Matrix Criteria for the Pseudo-P-Convexity of a Quadratic Form

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ABSTRACT

Except for the family of quasiconvex quadratic functions, no characterization of *generalized convex* quadratic forms on convex sets in $\mathbf{R}^k$ exists. We characterize pseudo-P-convex quadratic functions (i.e., bi-pseudoconvex quadratics for which every stationary point is a global minimum) on solid convex sets in $\mathbf{R}^n \times \mathbf{R}^m$.

1. INTRODUCTION

Let $\mathbf{R}^k$ denote the k-dimensional Euclidean space, and let $\mathscr{C} \subset \mathbf{R}^k$ be an open, convex set. A real valued function

$$\phi : \mathscr{C} \to \mathbf{R}$$

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is called *pseudoconvex* (Tuy [30]) if

$$(z_2 - z_1)^T \nabla\phi(z_1) \geqslant 0 \quad \text{implies} \quad \phi(z_2) \geqslant \phi(z_1) \tag{1.1}$$

for all column vectors $z_1, z_2 \in \mathscr{C}$. Here, T denotes the transpose, and ∇ denotes the gradient, i.e.,

$$\nabla\phi(z) = \left[\frac{\partial\phi(z)}{\partial\zeta_1}, \frac{\partial\phi(z)}{\partial\zeta_2}, \ldots, \frac{\partial\phi(z)}{\partial\zeta_k}\right]^T, \qquad z = [\zeta_1, \zeta_2, \ldots, \zeta_k]^T.$$

Pseudoconvex functions possess the *local-global* property, namely, every local minimum of ϕ in $\mathscr{C}$ is a global minimum. Consequently, such functions are used for modeling optimization problems, since virtually all nonlinear programming algorithms are designed to converge to a local minimum, and it is therefore desired that every local minimum be a global minimum. Mangasarian [23] has shown that pseudoconvex functions belong to the more general class of *quasiconvex* functions, whose lower level sets are convex. Quasiconvex functions play an important role in modeling various economic phenomena (see, for example, [3, 12, 28]).

A variety of optimization models involve *bifunctions*, that is, functions $f(x, y)$ of two vector variables defined on some open convex set $\mathscr{C}_1 \times \mathscr{C}_2 \subset \mathbf{R}^n \times \mathbf{R}^m$. Bifunctions which arise in modeling optimization problems are typically *bipseudoconvex*, i.e., $f(x, y_0)$ and $f(x_0, y)$ are each pseudoconvex for all fixed x_0 and y_0, but not necessarily pseudoconvex in (x, y) when considered as a single variable in $\mathbf{R}^{n+m}$. Bifunctions are used, for example, in bilinear programming, parametric optimization, and sensitivity problems [1, 2, 5, 22].

Bifunctions do not usually satisfy (1.1). Such functions, however, possess the abovementioned local-global property if they satisfy the following natural analogue of (1.1):

$$\left.\begin{aligned} (x_2 - x_1)^T \nabla_x f(x_1, y_1) \geqslant 0 \\ (y_2 - y_1)^T \nabla_y f(x_1, y_1) \geqslant 0 \end{aligned}\right\} \quad \text{imply} \quad f(x_2, y_2) \geqslant f(x_1, y_1) \tag{1.2}$$

for all $(x_1, y_1), (x_2, y_2) \in \mathscr{C}_1 \times \mathscr{C}_2$. A function $f(x, y)$ satisfying (1.2) is called *pseudo-P-convex* (Hackman and Passy [17]). Pseudo-P-convex functions arise quite naturally as nondecreasing superpositions of pseudoconvex functions, since for every two pseudoconvex functions $g : \mathscr{C}_1 \to \mathbf{R}$ and $h : \mathscr{C}_2 \to \mathbf{R}$, and every nondecreasing function $F : \mathbf{R}^2 \to \mathbf{R}$, the function $f(x, y) = F(g(x), h(y))$ is pseudo-P-convex.

From a practical point of view, it is difficult to verify whether a function is pseudoconvex or pseudo-P-convex, since (1.1) and (1.2) involve an infinite number of linear inequalities. This problem has motivated researchers to investigate nonconvex quadratic functions [7, 9, 21, 25, and 27].

From the modeling perspective (cf. [21, 29, 32]), it is important to determine whether a stationary point of a quadratic objective function, restricted to a given convex set, is a global minimum. This is related to the question of whether the quadratic function is convex, pseudoconvex, or pseudo-P-convex on the convex set in question. Most commonly, the set is the nonnegative orthant. Except for the family of quasiconvex quadratic functions, no matrix characterization of *generalized convex* quadratic functions on *solid* convex sets (i.e. convex sets with nonempty interior) in $\mathbf{R}^k$ exists.

The main purpose of this paper is to characterize quadratic functions (i.e. quadratic forms) f which are pseudo-P-convex in the sense of (1.2), on a solid convex subset $\mathscr{C}_1 \times \mathscr{C}_2$ of $\mathbf{R}^n \times \mathbf{R}^m$. We show that such quadratic forms have at most two negative eigenvalues. Quadratic forms with negative eigenvalues cannot be pseudo-P-convex on the whole space $\mathbf{R}^n \times \mathbf{R}^m$; i.e., (1.2) does not hold on the whole space. Instead, for a given quadratic form f we characterize a subset $\mathscr{D}$ of $\mathbf{R}^n \times \mathbf{R}^m$ such that f is pseudo-P-convex on the open convex set $\mathscr{C}_1 \times \mathscr{C}_2$ if and only if $\mathscr{C}_1 \times \mathscr{C}_2 \subseteq \mathscr{D}$. In this sense $\mathscr{D}$ is a *maximal domain of pseudo-P-convexity for f*. The set $\mathscr{D}$ is defined by a fourth order polynominal. While $\mathscr{D}$ is never convex, we extract three pairs of disjoint convex cones contained in $\mathscr{D}$. These pairs of convex cones are characterized by quadratic forms.

