$.

From results due to Kazhdan–Lusztig and KashiwaraTanisaki, one obtains the following decomposition of $\mathrm{char}(L(\lambda))$ on the basis of Weyl characters:

Theorem 5.1.1. *Consider $\lambda \in P_+$.*

1. *The character of $L(\lambda)$ decomposes on the form*

$$\mathrm{char}(L(\lambda)) = \sum_\mu (-1)^{l(w(\lambda+\rho))-l(w(\mu+\rho))} P_{\mu+\rho,\lambda+\rho}^-(1) s_\mu \tag{55}$$

where the sum runs over the dominant weights $\mu \in P_+$ such that $\mu + \rho \in \widehat{W} \cdot (\lambda + \rho)$.

2. *The parabolic Kazhdan–Lusztig polynomials $P_{\mu+\rho,\lambda+\rho}^-(q)$ have nonnegative integer coefficients.*

Remark. The decomposition (55) has been conjectured by Kazhdan–Lusztig and proved by Kashiwara–Tanisaki. In [5], Kashiwara and Tanisaki have also obtained that the parabolic Kazhdan–Lusztig polynomials have nonnegative integer coefficients as soon as the Coxeter system under consideration corresponds to the Weyl group of a Kac–Moody Lie algebra, as in the particular context of this paper.

Consider a nonnegative integer ℓ. The Frobenius map Fr_ℓ is the algebra homomorphism defined from $U_\zeta(\mathfrak{g})$ to $U(\mathfrak{g})$ by $\mathrm{Fr}_\ell(K_i)=1$ and

$$\mathrm{Fr}_\ell(E_i^{(k)})=\begin{cases} e_i^{(k/\ell)} & \text{if } \ell \text{ divides } k\\ 0 & \text{otherwise}\end{cases} \quad \text{and} \quad \mathrm{Fr}_\ell(F_i^{(k)})=\begin{cases} f_i^{(k/\ell)} & \text{if } \ell \text{ divides } k\\ 0 & \text{otherwise}\end{cases}$$

where $e_i, f_i, i \in I_n$ are the Chevalley generators of the enveloping algebra $U(\mathfrak{g})$. This permits us to endow each $U(\mathfrak{g})$-module M with the structure of a $U_\zeta(\mathfrak{g})$-module M^{Fr_ℓ}. Then we have

$$\mathrm{char}(M^{\mathrm{Fr}_\ell})=\psi_\ell(\mathrm{char}(M))$$

in particular for any $\lambda \in P_+$, $\mathrm{char}(V(\lambda)^{\mathrm{Fr}_\ell})=\psi_\ell(s_\lambda)$.

Each dominant weight $\lambda \in P_+$, can be uniquely decomposed in the form $\lambda=\overset{r}{\lambda}+\ell\overset{q}{\lambda}$ where $\overset{r}{\lambda},\overset{q}{\lambda}\in P_+$ and $\overset{r}{\lambda}=(\overset{r}{\lambda}_1,\ldots,\overset{r}{\lambda}_n)$ verifies $0\le \overset{r}{\lambda}_{i+1}-\overset{r}{\lambda}_i<\ell$ for any $i\in I_n$.

Theorem 5.1.2. *(Lusztig) The simple $U_\zeta(\mathfrak{g})$-module $L(\lambda)$ is isomorphic to the tensor product*

$$L(\lambda)\simeq L(\overset{r}{\lambda})\otimes V(\overset{q}{\lambda})^{\mathrm{Fr}_\ell}.$$

By replacing λ by $\ell\lambda$ in the previous theorem, we have $\overset{r}{\lambda}=0$ and $\overset{q}{\lambda}=\lambda$. Thus $L(\ell\lambda)\simeq V(\lambda)^{\mathrm{Fr}_\ell}$. Then one deduces from (55) the equality

$$\psi_\ell(s_\lambda)=\mathrm{char}(L(\ell\lambda))=\sum_{\mu+\rho\in\widehat{W}\cdot(\ell\lambda+\rho)}(-1)^{l(w(\lambda+\rho))-l(w(\mu+\rho))}P^-_{\mu+\rho,\ell\lambda+\rho}(1)s_\mu$$

which shows that the coefficients of the expansion of $\psi_\ell(s_\lambda)$ on the basis of Weyl characters are, up to a sign, parabolic Kazhdan–Lusztig polynomials specialized at $q=1$. This gives

$$\left|\langle\psi_\ell(s_\lambda),s_\mu\rangle\right|=\left|\langle s_\lambda,\varphi(s_\mu)\rangle\right|=P^-_{\mu+\rho,\ell\lambda+\rho}(1).$$

