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Difference Operators, Measuring Coalgebras, and Quantum Group-like Objects

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Quantum groups are Hopf algebras which are deformations of universal enveloping algebras. These Hopf algebras are usually defined in terms of generators and relations. This paper introduces the notion of deformed derivatives and uses the techniques of measuring coalgebras to generate objects resembling quantum groups given a set of deformed derivatives.

The appeal of this approach is that it couples the familiar and practical techniques of difference operators with the highly functorial properties of the universal measuring coalgebras. Difference operators have been used for years in the theory of differential equations, and their application to quantum groups has precedents [7]. The universal measuring coalgebra allows these operators to generate a Hopf algebra just as Lie algebras generate universal enveloping algebras. The resulting Hopf algebras resemble the quantum groups of Drinfeld and others [2–6].

The paper is organized as follows. Deformed derivatives are introduced in Section 1, and the corresponding single variable calculus is described. The analogous constructions with several variables are discussed in Section 2. Section 3 contains the necessary coalgebra theory and the central result, which specifies conditions under which a set of deformed derivatives generates a Hopf algebra which is a deformation of a given universal enveloping algebra. Section 4 contains explicit examples for the analogue of the Lie algebra of derivations on $\mathbb{C}^n$, the algebra of vector fields on a circle, and more general algebras of vector fields. Conclusions and geometric interpretations are given in Section 5.

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1. Deformed Derivations and Their Properties

For any algebra B (over $\mathbb{C}$) a *derivation* of B is a linear map $\gamma: B \to B$ with the property that

$$\gamma(ab) = \gamma(a)b + a\gamma(b) = \gamma(a)\, I(b) + I(a)\, \gamma(b),$$

where I is the identity, for all a, b in B. A deformed derivative is similarly a linear map with a coproduct where I is replaced by a more general isomorphism of B.

1.1. Definition. A general deformed derivation with respect to an algebra homomorphism $K: B \to B$ is a linear map

$$E: B \to B$$

with a product rule given by

$$E(ab) = E(a)\, I(b) + K(a)\, E(b).$$

In the example which motivates this paper, B is the algebra of polynomials in z with coefficients in the ring

$$A = \mathbb{C}[v, v^{-1}].$$

The algebra isomorphism $K: A[z] \to A[z]$ is defined by

$$K(v^r z^m) = v^{r+m} z^m.$$

An example of deformed derivatives is given by

$$E: A[z] \to A[z]$$
$$E(z^m) = [m]\, z^{m-1},$$

where $[m]$ is defined by

$$[m] = \frac{v^m - 1}{v - 1} = v^{m-1} + \cdots + 1.$$

It is evident that for $v = 1$, K is the identity and E is the usual derivative $\partial/\partial z$.

The Hopf algebras in this paper are all constructed by choosing a set of deformed derivatives of function rings tensored with A and by considering the algebra of endomorphisms thus generated. The properties of these deformed derivatives resemble those of ordinary derivations with the

Gaussian integer $[n]$ replacing the integer n throughout. The remainder of this section contains elementary results about E. Also see Exton [3] for further development of this idea.

1.2. PROPOSITION. *Let $F\colon A[z]\to A[z]$ be a deformed derivative with respect to K. Then*

$$F = F(z)E.$$

Proof. Compute $F(z^m)$ inductively.

$$\begin{aligned} F(z) &= F(z)1 = (F(z)\,E)(z) \\ F(z^m) &= (F(z))\,z^{m-1} + K(z)\,F(z^{m-1}) \\ &= F(z)\,z^{m-1} + vz(F(z)\,E(z^{m-1})) \\ &= F(z)(z^{m-1} + vz[m-1]\,z^{m-2}) \\ &= F(z)(1+v[m-1])\,z^{m-1} \\ &= F(z)[m]\,z^{m-1} = F(z)\,E(z^m), \end{aligned}$$

as desired.

If desired the deformed derivative can be expressed in limit-theoretic terms. The algebra isomorphism $K\colon A[z]\to A[z]$ is the A-linear extension of the restriction of K to $\mathbb{C}[z]$. Thus K is the map on function rings induced by the action

$$\begin{aligned} (\mathbb{C}\backslash\{0\})\times\mathbb{C} &\to \mathbb{C} \\ (v, z) &\to vz. \end{aligned}$$

The following proposition shows in what sense E may be regarded as a deformation of the usual complex derivative.

1.3. PROPOSITION. (i) *For f analytic on $\mathbb{C}$, the function Gf on $(\mathbb{C}\backslash\{0\})\times(\mathbb{C})$ defined by*

$$Gf(v, z) = \frac{f(K(v, z)) - f(z)}{K(v, z) - z}$$

is analytic on $(\mathbb{C}\backslash\{0\})\times\mathbb{C}$. Moreover

(ii) $Gf(1, z) = f(1, z)$.

