On Bernstein–Markov-type inequalities for multivariate polynomials in $L_q$-norm

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Abstract

Bernstein–Markov-type inequalities provide estimates for the norms of derivatives of algebraic and trigonometric polynomials. They play an important role in Approximation Theory since they are widely used for verifying inverse theorems of approximation. In the past decades these inequalities were extended to the multivariate setting, but the main emphasis so far was on the uniform norm. It is considerably harder to derive Bernstein–Markov-type inequalities in the $L_q$-norm, and it requires introduction of new methods. In this paper we verify certain Bernstein–Markov-type inequalities in $L_q$-norm on convex and star-like domains. Special attention is given to the question of how the geometry of the domain affects the corresponding estimates.

Let us introduce some basic notations used in this paper. We shall denote by $S^{d-1}$ the unit sphere in $\mathbb{R}^d$, $B^d(a, r) := \{ x \in \mathbb{R}^d : |a - x| \leq r \}$ stands for the ball centered at $a \in \mathbb{R}^d$ and radius $r$. For a convex body $K \in \mathbb{R}^d$ denote by $r_K$ the so-called width of $K$, which is defined as the radius of the largest ball contained in $K$. In case if $K$ is a convex body it is known that it contains a unique ellipsoid $E_K$ of maximal volume, which is called the maximal ellipsoid of $K$.

Keywords: Bernstein–Markov inequality; $L_q$-norm; Convex bodies; Maximal ellipsoid; Star-like domains

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The John Ellipsoid Theorem states that if \( c \) is the center of \( E_K \) then \( K \subset c + d(E_K - c) \), i.e. \( K \) is covered by a \( d \)-dilation of \( E_K \) around the center \( c \), see [7,2] for details. This result will play an important role in this paper. Furthermore, let \( P^d_n \) be the space of algebraic polynomials of \( d \) real variables and total degree at most \( n \). For any differentiable function \( f \) in \( d \) variables \( \partial f \) stands for its gradient,

\[
|\partial f| = \left( \sum_{1 \leq j \leq d} (\partial f/\partial x_j)^2 \right)^{1/2}
\]

is the Euclidean norm of the gradient, while \( D_{\mathbf{u}} f \) denotes its derivative in direction \( \mathbf{u} \in S^{d-1} \). Finally, \( \|f\|_{L_q(K)} \) denotes the usual \( L_q \)-norm on \( K \), \( 1 \leq q \leq \infty \).

Now we can introduce the \( n \)th order Markov Factor in \( L_q \)-norm on the compact set \( K \) as

\[
M_{n,q}(K) := \sup_{p \in P^d_n, p \neq 0} \frac{\| |\partial p| \|_{L_q(K)}}{\|p\|_{L_q(K)}}.
\]

The study of Markov Factors has a long and rich history. They play an important role in Approximation Theory since they are widely used for verifying inverse theorems of approximation. The first famous result here is due to A.A. Markov who verified that \( M_{n,\infty}([-1, 1]) = n^2 \). This gives the exact value of Markov Factors for univariate polynomials in supremum norm. It is known that \( M_{n,q}([-1, 1]) = O(n^2) \) for any \( 1 \leq q \leq \infty \) but finding the exact constants in \( L_q \)-case is rather difficult even for univariate polynomials. (Exact constant is known only in trigonometric case, see DeVore, Lorentz [6].)

In the multivariate case it has been known for sometime that the \( O(n^2) \) order of the Markov Factors is preserved on convex bodies (or “convex-like” sets), see Nikolskii, [10], or Daugavet [5]. A more delicate problem here consists in revealing the influence of the geometry of the underlying sets on the Markov Factors. In case of the uniform norm this problem was first studied by Wilhelmsen [15]. It was shown in [15] that if \( K \) is a convex body in \( \mathbb{R}^d \) then \( M_{n,\infty}(K) \leq 2n^2/r_K \). In addition, if \( K \) is central-symmetric, then the constant 2 above can be replaced by the exact constant 1, see Sarantopoulos, [13]. (For the case when \( K \) is a ball this was previously done by Kellogg.)

In this paper we shall study the question of how the geometry of the underlying sets affects the \( L_q \)-Markov Factors of these sets \( 1 \leq q < \infty, d \geq 2 \). This problem was recently raised by I. Babuska and communicated to the author by P. Oswald. Just as in the univariate case the question of multivariate \( L_q \)-Markov problem is much more complex. (Note that even in the univariate case the exact constant in \( L_q \)-Markov inequality is not known. A certain description of it can be found in [12] only for \( q = 2 \).)

First we present a Wilhelmsen-type estimate of the Markov Factors for the \( L_q \)-norm on convex bodies.

**Theorem 1.** Let \( K \) be a convex body in \( \mathbb{R}^d \), \( d \geq 2 \) and \( 1 \leq q < \infty \). Then with some absolute constant \( c > 0 \) we have

\[
M_{n,q}(K) \leq c \frac{d^{d+7} \ln d}{r_K} n^2,
\]

where \( r_K \) is the width of \( K \).

A standard approach to proving multivariate Markov-type inequalities in uniform norm on \( K \) consists in inscribing suitable polynomial curves into \( K \) and reducing the problem to univariate
setting on these curves. In case of $L_q$-norms with $1 \leq q < \infty$ this reduction of dimension technique does not work and a different geometric approach is needed. This leads to more complicated constants in Theorem 1 which depend on the dimension $d$ and are related to the John Maximal Ellipsoid theorem and certain covering constants (see [4]). Nevertheless, in terms of the factor $n^2$ and width $r_K$ Theorem 1 gives a Wilhelmsen-type upper bound for the Markov Factors in the $L_q$-norm for convex bodies.