By definition of the action of $\widehat{W}$ on P, we have $\widehat{W}\cdot(\ell\lambda+\rho)=\widehat{W}\cdot\rho$. This implies the

Corollary 5.1.3 (*Of* Theorems 4.2.2 *and* 5.1.2). *For any nonnegative integer ℓ*

$$\psi_\ell(s_\lambda)=\sum_{\mu+\rho\in\widehat{W}\cdot\rho}(-1)^{l(w(\lambda+\rho))-l(w(\mu+\rho))}P^-_{\mu+\rho,\ell\lambda+\rho}(1)s_\mu.$$

In particular $\varphi(s_\mu)\neq 0$ if and only if $\mu+\rho\in\widehat{W}\cdot\rho$, that is $\mu+\rho=w\cdot\rho-\ell\beta$ with $w\in W$ and $\beta\in P$.

Remark. The equivalence $\varphi(s_\mu)\neq 0 \iff \mu+\rho\in\widehat{W}\cdot\rho$ can also be obtained as a more elementary form from algorithms described in Section 3.2.

5.2. Parabolic K–L polynomials and branching coefficients

Warning: In the sequel of the paper, we will suppose that ℓ is odd when the Lie groups under consideration are of type C or D.

Under this hypothesis, we have for any $\mu \in \mathcal{P}_n \varphi_\ell(s_\mu) = 0$ or

$$\varphi_\ell(s_\mu) = \varepsilon(w_0) S_{\binom{\mu}{\ell},\mathcal{I}} \tag{56}$$

according to the results of Section 3.2.

Remark. According to the algorithms described in Section 3.2, when $\varphi_\ell(s_\mu) \neq 0$, the cardinalities of the sets $I^{(k)}$ or $X^{(k)}$ contained in $\mathcal{I}$ are determined by those of the sets $J^{(k)}$. In particular they depend only on n and ℓ and not on the partition μ considered. Thus in (56), the underlying subgroup of Levi type $G_{\mathcal{I}}$ is, up to isomorphism, independent on μ.

By using Proposition 2.5.2 and Theorems 3.2.1, 3.2.3, 3.2.6, 3.2.8 and 3.2.10, we deduce from Corollary 5.1.3 the

Theorem 5.2.1. *For any $\lambda, \mu \in \mathcal{P}_n$ such that $\mu + \rho \in \widehat{W} \cdot \rho$*

$$P^-_{\mu+\rho,\ell\lambda+\rho}(1) = \left[V(\lambda) : V_{\mathcal{I}}\binom{\mu}{\ell}\right]$$

where $\binom{\mu}{\ell}$ and $\mathcal{I}$ are obtained from μ and ℓ by applying the algorithms described in Section 3.2.

5.3. The functions H^ℓ_μ

For any $\mu \in \mathcal{P}_n$, we define the function G^ℓ_μ by setting

$$G^\ell_\mu = \sum_{\lambda \in \mathcal{P}_n} \left[V(\lambda) : V_{\mathcal{I}}\binom{\mu}{\ell}\right]_q s_\lambda \tag{57}$$

where for any $\lambda \in \mathcal{P}_n$, $[V(\lambda) : V_{\mathcal{I}}\binom{\mu}{\ell}]_q = P^-_{\mu+\rho,\ell\lambda+\rho}(q)$. We also consider the function H^ℓ_μ such that

$$H^\ell_\mu = G^\ell_{\ell\mu}. \tag{58}$$

Theorem 5.3.1. *Consider a partition $\mu \in \mathcal{P}_n$.*

1. *The coefficients of G^ℓ_μ and H^ℓ_μ on the basis of Weyl characters are polynomials in q with nonnegative integer coefficients.*
2. *We have $H^1_\mu = s_\mu$.*
3. *For ℓ sufficiently large, $H^\ell_\mu = Q'_\mu$, that is H^ℓ_μ coincide with the Hall–Littlewood function associated with μ.*

To prove our theorem, we need the following lemma:

Lemma 5.3.2. *Consider $\beta \in \mathbb{Z}^n$.*

- *In type A_{n-1}, suppose $\ell > n$. Then the weight $\ell\beta + \rho$ is regular.*
- *In type B_n, C_n or D_n, suppose $\ell > 2n$. Then the weight $\ell\beta + \rho$ is regular.*