Section 2 summarizes the necessary mathematical preliminaries. In particular, we discuss a characterization of pseudo-P-convexity; a characterization due to Chabrillac and Crouzeix [7] of when the restriction of a quadratic form on $\mathbf{R}^k$ to the null space of an $s \times k$ matrix is positive semidefinite; and some fundamental results on the Schur complement. Section 3 develops the criteria for pseudo-P-convexity. In Section 4 the maximal subset of pseudo-P-convex quadratics is defined, and illustrative examples are provided in Section 5.

2. MATHEMATICAL PRELIMINARIES

Pseudoconvex quadratic functions which are not convex are called *merely pseudoconvex* functions. Quadratic functions are never merely pseudoconvex on the whole of $\mathbf{R}^k$ [25]. Martos [25] was the first to characterize merely pseudoconvex functions on the semipositive orthant (we call a vector semipositive if it is entrywise nonnegative and at least one of its components is positive). For a given matrix Q, Cottle and Ferland [9], Crouzeix and Ferland [10], and Ferland [14] have provided matrix criteria for a quadratic form z^TQz to be merely pseudoconvex on a solid convex set. In general, Ferland [14] and Schaible [27] have shown that Q can have at most one negative eigenvalue. Moreover, a quadratic form with exactly one negative eigenvalue is pseudoconvex on any solid convex set[1] $\mathscr{C}$ if and only if $\mathscr{C}$ is contained in the union of the disjoint convex cones

$$\mathscr{D}^- = \{z \in \mathbf{R}^k : z^TQz \leqslant 0 \text{ and } v^Tz > 0\} \cup \{z \in \mathbf{R}^k : z^TQz \leqslant 0 \text{ and } v^Tz < 0\}, \tag{2.1}$$

where v is the eigenvector associated with the negative eigenvalue of Q. In this sense Equation (2.1) defines the *maximal domain of pseudoconvexity*.

If Q is nonsingular, then (2.1) is equivalent to

$$\mathscr{D}^- = \{z \in \mathbf{R}^k : z^TQz \leqslant 0,\ z \neq 0\}, \tag{2.2}$$

and it can be shown [14] that

$$\mathscr{D}^- = \{z \in \mathbf{R}^k : d^TQz = 0 \text{ for some } d \in \mathbf{R}^k \text{ implies } d^TQd \geqslant 0\}. \tag{2.3}$$

Diewert et al. [13] showed that a continously differentiable function ϕ defined on $\mathbf{R}^k$ is pseudoconvex on a convex subset $\mathscr{C} \subset \mathbf{R}^k$ if and only if for all $z_1, z_2 \in \mathscr{C}$

$$(z_2 - z_1)^T \nabla\phi(z_1) = 0 \quad \Rightarrow \quad \phi(z_1) = \min_{0 \leqslant \gamma \leqslant 1} \phi(z_1 + \gamma(z_2 - z_1)). \tag{2.4}$$

For quadratic pseudoconvex forms z^TQz, (2.4) becomes

$$(z_2 - z_1)^T Qz_1 = 0 \quad \Rightarrow \quad (z_2 - z_1)^T Q(z_2 - z_1) \geqslant 0. \tag{2.5}$$

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[1]This result does not necessarily hold for convex sets without interior. Consider the example $f(x, y) = x^TQ_1x - y^TQ_2y$ where Q_1 is positive definite but Q_2 is an arbitrary matrix. Then on the set $\{(x, y) : x \in \mathbf{R}^k,\ y = 0\}$ the function is convex and hence pseudoconvex.

Since $\mathscr{D}^-$ in (2.3) is the maximal domain of pseudoconvexity, it follows from (2.5) that a quadratic form is pseudoconvex on a solid convex set $\mathscr{C} \subset \mathbf{R}^k$ if and only if

$$d^T Qz = 0 \quad \Rightarrow \quad d^T Qd \geqslant 0 \qquad \text{for all} \quad z \in \mathscr{C}. \tag{2.6}$$

The question of when the restriction of a quadratic form on $\mathbf{R}^k$ to the null space of an $s \times k$ matrix is positive semidefinite has been extensively investigated by Hancock [18], Mann [24], Samuelson [26], Debreu [11], Bellman [6], and Hestenes [20]. Crouzeix and Ferland [10] and Schaible [27] dealt with the special case $s = 1$. Recently, Chabrillac and Crouzeix [7] unified all these results using Schur complement theory. Based on these results, it is possible to obtain (2.6) (see also Crouzeix and Ferland [10]).

The following result is used for the development of our criteria:

THEOREM 1 (Chabrillac and Crouzeix [7]). *Let K be a symmetric $k \times k$ matrix,m and let L be a $k \times s$ matrix of full column rank. Then $z^T Kz \geqslant 0$ for all z such that $z^T L = 0$ if and only if the bordered matrix*

$$\begin{bmatrix} K & \vdots & L \\ \cdots & & \cdots \\ L^T & \vdots & 0 \end{bmatrix}$$

has exactly s negative eigenvalues.

The following theorem characterizes pseudo-P-convex functions which are continuously differentiable and bi-pseudoconvex.

THEOREM 2. *Let $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ be a continuously differentiable function. Then f is pseudo-P-convex on the convex set $\mathscr{C}_1 \times \mathscr{C}_2$ if and only if*

$$\left.\begin{aligned} (x_2 - x_1)^T \nabla_x f(x_1, y_1) = 0 \\ (y_2 - y_1)^T \nabla_y f(x_1, y_1) = 0 \end{aligned}\right\} \quad \Rightarrow$$

$$f(x_1, y_1) = \min_{0 \leqslant \alpha, \beta \leqslant 1} f(x_1 + \alpha(x_2 - x_1), y_1 + \beta(y_2 - y_1)) \tag{2.7}$$

holds for all $(x_1, y_1), (x_2, y_2) \in \mathscr{C}_1 \times \mathscr{C}_2$.