Let us consider now the more general case of “star-like” domains in $\mathbb{R}^d$. Recall that $K$ is called star-like with respect to some $x \in K$ if any line $L$ passing through $x$ intersects $K$ along a line segment. Without restricting the generality we can assume that $K$ is star-like with respect to 0. We shall introduce a parametric representation for the star-like domain $K$ based on the usual spherical transformation of $S^{d-1}$. Let for $x = (x_1, \ldots, x_d) \in S^{d-1}$, $u = (u_1, \ldots, u_{d-1}) \in G_0 := \{|u_1| \leq \pi, |u_j| \leq \pi/2, 2 \leq j \leq d - 1\}$

$$x_j = F_j(u_1, \ldots, u_{d-1}) := \sin u_{j-1} \prod_{k=j}^{d-1} \cos u_k, \quad 1 \leq j \leq d,$$

be the usual spherical transformation. Set $F = (F_1, \ldots, F_d)$. Note that $S^{d-1} = \{F(u) : u \in G_0\}$.

Then given a positive real-valued function $r(u)$ continuous for $u \in G_0$ the star-like domain $K$ corresponding to $r(u)$ is given by $K := \{tF(u)r(u) : u \in G_0, 0 \leq t \leq 1\}$. We shall say that $K$ is a $C^\alpha$-domain, where $0 < \alpha \leq 1$ if $r \in \text{Lip} \alpha$ on $G_0$. When $\alpha < 1$ this allows the domain to have cusps at some points. The Markov Factors of cuspidal domains are relatively well studied in case of uniform norm, see e.g. [11,9]. In particular if $K$ is a star-like $C^\alpha$-domain then it is known that $M_{n,\infty}(K) = O(n^{2/\alpha})$. Now we give a similar result for the case of $L_q$-norm.

**Theorem 2.** Let $K \in \mathbb{R}^d$ be a $C^\alpha$ star-like domain, $0 < \alpha \leq 1$. Then for any $1 \leq q < \infty$ we have

$$M_{n,q}(K) \leq c(d, K)n^{1+2/\alpha},$$

where $c(d, K) > 0$ depends only on $d$ and $K$.

It is known that the $L_\infty$-result $M_{n,\infty}(K) = O(n^{2/\alpha})$ is in general asymptotically sharp for $C^\alpha$-domains, see [9]. This means that the $L_q$-Markov factors of these domains also cannot be of smaller order than $n^{2/\alpha}$. Note that the estimate of Theorem 2 is larger by a factor of $n$. The question if this extra factor can be omitted is open.

The methods used in the proof of Theorem 2 can be also applied for verifying Bernstein-type inequalities for spherical polynomials in $L_q$-norm. For a differentiable function $F$ and $x \in S^{d-1}$ let us denote by $D^TF(x)$ the tangential gradient of $F$ at $x$ given by $D^TF(x) := \max\{|D_wF(x)| : w \in S^{d-1}, \langle x, w \rangle = 0\}$. Moreover let $\|F\|_{L_q(S^{d-1})}$ stand for the $L_q$-norm with respect to the usual spherical Lebesgue measure on $S^{d-1}$. Then we have the next Bernstein-type inequality for spherical polynomials.

**Theorem 3.** For any $p \in P_n^d$, $d \geq 2$, $1 \leq q < \infty$ we have

$$\|D^T p\|_{L_q(S^{d-1})} \leq d^{3/2}(d - 1)n\|p\|_{L_q(S^{d-1})}.$$

In order to prove Theorem 1 we shall need several auxiliary lemmas. Let us assume that for some $r, R > 0$ the set $K$ satisfies the relations

$$B^d(0, r) \subset K \subset B^d(0, R).$$


easily yields the statement of the lemma.

Without loss of generality we may assume that

**Lemma 2.** Let \( K \in \mathbb{R}^d \) satisfy (3) and \( u \in S^{d-1} \).

Proof. Without loss of generality we may assume that \( u = (0, \ldots, 0, 1) \). For any \( x \in K \) denote by \( A_x, B_x \) the points of intersection with \( \partial K \) of the line in direction \( u \) passing through \( x \). Then clearly

\[
\| D_u p \|_{L_q(K,a,u)} = \int_{B^{d-1}(0,ar)} \int_{[A_x,B_x]} |D_u p|^q \, dx \, dt.
\]

Since \( K \) contains \( B^d(0, r) \) it follows that for any \( x \in B^{d-1}(0, ar) \) we have

\[
|A_x - B_x| \geq 2r(2 - |x|^2) \geq 2r\sqrt{1 - a^2}.
\]

Using this lower bound we have by the linear transformation of the segment \( [A_x, B_x] \) into \( [-1, 1] \)

\[
\int_{[A_x,B_x]} |D_u p|^q \, dt \leq \left( \frac{M_{n,q}([-1, 1])}{r\sqrt{1 - a^2}} \right)^q \int_{[A_x,B_x]} |p|^q \, dt. \quad (4)
\]

Finally, applying the last inequality together with (4) easily yields the statement of the lemma.