Proof. Consider $w \in W$ and t_γ such that $t_\gamma w \cdot (\ell\beta + \rho) = \ell\beta + \rho$. Then $\delta = \ell\beta + \rho - w \cdot (\ell\beta + \rho) \in (\ell\mathbb{Z})^\ell$. Set $\beta = (\beta_1, \ldots, \beta_n)$. For any $i = 1, \ldots, n$, the i-th coordinate of δ is $\delta_i = \ell\beta_i + \rho_i - \ell\beta_{w(i)} - \rho_{w(i)}$. Since $\delta_i \in \ell\mathbb{Z}$, we must have $\left|\rho_i - \rho_{w(i)}\right| \in \ell\mathbb{Z}$. One verifies easily that for type A_{n-1}, $\left|\rho_i - \rho_{w(i)}\right| \leq n-1$ and for types B_n, C_n, D_n $\left|\rho_i - \rho_{w(i)}\right| \leq 2n$. Hence when the conditions of the lemma are verified, $\left|\rho_i - \rho_{w(i)}\right| = 0$ for any $i = 1, \ldots, n$. This gives $w = 1$. The equality $t_\gamma w \cdot (\ell\beta + \rho) = \ell\beta + \rho$ implies then that $\gamma = 0$. Thus the stabilizer of $\ell\beta + \rho$ is reduced to $\{1\}$, that is $\ell\beta + \rho$ is regular. ■

Proof (*Of* Theorem 5.3.1).

1. Follows from Theorem 5.1.1 and (57).
2. When $\ell = 1$, we have seen that $G = G_{\mathcal{I}}$ and $\binom{\mu}{\ell} = \mu$. Thus $[V(\lambda) : V_{\mathcal{I}}\binom{\mu}{\ell}]_q \neq 0$ only if $\lambda = \mu$. In this case $H^1_\mu = s_\mu$ for $[V(\lambda) : V_{\mathcal{I}}\binom{\mu}{\ell}]_q = [V(\lambda) : V(\lambda)]_q = 1$.
3. Suppose ℓ as in the previous lemma. We have $[V(\lambda) : V_{\mathcal{I}}\left(\begin{smallmatrix}\ell\mu\\ \ell\end{smallmatrix}\right)]_q = P^-_{\ell\mu+\rho,\ell\lambda+\rho}(q)$. Since $\ell\lambda + \rho$ is regular for the action of $\widehat{W}$, we obtain by Theorem 4.2.2, $P^-_{\ell\mu+\rho,\ell\lambda+\rho}(q) = p_{w(\ell\mu+\rho),w(\ell\lambda+\rho)}(q)$. By using Lemma 4.1.1, we deduce $P^-_{\ell\mu+\rho,\ell\lambda+\rho}(q) = p_{n_\mu,n_\lambda}(q)$. Now by Theorem 4.2.1, this gives $P^-_{\ell\mu+\rho,\ell\lambda+\rho}(q) = K_{\lambda,\mu}(q)$. Finally

$$H^\ell_\mu = \sum_{\lambda\in\mathcal{P}_n} \left[V(\lambda) : V_{\mathcal{I}}\begin{pmatrix}\ell\mu\\ \ell\end{pmatrix}\right]_q s_\lambda = \sum_{\lambda\in\mathcal{P}_n} K_{\lambda,\mu}(q) s_\lambda = Q'_\mu. \quad ■ \tag{59}$$

Remarks. (i) By the previous theorem, the functions H^ℓ_μ interpolate between the Weyl characters and the Hall–Littlewood functions.

(ii) When ℓ is even for types C and D, one can also define the functions G^ℓ_μ and H^ℓ_μ by setting $G^\ell_\mu = \sum_{\lambda\in\mathcal{P}_n} P^-_{\mu+\rho,\ell\lambda+\rho}(q)s_\lambda$ and $H^\ell_\mu = G^\ell_{\ell\mu}$, respectively. When $\ell > 2n$, we have yet $H^\ell_\mu = Q'_\mu$, but the polynomials $P^-_{\mu+\rho,\ell\lambda+\rho}(q)$ cannot be interpreted as quantizations of branching coefficients.