Proof. It is immediate from the definition (1.2) that every pseudo-P-convex function satisfies (2.7) (here the assumption of continuous differentiability is not required.) As for the converse, pick $(x_1,y_1),(x_2,y_2)\in\mathscr{C}_1\times\mathscr{C}_2$ such that $(x_2-x_1)^T\nabla_x f(x_1,y_1)\geqslant 0$ and $(y_2-y_1)^T\nabla_y f(x_1,y_1)\geqslant 0$. Note that if $x_1=x_2$, then by identifying z with y and ϕ with $f(x_1,\cdot)$, (2.7) becomes (2.4) and it follows that for a given x_1, $f(x_1,\cdot)$ is pseudoconvex. Similarly, if $y_1=y_2$, (2.4) shows that $f(\cdot,y_1)$ is pseudoconvex. Thus, f is a bi-pseudoconvex function.

We must show that $f(x_2,y_2)\geqslant f(x_1,y_1)$. Define

$$h(\alpha,\beta)=f(x_1+\alpha(x_2-x_1),y_1+\beta(y_2-y_1)).$$

Note that h is continuously differentiable, bi-pseudoconvex function that satisfies (2.7). The proof will follow if we can argue that $h(1,1)\geqslant h(0,0)$.

Consider the set

$$\mathscr{D}=\{(\alpha,\beta)\in[0,1]\times[0,1]:\nabla h(\alpha,\beta)\geqslant 0 \text{ and } h(\alpha,\beta)\geqslant h(0,0)\}.$$

Clearly, $(0,0)\in\mathscr{D}$ and so $\mathscr{D}$ is nonempty. Maximize the sum $\alpha+\beta$ over $\mathscr{D}$. Since h is continuously differentiable and the unit square is compact, a maximal element (α^*,β^*) exists. If $\nabla h(\alpha^*,\beta^*)=0$, then (2.7) guarantees that $h(1,1)\geqslant h(0,0)$. Suppose, however, that both partial derivatives of h at (α^*,β^*) are positive. Since (α^*,β^*) is a maximal element, $(\alpha^*,\beta^*)=(1,1)$. Consequently, $h(1,1)\geqslant h(0,0)$. Without loss of generality, suppose finally that

$$\frac{\partial}{\partial\alpha}h(\alpha^*,\beta^*)=0 \quad\text{and}\quad \frac{\partial}{\partial\beta}h(\alpha^*,\beta^*)>0.$$

Since $h(\cdot,\beta^*)$ is pseudoconvex,

$$h(\alpha^*,\beta^*)=\min_{0\leqslant\alpha\leqslant 1}h(\alpha,\beta^*). \tag{2.8}$$

Suppose $\alpha^*<1$. Since h is continuously differentiable, (2.8) implies that there exists $\varepsilon>0$ for which both partial derivatives of h at $(\alpha^*+\varepsilon,\beta^*)$ are nonnegative, once again contradicting the maximality of (α^*,β^*). Therefore $\alpha^*=1$. Now, since $h(1,\beta)$ is pseudoconvex, it follows immediately that $h(1,1)\geqslant h(1,\beta^*)\geqslant h(0,0)$, as required. ∎

We conclude this section with a brief review of the Schur complement theory (see Chabrillac and Crouzeix [7] and Haynsworth [19]), which we shall use in the sequel. Given a real symmetric $k\times k$ matrix K, the *inertia* of K, denoted as usual by In(K) is the triple $(\pi(K), \nu(K), \delta(K))$ consisting of a number of positive, negative, and zero eigenvalues of K, so that $\pi(K)+\nu(K)+\delta(K)=k$. Given the partition

$$K=\begin{bmatrix} K_1 & K_3 \\ K_3^T & K_2 \end{bmatrix}$$

of K, where K_1, K_2, and K_3 are $n\times n$, $m\times m$, and $n\times m$, respectively, the *Schur complement* of K_1 in K is $K/K_1 \equiv K_2 - K_3^T K_1^{-1} K_3$, assuming that K_1 is nonsingular. Similarly, the Schur complement of K_2 in K is $K/K_2 \equiv K_1 - K_3 K_2^{-1} K_3^T$. In either case,

$$\det K = \det K_i \det K/K_i, \qquad i=1,2. \tag{2.9}$$

Finally, the inertia of K may be determined from the inertias of K_i and the Schur complement K/K_i by the equation

$$\mathrm{In}(K)=\mathrm{In}(K_i)+\mathrm{In}(K/K_i), \qquad i=1,2. \tag{2.10}$$

3. MATRIX CRITERIA FOR PSEUDO-P-CONVEXITY

Consider the following quadratic form:

$$f(u)=\tfrac{1}{2}u^TQu=\tfrac{1}{2}[x^T,y^T]\begin{bmatrix} Q_1 & Q_3 \\ Q_3^T & Q_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \qquad u=\begin{bmatrix} x \\ y \end{bmatrix}. \tag{3.1}$$

Throughout this paper we assume, unless otherwise stated, that Q_1, Q_2, and Q are symmetric and *nonsingular*. Consider the matrices

$$V=[Q_1 \,\vdots\, Q_3] \quad \text{and} \quad W=[Q_3^T \,\vdots\, Q_2].$$

Note that $Vu=\nabla_x f(u)$ and $Wu=\nabla_y f(u)$.

By Theorem 2, $f(u)$ in (3.1) is pseudo-P-convex on some solid convex domain $\mathscr{C}_1\times\mathscr{C}_2 \subset \mathbf{R}^n\times\mathbf{R}^m$ if and only if

$$\left.\begin{aligned} (x_2-x_1)^T V u_1 = 0 \\ (y_2-y_1)^T W u_1 = 0 \end{aligned}\right\}$$

$$\Rightarrow \big((x_2-x_1)^T,(y_2-y_1)^T\big)Q\big((x_2-x_1),(y_2-y_1)\big)\geqslant 0 \tag{3.2}$$

for all $u_1=(x_1,y_1)$, $u_2=(x_2,y_2)\in\mathscr{C}_1\times\mathscr{C}_2$. For a point u in the interior of $\mathscr{C}_1\times\mathscr{C}_2$ instead of (3.2) we can write

$$\left.\begin{array}{l} d_1^T\nabla_x f(u)=d_1^T Vu=0\\ d_2^T\nabla_y f(u)=d_2^T Wu=0\end{array}\right\} \Rightarrow \begin{bmatrix} d_1^T & d_2^T\end{bmatrix} Q\begin{bmatrix} d_1\\ d_2\end{bmatrix}\geqslant 0 \tag{3.3}$$

for all $(d_1,d_2)\in\mathbf{R}^n\times\mathbf{R}^m$.