**Lemma 3.** Let \( u, w \in S^{d-1} \) be such that with some \( r, R > 0 \) and \( 0 < a < 1 \) we have

\[
|u - w| \leq \frac{r^2(1 - a^2)}{2R^2}.
\]

Then it follows that

\[
B^d(0, R) \cap C(u, ar) \subset C(w, r).
\]

Proof. For any \( x \in B^d(0, R) \cap C(u, ar) \) we have by (5)

\[
|x|^2 \leq a^2r^2 + \langle u, x \rangle^2 = a^2r^2 + \langle w, x \rangle^2 + \langle u - w, x \rangle \langle u + w, x \rangle \\
\leq a^2r^2 + \langle w, x \rangle^2 + r^2(1 - a^2) = r^2 + \langle w, x \rangle^2.
\]

Hence \( x \in C(w, r) \) and thus the lemma is verified. \( \square \)

**Lemma 3.** Let \( u_0, \ldots, u_{d-1} \in S^{d-1} \), \( d \geq 2 \) be such that with some \( \epsilon > 0 \) we have

\[
\min_{u \in S^{d-1}} \max_{0 \leq j \leq d-1} |\langle u, u_j \rangle| \geq \epsilon.
\]

For \( u \in S^{d-1} \) and \( \gamma > 0 \) a cylinder centered at \( 0 \) with axis \( u \) and radius \( \gamma \) is given by

\[
C(u, \gamma) := \{ x \in \mathbb{R}^d : |x|^2 \leq \gamma^2 + \langle u, x \rangle^2 \}.
\]

Furthermore, for a convex body \( K \) satisfying (3) consider its cross-section with the cylinder of radius \( ar \), \( 0 < a < 1 \), given by

\[
K_{a,u} := K \cap C(u, ar), \quad u \in S^{d-1}.
\]

**Lemma 1.** For any convex body \( K \in \mathbb{R}^d \) satisfying (3), \( u \in S^{d-1} \), \( 0 < a < 1 \), and \( p \in P_n^d \) we have

\[
\| D_u p \|_{L_q(K,a,u)} \leq \frac{M_{n,q}([-1, 1])}{r\sqrt{1 - a^2}} \| p \|_{L_q(K,a,u)}.
\]

Proof. Without loss of generality we may assume that \( u = (0, \ldots, 0, 1) \). For any \( x \in K \) denote by \( A_x, B_x \) the points of intersection with \( \partial K \) of the line in direction \( u \) passing through \( x \). Then clearly

\[
\| D_u p \|_{L_q(K,a,u)} = \int_{B^{d-1}(0,ar)} \int_{[A_x,B_x]} |D_u p|^q \, dx \, dt.
\]

Since \( K \) contains \( B^d(0, r) \) it follows that for any \( x \in B^{d-1}(0, ar) \) we have

\[
|A_x - B_x| \geq 2r(2 - |x|^2) \geq 2r\sqrt{1 - a^2}.
\]

Using this lower bound we have by the linear transformation of the segment \( [A_x, B_x] \) into \( [-1, 1] \)

\[
\int_{[A_x,B_x]} |D_u p|^q \, dt \leq \left( \frac{M_{n,q}([-1, 1])}{r\sqrt{1 - a^2}} \right)^q \int_{[A_x,B_x]} |p|^q \, dt. \quad (4)
\]

Finally, applying the last inequality together with (4) easily yields the statement of the lemma.
Then for any function $F$ differentiable at a given $x \in \mathbb{R}^d$ we have

$$|\partial F(x)| \leq \frac{1}{\epsilon} \max_{0 \leq j \leq d-1} |D_{u_j} F(x)|.$$  \hfill (8)

**Proof.** We may assume that $w := \partial F(x) \neq 0$ since otherwise the statement of lemma is trivial. Let $u := w/|w|$ be the unit vector in direction of $w$. Using that by (7) for some $0 \leq j \leq d-1$ we have $|(u, u_j)| \geq \epsilon$ it follows that $|D_{u_j} F(x)| = |(w, u_j)| = |w||u, u_j| \geq \epsilon |w|$. Evidently this implies (8). \hfill □

**Lemma 4.** For arbitrary $0 < \epsilon < 1/2$ and $u_0 \in S^{d-1}$ there exist $u_1, \ldots, u_{d-1} \in S^{d-1}$ such that $|u_0 - u_j| \leq \epsilon$, $1 \leq j \leq d-1$, and

$$\min_{u \in S^{d-1}} \max_{0 \leq j \leq d-1} |(u, u_j)| \geq \frac{\epsilon}{d}.$$ \hfill (9)

**Proof.** We shall verify the lemma by induction on $d$. Let first $d = 2$. We may assume that $u_0 = (1, 0)$. Let $u_1 := (\cos t, \sin t)$ where $t := \arccos(1 - \epsilon^2/2)$. Clearly, $|u_0 - u_1| = \epsilon$ and the minimum in (9) is attained for $u := (-\sin(t/2), \cos(t/2))$, where $|(u, u_0)| = \sin(t/2) = \epsilon/2$. Thus (9) holds for $d = 2$.