(iii) The conditions $\ell > n$ for type A_{n-1} and $\ell > 2n$ for types B_n, C_n, D_n appear also naturally in the algorithms of Section 3.2. When they are fulfilled, one has $\varphi_\ell(s_{\ell\mu}) = 0$, or $J_k = I_k$ for any $k = 1, \ldots, n$ and $J_k = I_k = \emptyset$ for $k \notin \{1, \ldots, n\}$. Then $[(\ell\mu)/\ell] = \mu$ and $G_{\mathcal{I}} = H$. Hence $[V(\lambda) : V_{\mathcal{I}}\binom{\mu}{\ell}] = K_{\lambda,\mu}$ for any $\lambda \in \mathcal{P}_n$. This yields equality (59) specialized at $q = 1$.

6. Further remarks

6.1. *Quantization of tensor product coefficients*

Consider $\mu \in \mathcal{P}_n$ and set $\mu = (\mu^{(0)}, \ldots, \mu^{(\ell-1)})$ as in Theorem 3.2.1. For $G = GL_n$, the duality $c^\lambda_{(\mu^{(0)},\ldots,\mu^{(\ell-1)})} = [V(\lambda) : V_{\mathcal{I}}\binom{\mu}{\ell}]$ yields a q-analogue of the Littlewood–Richardson coefficient $c^\lambda_{(\mu^{(0)},\ldots,\mu^{(\ell-1)})}$ defined by setting

$$c^\lambda_{(\mu^{(0)},\ldots,\mu^{(\ell-1)})}(q) = \left[V(\lambda) : V_{\mathcal{I}}\begin{pmatrix}\mu\\ \ell\end{pmatrix}\right]_q = P^-_{\mu+\rho,\ell\lambda+\rho}(q). \tag{60}$$

By Theorem 5.1.1, $c^\lambda_{(\mu^{(0)},\ldots,\mu^{(\ell-1)})}(q)$ have then nonnegative integer coefficients.

In [10], we have shown that there also exists a duality between tensor product coefficients for types B, C, D defined as the analogues of the Littlewood–Richardson coefficients by counting the multiplicities of the isomorphic irreducible components in a tensor product of irreducible representations and branching coefficients. These branching coefficients correspond to the restriction of SO_{2n} to subgroups of the form $SO_{2r_0} \times \cdots SO_{2r_p}$, where the r_i's are positive integers summing n. These subgroups are not subgroups of Levi type; thus the Littlewood–Richardson coefficients for types B, C, D cannot be quantified as in (60) by using parabolic Kazhdan–Lusztig polynomials.

For $G = SO_{2n+1}, Sp_{2n}$ or SO_{2n} and $\lambda \in \mathcal{P}_n$, denote by $\mathfrak{V}(\lambda)$ the restriction of the irreducible finite dimensional GL_N-module of highest weight λ to G. Consider a p-tuple $(\mu^{(0)}, \ldots, \mu^{(p-1)})$ of partitions such that $\mu^{(k)} \in \mathcal{P}_{r_k}$ for any $k = 0, \ldots, p-1$. One can define the coefficients $\mathfrak{D}^{\lambda}_{\mu^{(0)},\ldots,\mu^{(p-1)}}$ as the multiplicity of $V(\lambda)$ in $\mathfrak{V}(\mu^{(0)}) \otimes \cdots \otimes \mathfrak{V}(\mu^{(p-1)})$, that is such that

$$\mathfrak{V}(\mu^{(0)}) \otimes \cdots \otimes \mathfrak{V}(\mu^{(p-1)}) \simeq \coprod_{\lambda \in \mathcal{P}_n} V(\lambda)^{\oplus \mathfrak{D}^{\lambda}_{\mu^{(0)},\ldots,\mu^{(p-1)}}}.$$

We have also obtained in [10] a duality result between the coefficients $\mathfrak{D}^{\lambda}_{\mu^{(0)},\ldots,\mu^{(p-1)}}$ and branching coefficients corresponding to the restriction of G to the subgroup of Levi type $GL_{r_0} \times \cdots \times GL_{r_{p-1}}$. The coefficients $\mathfrak{D}^{\lambda}_{\mu^{(0)},\ldots,\mu^{(p-1)}}$ can be expressed by using a partition function similarly to Proposition 2.5.1. By quantifying this partition function, one shows that they admit nonnegative q-analogues. It is conjectured that stable one-dimensional sums defined in [4] from affine crystals obtained by considering the affinizations of the classical root systems are special cases of the q-analogues obtained in this way. Recall that the subgroups of Levi type $G_{\mathcal{I}}$ obtained in the theorems of Section 3.2 are, up to isomorphism, determined only by G and ℓ. This implies that there exist subgroups of Levi type L in G which are not isomorphic to a subgroup $G_{\mathcal{I}}$. This is, for instance, the case when $G = Sp_{2n}$ for the subgroups of Levi type $G_{\mathcal{I}} \simeq GL_{r_0} \times \cdots \times GL_{r_{p-1}}$ such that $r_k > 1$ for any $k = 0, \ldots, p-1$. Indeed, by Theorem 3.2.3, when $r_0 = \mathrm{card}(I^{(0)}) > 1$, $G_{\mathcal{I}}$ is isomorphic to