It will be shown in Theorem 3 that (3.3) holds even for boundary points of $\mathscr{C}_1\times\mathscr{C}_2$, except possibly for points where the partial gradients $\nabla_x f(u), \nabla_y f(u)$ vanish.

If we identify

$$z=\begin{bmatrix} d_1\\ d_2\end{bmatrix},\qquad L=\begin{bmatrix} Vu & \vdots & 0\\ \cdots & \cdots & \cdots\\ 0 & \vdots & Wu\end{bmatrix},\qquad K=Q,\qquad s=2,\quad\text{and}\quad k=n+m,$$

then Theorem 1 implies that the matrix

$$\begin{bmatrix} K & \vdots & L\\ \cdots & \cdots & \cdots\\ L^T & \vdots & 0\end{bmatrix}$$

has exactly two negative eigenvalues; hence Q has at most two negative eigenvalues, and this is a necessary condition for the pseudo-P-convexity of f. If Q has no negative eigenvalues, then clearly f is convex on the whole of $\mathbf{R}^n\times\mathbf{R}^m$ and no further discussion is required.

We now turn to the case when Q has one negative eigenvalue. The case when Q has two negative eigenvalues will be examined later.

Hereafter, the expression *f is pseudo-P-convex at a point* $u=(x,y)\in\mathscr{C}_1\times\mathscr{C}_2$ means that (3.3) holds. Pseudoconvexity implies pseudo-P-convexity; thus f is pseudo-P-convex at each $u\in\mathscr{D}^-$ (2.6). Since Q has a negative eigenvalue, f is not pseudo-P-convex at the origin. Therefore, we have to consider only the set

$$\mathscr{D}^+=\{u=(x,y):u^TQu>0\}. \tag{3.4}$$

The following well-known *interlacing* lemma is critical to the proof of the next theorem.

LEMMA 1 [15, p. 269]. *Let A be an $n \times n$ symmetric matrix, let A_r denote its leading $r \times r$ principle submatrix, and let $\lambda_i(A_r)$, $1 \leqslant i \leqslant r$, denote the value of the ith largest eigenvalue of the matrix A_r. Then*

$$\lambda_{r+1}(A_{r+1}) \leqslant \lambda_r(A_r) \leqslant \lambda_r(A_{r+1}) \leqslant \cdots$$
$$\leqslant \lambda_2(A_r) \leqslant \lambda_2(A_{r+1}) \leqslant \lambda_1(A_r) \leqslant \lambda_1(A_{r+1}).$$

THEOREM 3. *The quadratic function f is pseudo-P-convex at $u \in \mathscr{D}^+$ if and only if $\nabla_x f(u) \neq 0$, $\nabla_y f(u) \neq 0$, and*

$$u^T Qu\{u^T W^T (Q/Q_1)^{-1} Wu\} - (u^T W^T y)^2$$
$$= u^T Qu\{u^T V^T (Q/Q_2)^{-1} Vu\} - (u^T V^T x)^2 \leqslant 0. \tag{3.5}$$

REMARK 1. *Neither of the partial gradients $\nabla_x f(u)$ and $\nabla_y f(u)$ vanishes at points where the inequalities* (3.5) *are strict.*

Proof. We first argue that a necessary condition for the pseudo-P-convexity of f at a point $u \in \mathscr{D}^+$ is that $\nabla_x f(u) \neq 0$ and $\nabla_y f(u) \neq 0$. Indeed, when $\nabla_x f(u)$ or $\nabla_y f(u)$ vanishes, the condition (3.3) is equivalent to (2.6); hence $u \in \mathscr{D}^-$, contradicting the assumption that $u \in \mathscr{D}^+$. Consequently, the statement of the theorem is trivially true in this case. We now assume that either $\nabla_x f(u) \neq 0$ and $\nabla_y f(u) \neq 0$. By Theorem 1, the function f is pseudo-P-convex at $u \in \mathscr{D}^+$ if and only if the bordered matrix

$$B(u) = \left[\begin{array}{cc:cc} Q_1 & Q_3 & \nabla_x f(u) & 0 \\ Q_3^T & Q_2 & 0 & \nabla_y f(u) \\ \hdashline \nabla_x^T f(u) & 0 & 0 & 0 \\ 0 & \nabla_y^T(u) & 0 & 0 \end{array}\right]$$

has exactly two negative eigenvalues. Add the last row and column of $B(u)$ to its $(n+m+1)$st row and column to obtain the matrix

$$D(u) = \left[\begin{array}{cc:c:c} Q_1 & Q_3 & \nabla f(u) & 0 \\ Q_3^T & Q_2 & & \nabla_y f(u) \\ \hdashline \multicolumn{2}{c:}{\nabla^T f(u)} & 0 & 0 \\ 0 & \nabla_y^T f(u) & 0 & 0 \end{array}\right].$$

Of course, $\nu(D(u)) = \nu(B(u))$. We shall think of $D(u)$ as a matrix obtained from

$$Q = \begin{bmatrix} Q_1 & Q_3 \\ Q_3^T & Q_2 \end{bmatrix}$$

in two steps, as follows:

$$Q \to C(u) = \left[\begin{array}{cc:c} Q_1 & Q_3 & \multirow{2}{*}{$\nabla f(u)$} \\ Q_3^T & Q_2 & \\ \hdashline \multicolumn{2}{c:}{\nabla^T f(u)} & 0 \end{array}\right] \to D(u). \tag{3.6}$$