Assume now that the statement of the lemma is true for $d - 1, d \geq 3$. Again without loss of generality we may set $u_0 := (1, 0, \ldots, 0)$. By the induction hypothesis there exist $u_1, \ldots, u_{d-2} \in S^{d-2} \subset \mathbb{R}^{d-1}$ such that $|u_0 - u_j| \leq \epsilon$, $1 \leq j \leq d - 2$, and

$$\min_{u \in S^{d-2}} \max_{0 \leq j \leq d-2} |(u, u_j)| \geq \frac{\epsilon}{d - 1}.$$ \hfill (10)

Set now

$$w := (0, \ldots, 0, 1) \in S^{d-1}, u_{d-1} := (u_0 + \epsilon w)/\sqrt{1 + \epsilon^2} \in S^{d-1}.$$ \hfill (11)

It can be easily verified that $|u_0 - u_{d-1}| \leq \epsilon$, and hence by above $|u_0 - u_j| \leq \epsilon$ for every $1 \leq j \leq d - 1$.

It remains to verify now that (9) holds. Consider an arbitrary $u = u' + \epsilon w \in S^{d-1}$ where $u' \in \mathbb{R}^{d-1}$ and $c \in \mathbb{R}$, $|c| = \sqrt{1 - |u'|^2}$. We may assume that $u' \neq 0$ since otherwise (9) is trivial. Using the induction hypothesis (10) for $u'/|u'| \in S^{d-2}$ we obtain

$$\max_{0 \leq j \leq d-2} |(u, u_j)| = \max_{0 \leq j \leq d-2} |(u', u_j)| \geq \frac{\epsilon |u'|}{d - 1}.$$ \hfill (12)

Consider now the following cases.

Case 1. $|u'| \geq (d - 1)/d$. Then (9) follows immediately from (12).

Case 2. $|(u, u_0)| = |(u, u_0)| \geq \epsilon/d$. Again, (9) is evident in this case.

Case 3. $|u'| < (d - 1)/d$ and $|(u', u_0)| < \epsilon/d$. In this case using (11) we have

$$\max_{0 \leq j \leq d-1} |(u, u_j)| \geq |(u, u_{d-1})| = \frac{1}{\sqrt{1 + \epsilon^2}} |c\epsilon + (u', u_0)|$$

$$> \frac{1}{\sqrt{1 + \epsilon^2}}\epsilon |c| - \epsilon/d = \frac{\epsilon}{\sqrt{1 + \epsilon^2}}(\sqrt{1 - |u'|^2} - 1/d)$$

$$\geq \frac{\epsilon}{\sqrt{1 + \epsilon^2}}(\sqrt{1 - (1 - 1/d)^2} - 1/d) > \frac{\epsilon}{d},$$
where in the last inequality we have used $\epsilon < 1/2$ and $d \geq 3$. The proof of the lemma is complete.

\begin{proof}
Consider now an arbitrary $u_0 \in S^{d-1}$ and $0 < a < 1$. Using Lemma 2 with $r := ar$ and $R$ it follows that for arbitrary $u_j \in S^{d-1}$ such that

$$|u_0 - u_j| \leq \epsilon := \frac{a^2 r^2 (1 - a^2)}{2 R^2}, \quad 1 \leq j \leq d - 1$$

we have

$$K_{a^2, u_0} \subset K_{a, u_j}, \quad 1 \leq j \leq d - 1. \tag{15}$$

In addition by Lemma 4 $u_j, 1 \leq j \leq d - 1$ can be chosen so that (9) holds. (Note that in view of (14) $\epsilon \leq 1/8$, i.e. Lemma 4 is applicable.) Using now Lemma 1 together with (15) yields for an arbitrary $p \in P_n^{d}$

$$\|D_{u_j} p\|_{L_q(K_{a^2, u_0})} \leq \|D_{u_j} p\|_{L_q(K_{a, u_j})} \leq \frac{M_{n,q}([-1, 1])}{r \sqrt{1 - a^2}} \frac{d}{\epsilon \sqrt{1 - a^2}} \frac{d M_{n,q}([-1, 1])}{\epsilon r \sqrt{1 - a^2}} \|p\|_{L_q(K)}, \quad 1 \leq j \leq d - 1.$$ 

Applying the above inequality and recalling that $u_j$’s satisfy (9) we obtain by Lemma 3 (used with $\epsilon / d$ instead of $\epsilon$)

$$\|\partial p\|_{L_q(K_{a^2, u_0})} \leq \sum_{j=0}^{d-1} \|D_{u_j} p\|_{L_q(K_{a^2, u_0})} \leq \frac{d^2 M_{n,q}([-1, 1])}{\epsilon r \sqrt{1 - a^2}} \|p\|_{L_q(K)}. \tag{16}$$

The above estimate provides an upper bound on the cylindrical section $K_{a^2, u_0}$ for an arbitrary $u_0 \in S^{d-1}$. So now it remains to take into account how many such cylindrical sections will cover the convex body $K$. In order to estimate this we shall use a result by Böröczky and Wintsche [4], Corollary 1.2. According to this result $S^{d-1}$ can be covered by $c_0^3 \frac{1}{d^{3/2}} \ln d$ balls of radius $\delta$, where $c_0$ is an absolute constant. Since in our case we need to cover $K$ by cylindrical sections of radius $a^2 r$ and $K$ is imbedded into a ball of radius $R$ by the above result this can be accomplished by at most $c (R/a^2 r)^{d-1} d^{3/2} \ln d$ such sections. Finally, by using (16) with $\epsilon$ defined in (14) and $a^2 := d/(d + 3/2)$, and taking into account the covering constant mentioned above we complete the proof of the lemma.