$$Sp_{2r_0} \times GL_{r_1} \times \cdots \times GL_{r_{p-1}}.$$

This implies that one cannot obtain in general a quantization of the tensor product coefficients $\mathfrak{D}^{\lambda}_{\mu^{(0)},\ldots,\mu^{(p-1)}}$ by using parabolic Kazhdan–Lusztig polynomials as in (60).

6.2. *Combinatorial description of the functions G^{ℓ}_{μ}*

When $G = GL_n$, the functions G^{ℓ}_{μ} defined in (57) admit the following combinatorial description:

$$G^{\ell}_{\mu} = \sum_{T \in \mathrm{Tab}_{\ell}(\mu)} q^{s(T)} x^T$$

where $\mathrm{Tab}_{\ell}(\mu)$ is the set of ℓ-ribbon tableaux of shape μ on I_n and s the spin statistic defined on ribbon tableaux (see [8] page 1057). Recently, Haglund, Haiman and Loehr have obtained the expansion of the Macdonald polynomials in terms of simple renormalizations of the LLT polynomials G^{ℓ}_{μ}. This expansion yields a combinatorial formula for the Macdonald polynomials [3].

This suggests investigating the following combinatorial problem:

Problem 6.2.1. Find a combinatorial description of the polynomials G_μ^ℓ and the q-analogues $[V(\lambda) : V_{\mathcal{I}}\binom{\mu}{\ell}]_q$ related to the roots systems of type B, C or D.

6.3. Exceptional root systems

It is also possible to define the plethysm ψ_ℓ and the dual plethysm φ_ℓ for exceptional root systems. Consider such an exceptional root system R and μ a dominant weight for R. Denote also by s_μ the Weyl character of the irreducible finite dimensional module of highest weight λ. When ℓ is sufficiently large (the bound depends on R), we have $\varphi_\ell(s_\mu) = s_\mu$. For the other values of ℓ, one shows that the polynomials $\varphi_\ell(\mathrm{e}^\mu \prod_{\alpha\in R_+}(1-\mathrm{e}^\alpha))$ do not factorize in general as a product of factors $(1-x^\beta)$, where β is a positive root. This implies that one cannot define generalized Hall–Littlewood functions for exceptional types by proceeding as in (58).

6.4. Stabilized plethysms

When $G = Sp_{2n}$ or SO_{2n} and ℓ is even, we have seen that the combinatorial methods of Section 3 do not permit us to obtain the coefficients of the expansion of the plethysms $\varphi(s_\lambda)$ on the basis of the Weyl characters. In [12], we show that this difficulty can be overcome by considering stabilized power sum plethysms, i.e. by assuming $n \geq \ell\,|\lambda|$. Under this hypothesis, one can indeed prove that the coefficients in the expansion of $\varphi(s_\lambda)$ coincide for $G = SO_{2n+1}, Sp_{2n}$ and SO_{2n}. So it suffices to compute them in type B_n, for which we have a relevant combinatorial procedure in both cases ℓ even and ℓ odd.
Note: *While revising a previous version of this work* [11], *I was informed that Grojnowski and Haiman* [2] *also define, in a paper in preparation, generalized Hall–Littlewood polynomials for reductive Lie groups. Their polynomials are introduced as formal q-characters depending on a subgroup of Levi type. The coefficients of the corresponding expansion on the basis of the Weyl characters are also affine parabolic Kazhdan–Lusztig polynomials. As far as the author can see, the generalization of the Hall–Littlewood polynomials presented in the present paper satisfies the general definition given in* [2] *(see Definition 5.12). Nevertheless, our combinatorial results based on the study of the power sum plethysms on Weyl characters are completely independent of the approach of Grojnowski and Haiman. It also naturally yields the family of polynomials* $\{G_\mu^\ell \mid \ell \in \mathbb{N}\}$ *in the spirit of the original work by Lascoux, Leclerc and Thibon* [8].

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