Using (2.9),

$$\det C(u) = (\det Q)\left[0 - \nabla^T f(u) Q^{-1} \nabla f(u)\right] = -(\det Q) u^T Q u,$$

which is positive, since by assumption $\det Q < 0$ and $u^T Q u > 0$. From Lemma 1 it follows that the first step, $Q \to C(u)$, adds one negative eigenvalue, i.e.,

$$\nu(C(u)) = 2 \quad \text{and} \quad \delta(C(u)) = 0. \tag{3.7}$$

Lemma 1 together with (3.7) establishes that

$$2 \leqslant \nu(D(u)) + \delta(D(u)) \leqslant 3.$$

Consequently, $\nu(D(u)) = 2$ if and only if

$$\det D(u) = (\det Q)\left[u^T Q u \left(u^T W^T (Q/Q_1)^{-1} W u - \frac{1}{u^T Q u}\left(u^T W^T y\right)^2\right)\right] \geqslant 0. \tag{3.8}$$

Here we have used (2.9) to calculate $\det D(u)$ from $\det C(u)$.

Interchanging the roles of x and y in Theorem 2, f is pseudo-P-convex at $u \in \mathscr{D}^+$ if and only if the bordered matrix

$$E(u) = \left[\begin{array}{cc:cc} Q_2 & Q_3^T & 0 & \nabla_x f(u) \\ Q_3 & Q_1 & \nabla_y f(u) & 0 \\ \hdashline 0 & \nabla_y^T f(u) & 0 & 0 \\ \nabla_x^T f(u) & 0 & 0 & 0 \end{array}\right]$$

has exactly two negative eigenvalues. And, by similar reasoning to the above, this is equivalent to

$$\det E(u) = (\det Q)\left[u^TQu\left(u^TV^T(Q/Q_2)^{-1}Vu - \frac{1}{u^TQu}(u^TV^Tx)^2\right)\right] \geqslant 0. \tag{3.9}$$

The result now follows from (3.8) and (3.9), since $\det E(u) = \det D(u)$ and $\det Q < 0$. ■

Note that (3.8) holds when $u^TQu > 0$ and $u^TW^T(Q/Q_1)^{-1}Wu \leqslant 0$. Similarly, (3.9) holds when $u^TQu > 0$ and $u^TV^T(Q/Q_2)^{-1}Vu \leqslant 0$. Note further that if $\nabla_x f(u) = 0$ then $u^TQu = u^TW^T(Q/Q_1)^{-1}Wu$, and similarly, if $\nabla_y f(u) = 0$ then $u^TQu = u^TV^T(Q/Q_2)^{-1}Vu$.

Motivated by these observations, we have the following corollary:

Corollary 1. *The quadratic function f is pseudo-P-convex at u if at least one of the following inequalities holds*:

(i) $u^TQu \leqslant 0$, $u \neq 0$;
(ii) $u^TV^T(Q/Q_2)^{-1}Vu \leqslant 0$, $\nabla_x f(u) \neq 0$;
(iii) $u^TW^T(Q/Q_1)^{-1}Wu \leqslant 0$, $\nabla_y f(u) \neq 0$.

When Q is nonsingular and has exactly two negative eigenvalues, then we have the following:

Remark 2. *If Q has two negative eigenvalues and either one of the partial gradients $\nabla_x f(u)$, $\nabla_y f(u)$ vanishes, then the condition* (3.3) *cannot hold.*

This is a direct consequence of Theorem 1. It follows therefore, that in this case pseudo-P-convexity does not hold at points where a partial gradient vanishes.

Theorem 4. *Suppose Q is nonsingular and has exactly two negative eigenvalues. The quadratic form f is pseudo-P-convex at u if and only if the following hold*:

(i) $\nabla_x f(u) \neq 0$, $\nabla_y f(u) \neq 0$,
(ii) $u^TQu[u^TW^T(Q/Q_1)^{-1}Wu] - (u^TW^Ty)^2$
$= u^TQu[u^TV^T(Q/Q_2)^{-1}Vu] - (u^TV^Tx)^2 \geqslant 0$,
(iii) $u^TQu \leqslant 0$.

Proof. By Remark 2, f is pseudo-P-convex at u only if $\nabla_x f(u) \neq 0$ and $\nabla_y f(u) \neq 0$. In this case $\det Q > 0$ and $C(u)$ (3.6) has two negative eigenvalues. Thus,

$$\det C(u) = (\det Q)\left[0 - \nabla^T f(u) Q^{-1} \nabla f(u)\right] = -\det(Q) u^T Q u \geqslant 0;$$

hence $u^T Q u \leqslant 0$. Since $\nabla_x f(u) \neq 0$ and $\nabla_y f(u) \neq 0$, the matrix $D(u)$ has also two negative eigenvalues. The rest of the proof is identical to the steps taken in Theorem 3. ■

As an immediate result we have the following corollary:

COROLLARY 2. *Suppose Q is nonsingular and has exactly two negative eigenvalues. If f is pseudo-P-convex at u, then each of the three inequalities*

(i) $u^T Q u \leqslant 0$,
(ii) $u^T W^T (Q/Q_1)^{-1} W u > 0$,
(iii) $u^T V^T (Q/Q_2)^{-1} V u \leqslant 0$

holds.

Proof. From Theorem 4 it follows that $u^T Q u \leqslant 0$. Conditions (ii) and (iii) can be obtained if in (3.6) $\nabla f(u)$ is replaced with $\nabla_x f(u)$ or with $\nabla_y f(u)$. ■

Compare this result with Corollary 1.

4. DOMAINS OF PSEUDO-P-CONVEXITY OF QUADRATIC FUNCTIONS

We now suppose that a quadratic form f is given, and we wish to find whether f is pseudo-P-convex on a given solid convex domain $\mathscr{C}_1 \times \mathscr{C}_2$. A similar question for pseudoconvex functions is addressed by Cottle and Ferland in [9].