\end{proof}

\begin{corollary}
If $K$ is an ellipsoid then

$$M_{n,q}(K) \leq \frac{d^5 \ln d}{r_K^2} n^2,$$

where $r_K$ is the width of $K.$

\end{corollary}

\begin{proof}[Proof of Corollary 1]
Recall that ellipsoids are images of nonsingular affine maps of the unit ball in $\mathbb{R}^d$. Thus after a proper shift we may assume that $K = AB^d(0, 1)$ where $A$ is a nonsingular linear transformation in $\mathbb{R}^d$. For any $p \in P_n^d$ set $g(y) := p(Ay), \quad y \in \mathbb{R}^d, \quad g \in P_n^d.$ Then clearly

$$|

\end{proof}
\[ \partial g = A^T \partial p, \text{ i.e., } (A^T)^{-1} \partial g = \partial p. \] Let us denote by \(|A|^*\) the \(2\)-norm of the transformation \(A\). Evidently, \(|(A^T)^{-1}|^* = |(A^{-1})^T|^* = |A^{-1}|^*\). Now we give an upper bound for \(|A^{-1}|^*\).

Since \(K\) contains a ball of radius \(r_K\) and \(A^{-1}K = B^d(0, 1)\) it follows that \(|A^{-1}z| \leq 2\) for any \(z \in 2r_K S^{d-1}\), i.e., \(|A^{-1}|^* \leq 1/r_K\). This easily yields

\[ \frac{\| \partial p \|_{L_q(K)}}{\| p \|_{L_q(K)}} \leq |A^{-1}|^* \frac{\| \partial g \|_{L_q(B^d(0,1))}}{\| g \|_{L_q(B^d(0,1))}} \leq \frac{1}{r_K} \frac{\| \partial g \|_{L_q(B^d(0,1))}}{\| g \|_{L_q(B^d(0,1))}}. \]

Finally, estimating the right-hand side of the above inequality using (13) with \(K = B^d(0, 1)\), i.e., \(r = R = 1\) and applying the well known bound \(M_{n,q}([-1,1]) \leq cn^2\) (see [12], p.611) completes the proof of the Corollary.

**Proof of Theorem 1.** In order to complete the proof of Theorem 1 we need to recall the John Ellipsoid Theorem (see [2,7]). According to this theorem for any convex body \(K \subset \mathbb{R}^d\) there is an ellipsoid \(E_K\) so that if \(e\) is the center of \(E_K\) then the inclusions

\[ E_K \subset K \subset c + d(E_K - c) \]

hold. (Note that \(c + d(E_K - c)\) is the dilation of \(E_K\) by a factor of \(d\) with center \(c\).) This ellipsoid (called John maximal ellipsoid) is the unique ellipsoid of maximal volume embedded into \(K\). Again we may assume that the center of this ellipsoid is \(0\) that is \(E_K = AB^d(0, 1)\) where \(A\) is a nonsingular linear transformation in \(\mathbb{R}^d\). Then setting \(K_0 := A^{-1}K\) we obtain by the above inclusions that

\[ B^d(0, 1) \subset K_0 \subset B^d(0, d). \]  

(17)

Setting as in the proof of above corollary for any \(p \in P_n^d, g(y) := p(Ay), y \in \mathbb{R}^d, g \in P_n^d\), we clearly have again \(\partial g = A^T \partial p\), i.e., \((A^T)^{-1} \partial g = \partial p\). Now we need to estimate \(|A^{-1}|^*\). Since \(K\) contains a ball of radius \(r_K\) and by (17) \(A^{-1}K = K_0 \subset B^d(0, d)\) it follows that \(|A^{-1}z| \leq 2d\) for any \(z \in 2r_K S^{d-1}\), i.e., \(|A^{-1}|^* \leq d/r_K\). Thus we have

\[ \frac{\| \partial p \|_{L_q(K)}}{\| p \|_{L_q(K)}} \leq |A^{-1}|^* \frac{\| \partial g \|_{L_q(K_0)}}{\| g \|_{L_q(K_0)}} \leq \frac{d}{r_K} \frac{\| \partial g \|_{L_q(K_0)}}{\| g \|_{L_q(K_0)}}. \]

Hence estimating again the right-hand side of the above inequality using (13) with \(K = K_0\), where in view of (17) \(r \geq 1, R \leq d\), and applying again the bound \(M_{n,q}([-1, 1]) \leq cn^2\) from [12], p. 611 completes the proof of Theorem 1.

Now we turn our attention to the proof of Theorem 2. In contrast to the above proof of Theorem 1 which was essentially based on geometric considerations the proof of Theorem 2 is using primarily analytic methods. Again we start with several auxiliary lemmas.

**Lemma 6.** Let \(K \subset \mathbb{R}^d\) be a star-like domain with respect to 0, \(1 \leq q \leq \infty, d \geq 2\). Set \(\delta K := \{\delta x : x \in K\}\) where \(\delta := 1 + 1/(n + d - 1)^2\). Then for any \(p \in P_n^d\) we have

\[ \| p \|_{L_q(\delta K)} \leq c \| p \|_{L_q(K)}, \]

(18)

where \(c > 0\) is an absolute constant.

**Proof.** Passing to spherical coordinates in \(\mathbb{R}^d\) we have

\[ K = \{(\rho, u) : u \in G_0, 0 \leq \rho \leq r(u)\}, \quad \delta K = \{(\rho, u) : u \in G_0, 0 \leq \rho \leq \delta r(u)\}. \]
Denoting by \( \rho^{d-1} J(\mathbf{u}) \) the Jacobian of this transformation we have
\[
\| p \|^q_{L^q(\delta K)} = \int_{\delta K} |p(\mathbf{x})|^q \, d\mathbf{x} = \int_{G_0} \int_{[0, \delta r(\mathbf{u})]} |p|^q \rho^{d-1} \, d\rho \, d\mathbf{u}.
\] (19)

Consider now the function \( g(\rho) := |p|^q \rho^{d-1} \). This function is a positive generalized algebraic polynomial of degree \( m := nq + d - 1 \leq q(n + d - 1) \) of the variable \( \rho \) (see [3], p.392 for the corresponding definition). Applying the \( L_1 \) Remez-type inequality for generalized algebraic polynomials ([3], Theorem A.4.10) it follows that
\[
\int_{[0, \delta r(\mathbf{u})]} g(\rho) \, d\rho \leq (1 + e^{cm}) \int_{[0, r(\mathbf{u})]} g(\rho) \, d\rho \leq e^{cq} \int_{[0, r(\mathbf{u})]} g(\rho) \, d\rho
\]
with some absolute constant \( c > 0 \). Applying this inequality together with (19) clearly yields (18). This completes the proof of the lemma. \( \square \)

Our next lemma provides an estimate for the gradient of a function via certain partial differential operators. Set
\[
Df(\mathbf{x}) := \langle \partial f(\mathbf{x}), \mathbf{x} \rangle,
\]
\[
D_{i,j}f(\mathbf{x}) := -\frac{\partial f}{\partial x_i}(\mathbf{x})x_j + \frac{\partial f}{\partial x_j}(\mathbf{x})x_i, \quad \mathbf{x} = (x_1, \ldots, x_d), \quad 1 \leq i, j \leq d.
\] (20)

Then we have the following.

**Lemma 7.** Let \( 0 < \delta < R \). Then for any \( \delta < |\mathbf{x}| < R \) and any function \( f \) differentiable at \( \mathbf{x} \)
\[
|\partial f(\mathbf{x})| \leq c_1(d) R^{d-1} \delta^{-d} \max_{1 \leq i, j \leq d} \{ |Df(\mathbf{x})|, |D_{i,j}f(\mathbf{x})| \},
\] (21)
where the constant \( c_1(d) \) depends only on \( d \).

**Proof.** Since \( \mathbf{x} = (x_1, \ldots, x_d) \) satisfies \( |\mathbf{x}| > \delta \) it follows that \( \max_{1 \leq j \leq d} |x_j| \geq \delta/\sqrt{d} \). Thus without loss of generality we may assume that \( |x_1| \geq \delta/\sqrt{d} \). Set now
\[
M_1 := Df(\mathbf{x}), \quad M_j := D_{1,j}f(\mathbf{x}), \quad 2 \leq j \leq d.
\]

The above relations can be considered as a system of \( d \) linear equations with respect to \( d \) unknowns \( \partial f/\partial x_j(\mathbf{x}) \), \( 1 \leq j \leq d \). It can be verified by induction (we omit the details) that the determinant of this system equals \( x_1^{d-2} |\mathbf{x}|^2 \). The supplementary determinants of the system can be easily estimated from above by \( c(d) R^{d-1} \max_{1 \leq j \leq d} |M_j| \), where \( c(d) \) is a positive constant depending only on \( d \). Hence we obtain by the Cramer’s rule that
\[
|\partial f/\partial x_j(\mathbf{x})| \leq \frac{c(d) R^{d-1} \max_{1 \leq j \leq d} |M_j|}{|x_1|^{d-2} |\mathbf{x}|^2} \leq c_1(d) R^{d-1} \delta^{-d} \max_{1 \leq j \leq d} |M_j|.
\]

Clearly this leads to the needed statement. \( \square \)

**Proof of Theorem 2.** Let \( K := \{ z\mathbf{F}(\mathbf{u})r(\mathbf{u}) : \mathbf{u} \in G_0, 0 \leq z \leq 1 \} \) be a star-like \( C^\alpha \)-domain and consider the transformation of \( K \) into the parallelepiped \( G := G_0 \times [0, 1] \). \( G_0 := \{|u_1| \leq \pi, |u_j| \leq \pi/2, 2 \leq j \leq d - 1 \} \) given for \( \mathbf{x} = (x_1, \ldots, x_d) \in K \) by
\[
x_j = z r(\mathbf{u}) \sin u_{j-1} \prod_{k=j}^{d-1} \cos u_k, \quad 1 \leq j \leq d, \quad \mathbf{u} = (u_1, \ldots, u_{d-1}) \in G_0, z \in [0, 1].
\] (22)
Here by the assumption of Theorem 2 \( r(\mathbf{u}) \in \text{Lip } \alpha \) on \( G_0 \). Note that apart from the factor \( zr(\mathbf{u}) \) this is the usual spherical transformation (1) on \( S^{d-1} \). It can be easily verified that the Jacobian of this transformation equals \( J^* := r^d(\mathbf{u})z^{d-1}J(\mathbf{u}) \) where \( J(\mathbf{u}) \) is the Jacobian of the spherical transformation (1). It is important to note that for \( \mathbf{u} = (u_1, \ldots, u_d) \) the function \( J(\mathbf{u}) \) which is the Jacobian of spherical transformation (1) is independent of \( u_1 \), i.e. \( J(\mathbf{u}) = J(u_2, \ldots, u_d) \), see [8] for details. Clearly, \( r(\mathbf{u}) \) can be extended to \( T^{d-1} := [-\pi, \pi]^{d-1} \) so that it is \( \text{Lip}_\alpha \) and periodic on \( T^{d-1} \). Then by the multivariate trigonometric Jackson Theorem (see [14], p. 288) there exists a trigonometric polynomial \( t_m(\mathbf{u}) \) of degree \( m \) in each variable such that

\[
\max_{\mathbf{u} \in T^{d-1}} |r(\mathbf{u}) - t_m(\mathbf{u})| = O(m^{-\alpha}).
\]