Let $\mathscr{D}$ be the set of all points at which f is pseudo-P-convex. In view of the following corollary we call $\mathscr{D}$ *the maximal domain of pseudo-P-convexity of f*. If $\nu(Q) = 1$, then $\mathscr{D}$ is defined via Equation (2.3) and Theorem 3. If $\nu(Q) = 2$, then $\mathscr{D}$ is defined via Theorem 4. Denote by $\bar{\mathscr{D}}$ the closure of $\mathscr{D}$; it follows from Theorems 3 and 4 that

$$\bar{\mathscr{D}} = \mathscr{D} \cup \{u : \nabla_x f(u) = 0 \text{ or } \nabla_y f(u) = 0\}. \tag{4.1}$$

COROLLARY 3. *Let $\mathscr{D}$ be the maximal domain of pseudo-P-convexity.*

(i) *If $\mathscr{C}_1 \times \mathscr{C}_2 \subseteq \mathscr{D}$, then f is pseudo-P-convex on $\mathscr{C}_1 \times \mathscr{C}_2$.*

(ii) *If f is pseudo-P-convex on $\mathscr{C}_1 \times \mathscr{C}_2$ then* $\operatorname{int}(\mathscr{C}_1 \times \mathscr{C}_2) \subseteq \mathscr{D}$, *where* int *denotes the interior of a set.*

Proof. These results follow from Equation (2.2) and Theorems 2, 3, and 4. ∎

Note that when $\mathscr{C}_1 \times \mathscr{C}_2$ is open, f is pseudo-P-convex if and only if $\mathscr{C}_1 \times \mathscr{C}_2 \subseteq \mathscr{D}$. In general, if $\operatorname{int}[\mathscr{C}_1 \times \mathscr{C}_2] \subseteq \mathscr{D}$, to determine whether f is pseudo-P-convex on $\mathscr{C}_1 \times \mathscr{C}_2$ one must check separately the boundary points, i.e., the points in $\mathscr{C}_1 \times \mathscr{C}_2 \cap \overline{\mathscr{D}}$ which do not belong to $\mathscr{D}$, using the relation (3.2).

Recall that $\mathscr{D}$ is characterized by a fourth order polynomial. However, if $\nu(Q) = 1$ and Q_1 and Q_2 are positive definite, we can extract three pairs of disjoint convex cones contained in $\mathscr{D}$ which are characterized by a quadratic form.

Let v denote a normalized eigenvector associated with the single negative eigenvalue of Q. Since Q_1 and Q_2 have only positive eigenvalues, it follows from (2.10) that the Schur complements Q/Q_1 and Q/Q_2 each have exactly one negative eigenvalue. Let v_1 and v_2 denote the eigenvectors associated with the single negative eigenvalues of Q/Q_1 and Q/Q_2, respectively. Define

$$\mathscr{T}_1^{<} = \{y \in \mathbf{R}^m : y^T(Q/Q_1)^{-1}y \leqslant 0 \text{ and } y^T v_1 < 0\},$$

$$\mathscr{T}_1^{>} = \{y \in \mathbf{R}^m : y^T(Q/Q_1)^{-1}y \leqslant 0 \text{ and } y^T v_1 > 0\},$$

$$\mathscr{T}_2^{<} = \{x \in \mathbf{R}^n : x^T(Q/Q_2)^{-1}x \leqslant 0 \text{ and } x^T v_2 < 0\},$$

$$\mathscr{T}_2^{>} = \{x \in \mathbf{R}^n : x^T(Q/Q_2)^{-1}x \leqslant 0 \text{ and } x^T v_2 > 0\}.$$

The sets $\mathscr{T}_1^{<}$, $\mathscr{T}_1^{>}$, $\mathscr{T}_2^{<}$, and $\mathscr{T}_2^{>}$ are convex cones with nonempty interior [16, p. 270]. By Corollary 1, f is pseudo-P-convex on any solid convex set $\mathscr{C}_1 \times \mathscr{C}_2$ if this set lies in the union of the following three cones, each a union of two disjoint convex sets:

$$\mathscr{T}_1 \equiv \{u : Wu \in \mathscr{T}_1^{>}\} \cup \{u : Wu \in \mathscr{T}_1^{<}\},$$

$$\mathscr{T}_2 \equiv \{u : Vu \in \mathscr{T}_2^{>}\} \cup \{u : Vu \in \mathscr{T}_2^{<}\},$$

$$\mathscr{T}_3 \equiv \{u \in \mathscr{D}^- : u^T v < 0\} \cup \{u \in \mathscr{D}^- : u^T v > 0\}.$$

From the modeling perspective (cf. [21, 29, 32]), it is important to determine whether a Kuhn-Tucker stationary point of a quadratic objective function defined on the nonnegative orthant is a global minimum. This question is related to the question of whether the quadratic function is convex, pseudoconvex, or pseudo-P-convex on the nonnegative orthant. The next paragraph deals with this question. The following definition is required.

DEFINITION 1 [4]. An $n \times n$ symmetric matrix A is *conegative* (*copositive*) if $p^TAp \leqslant 0$ ($p^TAp \geqslant 0$) for all $p \geqslant 0$.

A test for conegativity is provided in [8]. Clearly, a matrix A is conegative if all its entries are nonpositive. A test for conegativity on the intersection of the nonnegative orthant with any polyhedral cone is given in [31].

COROLLARY 4. *Let Q be a nonsingular symmetric matrix with a single negative eigenvalue. Then the quadratic form u^TQu is pseudo-P-convex on the semipositive orthant if any of the following three conditions holds*:

(a) *Q is conegative.*

(b) *The matrix $W^T(Q/Q_1)^{-1}W$ is conegative, and Q is also conegative but only on the cone $\{u=(x,y):\nabla_y^Tf(u)=[x^T,y^T]W^T=0\}\cap\{u=(x,y)\geqslant 0\}$.*

(c) *The matrix $V^T(Q/Q_2)^{-1}V$ is conegative, and Q is also conegative but only on the cone $\{u=(x,y):\nabla_x^Tf(u)=[x^T,y^T]V^T=0\}\cap\{u=(x,y)\geqslant 0\}$.*

Proof. The proof follows immediately from Corollary 1 by observing that whenever the partial gradients vanish, conditions (b) and (c) in Corollary 4 imply $u^TQu \leqslant 0$, i.e., condition (i) in Corollary 1 holds. ■

For a vector z the expression $z \nleqslant 0$ denotes that some components of z are positive. The partial gradients of the quadratic form do not vanish on the semipositive orthant when the following holds.

COROLLARY 5. *If $Q_2^{-1}Q_3^Tx \nleqslant 0$ ($Q_1^{-1}Q_3y \nleqslant 0$) on the semipositive orthant and if $W^T(Q/Q_1)^{-1}W$ ($V^T(Q/Q_2)^{-1}V$) is conegative, then Q is pseudo-P-convex on the semipositive orthant.*

Proof. The proof is a direct consequence of Corollary 4. The partial gradients of $f(u)$, $\nabla_y^Tf(u)=[x^T,y^T]W^T=x^TQ_3+y^TQ_2$ and $\nabla_x^Tf(u)=[x^T,y^T]V^T=x^TQ_1+y^TQ_3^T$, do not vanish on the semipositive orthant if $Q_2^{-1}Q_3^Tx \nleqslant 0$ or $Q_1^{-1}Q_3y \nleqslant 0$, respectively. ■

5. PSEUDO-P-CONVEX QUADRATIC FUNCTIONS ON THE SEMIPOSITIVE ORTHANT

The examples in this section illustrate the above results. More specifically, we characterize $(n+1)\times(n+1)$ matrices of the form

$$Q=\begin{bmatrix} Q_1 & q \\ q^T & \alpha \end{bmatrix},$$

with one or two negative eigenvalues, whose associated quadratic form

$$f(u)=\tfrac{1}{2}\left[x^TQ_1x+2x^Tqy+\alpha y^2\right] \tag{5.1}$$

is pseudo-P-convex on the semipositive orthant of $\mathbf{R}^n\times\mathbf{R}$. Here, Q_1 is an $n\times n$ matrix and q is an n-vector.

We begin by examining the case when Q has exactly one negative eigenvalue. Let $\mathscr{D}^-$ be defined as in (2.2), and let $u\in\mathscr{D}^-$. Then f is pseudoconvex and therefore pseudo-P-convex. When $u\in\mathscr{D}^+=\{u:u^TQu>0\}$ we know from Theorem 3 that f is pseudo-P-convex at u if and only if

$$\nabla_y^Tf(u)=\left[x^T,y\right]\begin{bmatrix} q \\ \alpha \end{bmatrix}\neq 0, \tag{5.2}$$

$$\nabla_x^Tf(u)=\left[x^T,y\right]\begin{bmatrix} Q_1 \\ q^T \end{bmatrix}\neq 0, \tag{5.3}$$

$$u^TQu\left(\left[x^T,y\right]\begin{bmatrix} q \\ \alpha \end{bmatrix}\left(\alpha-q^TQ_1^{-1}q\right)^{-1}\left[q^T,\alpha\right]\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$-\left(\left[x^T,y\right]\begin{bmatrix} q \\ \alpha \end{bmatrix}y\right)^2\leqslant 0. \tag{5.4}$$

When (5.2) holds, (5.4) simplifies to

$$\frac{1}{\alpha-q^TQ_1^{-1}q}-\frac{y^2}{u^TQu}\leqslant 0. \tag{5.5}$$

We should consider four cases:

(a) the biconvex case, i.e., Q_1 is positive definite and $\alpha > 0$;
(b) the convex-pseudoconvex case, i.e., Q_1 is positive definite and $\alpha < 0$;
(c) the pseudoconvex-convex case, i.e., the matrix Q_1 defines a merely pseudoconvex quadratic and $\alpha > 0$; and
(d) the bi-pseudoconvex case, i.e., the matrix Q_1 defines a merely pseudoconvex quadratic and $\alpha < 0$.

We examine each case separately.

(a) The Biconvex Case

Since Q has exactly one negative eigenvalue, we know from Schur complement theory that

$$\alpha - q^T Q_1^{-1} q < 0. \tag{5.6}$$

It is immediate from (5.5) and (5.6) that f is pseudo-P-convex on the semipositive orthant except at points where $\nabla_y f(u)$ or $\nabla_x f(u)$ vanishes.

When $\nabla_y f(u) = 0$, then

$$x = -Q_1^{-1} q y; \tag{5.7}$$

consequently,

$$u^T Q u = \tfrac{1}{2} y^2 \left(\alpha - q^T Q_1^{-1} q\right), \tag{5.8}$$

which is negative by (5.6). Thus when $\nabla_x f(u) = 0$, we have $u \in \mathscr{Q}^-$, and f is pseudoconvex and hence pseudo-P-convex.

When $\nabla_y f(u) = 0$, then

$$y = -x^T q / \alpha. \tag{5.9}$$

Here we have to consider two possibilities:

(i) If $q \geqslant 0$, then f is pseudo-P-convex on the semipositive orthant. An example of a biconvex quadratic with $q \geqslant 0$ is given by the matrix[2]

$$Q = \left[\begin{array}{cc:c} 1 & 0 & 3 \\ 0 & 1 & 3 \\ \hdashline 3 & 3 & 1 \end{array}\right].$$

[2] If f is pseudo-P-convex on the semipositive orthant, then it is pseudo-P-convex on the nonnegative orthant if and only if the relation (1.2) holds at the origin. Here this condition is

(ii) If $q \ngeqslant 0$, then it follows by substituting (5.9) in (5.1) that f is pseudo-P-convex on the semipositive orthant if

$$u^T Q u = \tfrac{1}{2} x^T \left(Q_1 - \frac{qq^T}{\alpha} \right) x \leqslant 0 \qquad \text{on the cone } \{x : x \geqslant 0,\, x^T q \leqslant 0\}. \quad (5.10)$$

A test for (5.10) is given in [31]. Note that when $q \leqslant 0$, (5.10) simplifies to the conegativity of $Q_1 - qq^T/\alpha$. Since $Q_1 - qq^T/\alpha$ is the Schur complement of Q, it is a nonsingular matrix with a single negative eigenvalue. By a simple inductive argument using Lemma 1, it can be shown that a nonsingular matrix with a single negative eigenvalue is conegative if and only if all its elements are nonpositive. An example of a biconvex quadratic with $q \leqslant 0$ is given by the matrix

$$Q = \left[\begin{array}{cc:c} 1 & 0 & -3 \\ 0 & 1 & -3 \\ \hdashline -3 & -3 & 1 \end{array}\right].$$

(b) The Convex-Pseudoconvex Case

The mere pseudoconvexity of $f(x^*, y)$ for a fixed x^* assures that the partial gradient with respect to y, $\nabla_y f(u)$, does not vanish [25]. As shown in case (a), it is still true that when $\nabla_x f(u)$ vanishes, $u \in \mathscr{Q}^-$. Thus, in the present case, f is pseudo-P-convex on the semipositive orthant.