Thus choosing \( m := \lceil n^{2/\alpha} \rceil \) where \( n \) is the degree of the Markov Factor \( M_{n,q}(K) \) and \([...]\) denotes the integer part, we can ensure that \( t_m \) deviates from \( r(\mathbf{u}) \) by at most \( O(n^{-2}) \) on \( T^{d-1} \). Moreover, \( c_1 \leq t_m \leq c_2 \) on \( T^{d-1} \) and hence by the Stechkin inequality (see [14], p.228)

\[
\left\| \frac{\partial t_m}{\partial u_j} \right\|_{C(T^{d-1})} \leq m\omega(t_m, 2\pi/m) \leq 2m\|r - t_m\|_{C(T^{d-1})} + m\omega(r, 2\pi/m) \leq cm^{1-\alpha} = O(n^{2/\alpha-2}),
\]

where \( \omega(r, \cdot) \) is the usual modulus of continuity of the function \( r \). On the other hand in view of Lemma 6 dilating the set \( K \) by \( O(n^{-2}) \) can change the \( L_q \)-norm of a polynomial of degree \( n \) at most by a constant factor. Thus replacing \( r(\mathbf{u}) \) in the definition of the star-like domain \( K \) by \( t_m \) can modify the Markov Factor \( M_{n,q}(K) \) at most by a constant independent of \( n \). Therefore without loss of generality we may assume that \( r(\mathbf{u}) \) is a multivariate trigonometric polynomial of \( \mathbf{u} \) of degree \( m = \lceil n^{2/\alpha} \rceil \) in each of its \( d \) variables. Moreover by the previous estimate

\[
\left\| \frac{\partial r}{\partial u_j} \right\|_{C(T^{d-1})} = O(n^{2/\alpha-2}), \quad 1 \leq j \leq d.
\]  

(23)

Now for an arbitrary \( p \in P_n^d, \|p\|_{L_q(K)} = 1 \) after transformation (22) we obtain \( p(\mathbf{x}) := T(z, \mathbf{u}) \) where \( T \in P_n^1 \) with respect to the real variable \( z \). Moreover, recalling that \( r(\mathbf{u}) \) is a trigonometric polynomial in \( d \) variables of degree \( m = \lceil n^{2/\alpha} \rceil \) in each variable it follows that \( T \) is a trigonometric polynomial of degree \( \leq nm + n \) in each variable \( u_j, 1 \leq j \leq d - 1 \). Furthermore by (20) and (22) we easily derive

\[
\frac{\partial T}{\partial z} = \langle \partial p, \mathbf{x} \rangle = Dp(\mathbf{x}).
\]  

(24)

Moreover using relations (22) we also obtain that

\[
\frac{\partial x_j}{\partial u_1} = \frac{1}{r(\mathbf{u})} \frac{\partial r}{\partial u_1} x_j + y_j, \quad y_1 := -x_2, y_2 := x_1, y_j := 0, 3 \leq j \leq d.
\]

Applying above relations together with notations (20) yields

\[
\frac{\partial T}{\partial u_1} = \frac{1}{r(\mathbf{u})} \frac{\partial r}{\partial u_1} Dp + D_{1.2}p.
\]  

(25)

Now we proceed by estimating the partial derivatives of \( T \) appearing on the left-hand side of (24) and (25). We shall apply the \( L_q \)-Bernstein inequality for the univariate trigonometric polynomials \( t \) of degree at most \( n \) which is due to Arestov [1]:

\[
\|t\|_{L_q[0,2\pi]} \leq n\|t\|_{L_q[0,2\pi]}.
\]
Using this inequality for $T$ considered as a univariate polynomial of $u_1$ and recalling that $c_1 \leq r(u) \leq c_2$, $u \in G_0$ we obtain
\[
\left\| \frac{\partial T}{\partial u_1} \right\|_{L_q(K)}^q \leq \int_G \left| \frac{\partial T}{\partial u_1} \right|^q \, dx
\]
\[
= \int_0^1 \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} r^d(u) \rho^{d-1} J(u_2, \ldots, u_{d-1}) \left| \frac{\partial T}{\partial u_1} \right|^q \, du_1 \, du_2 \cdots du_{d-1} \, dz
\]
\[
\leq c(nm+n)q \int_0^1 \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} r^d(u) \rho^{d-1} J(u_2, \ldots, u_{d-1}) \left| T \right|^q \, du_1 \, du_2 \cdots du_{d-1} \, dz
\]
\[
\leq c_3(nm+n)^q \|p\|_{L_q(K)}^q = c_3(nm+n)^q. \quad (26)
\]
Applying this together with (23) and (25) we obtain
\[
\| D_{1,2} p \|_{L_q(K)} \leq c_4(n^{2/\alpha-2} \|Dp\|_{L_q(K)} + (nm+n)).
\]
Recalling that $m = [n^{2/\alpha}]$ and using the symmetry of variables
\[
\| D_{i,j} p \|_{L_q(K)} \leq c_5(n^{2/\alpha-2} \|Dp\|_{L_q(K)} + n^{2/\alpha+1}), \quad 1 \leq i, j \leq d, i \neq j. \quad (27)
\]
It remains now to estimate the quantity $\|Dp\|_{L_q(K)} = \|z \frac{\partial T}{\partial z}\|_{L_q(K)}$, see (24). First we estimate this norm on the set $K_\delta := \{ x \in K : |x| > \delta \}$ where $\delta > 0$ is chosen sufficiently small so that $B^d(0, \delta) \subset K$. Then using this time the $L_q$-Markov inequality on $[\delta, 1]$ for the univariate algebraic polynomial $T$ of degree $\leq n$ with respect to variable $z$ yields
\[
\| Dp \|_{L_q(K_\delta)}^q = \left| \frac{\partial T}{\partial z} \right|_{L_q(K_\delta)}^q
\]
\[
\leq \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} r^d(u) J(u_2, \ldots, u_{d-1}) \int_{\delta}^{1} \left| \frac{\partial T}{\partial z} \right|^q \, dz \, du_1 \, du_2 \cdots du_{d-1} \, dz
\]
\[
\leq \frac{c_2 n^{2q}}{\delta^{d-1}} \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} r^d(u) J(u_2, \ldots, u_{d-1}) \int_{\delta}^{1} \rho^{d-1} |T|^q \, dz \, du_1 \, du_2 \cdots du_{d-1} \, dz
\]
\[
= c_6 n^{2q} \|p\|_{L_q(K_\delta)}^q \leq c_6 n^{2q}. \quad (28)
\]
Combining estimate (28) with (27) yields
\[
\| D_{i,j} p \|_{L_q(K_\delta)} \leq c_7 n^{2/\alpha+1}, \quad 1 \leq i, j \leq d, i \neq j. \quad (29)
\]
Finally using relations (28)–(29) together with Lemma 7 yields that $\|Dp\|_{L_q(K_\delta)} = O(n^{2/\alpha+1})$. This provides the needed estimate on the set $K_\delta$. By Theorem 1 on $B^d(0, \delta)$ even the stronger estimate $O(n^2)$ is true. Thus the proof of Theorem 2 is completed.

**Proof of Theorem 3.** Let us first prove that for any differentiable function $F$ and $x \in S^{d-1}$ we have
\[
D^T F(x) \leq \sqrt{d} \sum_{i < j} |D_{i,j} F(x)|, \quad (30)
\]
where the differential operator $D_{i,j}$ is defined in (20). Since $|x| = 1, x = (x_1, \ldots, x_d)$ without loss of generality we may assume that $|x_1| \geq 1/\sqrt{d}$. Set $u_j := (-x_j, 0, \ldots, 0, x_1, 0, \ldots, 0)$,
where $x_1$ is the $j$th coordinate of $u_j$. Then clearly $u_j, 2 \leq j \leq d$, are linearly independent and orthogonal to $x$. Since tangent planes to $S^{d-1}$ are $d-1$-dimensional for any $w \in S^{d-1}$ which is orthogonal to $x$ we have

$$w = (w_1, \ldots, w_d) = \sum_{2 \leq j \leq d} c_j u_j,$$

(31)

with some $c_j \in \mathbb{R}^d$. This clearly implies that

$$w_1 = -\sum_{2 \leq j \leq d} c_j x_j, \quad w_i = c_i x_1, \quad 2 \leq i \leq d.$$

Therefore

$$1 = |w|^2 = \left( \sum_{2 \leq j \leq d} c_j x_j \right)^2 + x_1^2 \sum_{2 \leq j \leq d} c_j^2 \geq \frac{1}{d} \sum_{2 \leq j \leq d} c_j^2,$$

i.e.

$$\max_{2 \leq j \leq d} |c_j| \leq \sqrt{d}.$$

(32)

It follows from (31) and (20) that for given $x$ and any $w \in S^{d-1}$ orthogonal to $x$

$$D_w F(x) = \langle \partial F(x), w \rangle = \sum_{2 \leq j \leq d} c_j \langle \partial F(x), u_j \rangle = \sum_{2 \leq j \leq d} c_j D_{1,j} F(x).$$

This relation together with (32) yields that for every $x \in S^{d-1}$ and $w \in S^{d-1}$ orthogonal to $x$

$$|D_w F(x)| \leq \sqrt{d} \sum_{i < j} |D_{i,j} F(x)|,$$

i.e., (30) holds.

Now we shall follow the proof of Theorem 2. First we use again the transformation of variables (22). Note that since now we work on the unit sphere $r(u)$ is identically equal to 1, i.e. (22) is the usual spherical transformation. In particular, (25) can be rewritten now as

$$\frac{\partial T}{\partial u_1} = D_{1,2} p.$$  

Moreover after the spherical transformation $p \in P^n_d$ becomes a trigonometric polynomial $T$ of degree at most $n$ in each variable. Thus estimating again as in (26) (with $r(u) = 1$ and $\deg T = n$) yields that

$$\|D_{i,j} p\|_{L_q(S^{d-1})} \leq n \|p\|_{L_q(S^{d-1})}.$$

Finally, this last estimate together with (30) easily yields the statement of Theorem 3.

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