An example of a convex-pseudoconvex quadratic is given by the matrix

$$Q = \left[\begin{array}{cc:c} 1 & 0 & -3 \\ 0 & 1 & -3 \\ \hdashline -3 & -3 & -1 \end{array}\right].$$

(c) The Pseudoconvex-Convex Case

For each fixed nonnegative y^*, $f(x, y^*)$ is merely pseudoconvex; therefore $Q_1 \leqslant 0$, $q \leqslant 0$, and $q^T Q_1^{-1} q \leqslant 0$ [25]. Assume that the partial gradients $\nabla_x f(u)$, $\nabla_y f(u)$ do not vanish, and $u \notin \mathscr{Q}^-$. With these assumptions (5.5)

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equivalent to the copositivity of Q. The following example satisfies this condition and is therefore pseudo-P-convex on the nonnegative orthant, including the origin.

becomes

$$u^T Q u - \alpha y^2 + q^T Q_1^{-1} q y^2 \leqslant 0.$$

It follows that

$$x^T Q_1 x + 2x^T q y + \alpha y^2 - \alpha y^2 + q^T Q_1^{-1} q y^2 \leqslant 0,$$

or equivalently

$$[x^T, y^T] \begin{bmatrix} Q_1 & q \\ q^T & q^T Q_1^{-1} q \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leqslant 0. \tag{5.11}$$

[The elements of the matrix in (5.11) are nonpositive; thus it is conegative and the inequality holds.] Therefore, the function $f(u)$ is pseudo-P-convex on the semipositive orthant, except possibly at points where $\nabla_y f(u)$ or $\nabla_x f(u)$ vanish. Mere pseudoconvexity of $f(x, y^*)$ for any fixed y^* assures that $\nabla_x f(u)$ does not vanish on this orthant [25]. It can also be shown that if $\nabla_y f(u)$ vanishes, then $u \in \mathscr{D}^-$. Thus, in the present case, f is pseudo-P-convex on the semipositive orthant.

An example of a pseudoconvex-convex quadratic is given by the matrix

$$Q = \left[\begin{array}{cc:c} 0 & -1 & -1 \\ -1 & 0 & -1 \\ \hdashline -1 & -1 & 2 \end{array}\right].$$

(d) The Bi-Pseudoconvex Case

Mere bi-pseudoconvexity of $f(x, y^*)$ and $f(x^*, y)$ on the corresponding semipositive orthants assures that all the entries of Q_1 and q are nonpositive, and $\alpha \leqslant 0$. Hence, f satisfies Martos's criteria in [25], so f is merely pseudoconvex and therefore pseudo-P-convex on the semipositive orthant.

We now turn to consider the case when Q has two negative eigenvalues. First, assume that f is pseudoconvex-convex. Then by Martos's criteria in [25] it follows that $q \leqslant 0$, and since $\alpha > 0$, $\nabla_y f(x^*, y) = q^T x^* + \alpha y$ vanishes on the positive orthant. By Remark 2, such a quadratic function is never pseudo-P-convex, so the possibility that f is pseudoconvex-convex is excluded.

It remains to consider the case when f is bi-pseudoconvex. Again, by Remark 2 bi-pseudoconvexity assures that the partial gradients do not vanish on the semipositive orthant. Here, all the elements of Q_1 and q are nonpositive and $q^TQ_1^{-1}q \leqslant 0$ [25]. In this case, by Corollary 2, $u^TQu \leqslant 0$ and (5.5) must hold. Equation (5.5) can be written as

$$x^TQ_1x + 2q^Txy + q^TQ_1^{-1}qy^2 \leqslant 0,$$

or equivalently

$$\left[x^T, y^T\right]\begin{bmatrix} Q_1 & q \\ q^T & q^TQ_1^{-1}q \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \leqslant 0.$$

The above matrix is conegative and the inequality holds on the semipositive orthant. Hence the bi-pseudoconvex function f is pseudo-P-convex on the semipositive orthant.

An example of a bi-pseudoconvex quadratic is given by the matrix

$$Q = \left[\begin{array}{cc:c} 0 & -1 & -1 \\ -1 & 0 & -1 \\ \hdashline -1 & -1 & -3 \end{array}\right].$$

We conclude the paper by considering a 4×4 quadratic

$$f(u) = \tfrac{1}{2}\left[x^T, y^T\right]Q\begin{bmatrix} x \\ y \end{bmatrix}$$

defined on the positive orthant of $\mathbf{R}^2 \times \mathbf{R}^2$, where

$$Q_1 = Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad Q_3 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

The Schur complement

$$(Q/Q_2) = (Q/Q_2)^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

is conegative. Since the entires of $V = [Q_1 \vdots Q_3]$ are nonnegative, the matrix $V^T(Q/Q_2)^{-1}V$ is also conegative. Moreover, only $\nabla_x f(u_1)$ vanishes on the boundary of the semipositive orthant, i.e., at the points of the form $u_1 = (x_1, y_1) = (0, 0, \eta_1, 0)$. It can easily be checked that such points satisfy (3.2) for all u_2 in the nonnegative orthant. Thus, by Corollary 5 and footnote 2, f is pseudo-P-convex on the nonnegative orthant.

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