(iii) $G\colon C^{\omega}(\mathbb{C})\to C^{\omega}(\mathbb{C}\backslash\{0\}\times\mathbb{C})$ *is a deformed derivative with respect to K.*

(iv) $G(z^n) = [n]\,z^{n-1}$.

Proof. (i) and (ii): By definition

$$(Gf)(v, z) = \frac{f(vz) - f(z)}{(v-1)z}.$$

Write $vz = z + (v-1)z$, so that Taylor's theorem implies

$$f(vz) = f(z) + (f'(z) + \eta(v, z))(v-1)z,$$

where $\eta()v, z)$ is analytic in both v and z, and

$$\lim_{v \to 1} \eta(v, z) = 0.$$

This establishes (i) and (ii).

(iii) Using the formula,

$$\begin{aligned} G(fg)(v, z) &= \frac{fg(vz) - fg(z)}{vz - z} \\ &= \frac{fg(vz) - f(vz)\, g(z) + f(vz)\, g(z) - f(z)\, g(z)}{vz - z} \\ &= f(vz)(Gg)(z) - g(z)\, G_f(z) \\ &= ((Gf)\, g - (Kf)(Gg))(v, z). \end{aligned}$$

(iv) Compute directly

$$(Gz^n)(v, z) = \frac{v^n z^n - z^n}{(v-z)z} = [n]\, z^{n-1}.$$

Note that the maps

$$K, G\colon C^{\omega}(\mathbb{C}) \to C^{\omega}((\mathbb{C}\setminus\{0\}) \times (\mathbb{C})$$

can be extended to maps

$$K, G\colon C^{\omega}((\mathbb{C}\setminus\{0\}) \times (\mathbb{C}) \to C^{\omega}((\mathbb{C}\setminus\{0\}) \times (\mathbb{C}))$$

via $(Kf)(v, z) = f(v, vz) = Kf_v(v, z)$ and $Gf(v, z) = Gf_v(v, z)$ where f_v in $C^{\omega}(\mathbb{C}\setminus\{0\})$ is given by $f_v(z) = f(v, z)$.

By part ii of the previous proposition then, E as defined in [1.1] coincides with the restriction of G to the subalgebra $A \otimes \mathbb{C}[z]$ in $C^{\omega}((\mathbb{C}\setminus\{0\}- \times (\mathbb{C}))$. The following proposition, which generates results familiar from calculus, applies to this more general definition of the deformed derivative.

1.4. DEFINITION. Define $K, E: C^{\omega}(\mathbb{C}\backslash\{0\}\times\mathbb{C})\to C^{\omega}(\mathbb{C}\backslash\{0\}\times\mathbb{C})$ by

$$Kf(v, z)=f(v, vz)$$

$$Ef(v, z)=\frac{Kf(v, z)-f(v, z)}{(v-1)z}.$$

1.5. PROPOSITION. *With K, E as above,*

(i) $zE=(K-1)/(v-1)$,

(ii) $EK=vKE$,

(iii) $[E, z]=K$,

(iv) $E^n fg=\sum_{i=0}^{n}\left[\begin{smallmatrix}n\\ i\end{smallmatrix}\right]K^iE^{n-i}(f)\,E^ig$,

where $\left[\begin{smallmatrix}n\\ i\end{smallmatrix}\right]$ *is defined as*

$$[n]!=[n][n-1]\cdots[1],$$

$$\begin{bmatrix}n\\ i\end{bmatrix}=\frac{[n]!}{[n-i]!\,[i]}.$$

Proof. (i) By inspection,

$$zEf(v, z)=z\cdot\frac{f(v, vz)-f(v, z)}{(v-1)z}$$
$$=\frac{Kf(v, z)-f(v, z)}{v-1}$$

as desired.

(ii) Again compute directly.

$$EKf(v, z)=\frac{Kf(v, vz)-Kf(v, z)}{(v-1)z}$$
$$=\frac{f(v, v^2z)-f(v, vz)}{(v-1)z},$$

whereas

$$KEf(v, z)=Ef(v, vz)$$
$$=\frac{f(v, v^2z)-f(v, vz)}{(v-1)\,vz}.$$

(iii) Compute

$$\begin{aligned}[E, z]\, f(v, z) &= E(zf)(v, z) - z(Ef)(v, z)\\ &= \frac{vzf(v, vz) - zf(v, z)}{(v-1)z} - z\left(\frac{f(v, vz) - f(v, z)}{(v-1)z}\right)\\ &= (v-1)z\left(\frac{f(v, vz)}{(v-1)z}\right)\\ &= Kf(v, z).\end{aligned}$$

(iv) Prove this by induction on n. The base case is part (i) of Proposition 1.3. Compute

$$\begin{aligned}E^n fg &= E^{n-1}(Efg)\\ &= E^{n-1}((Ef)\, g + (Kf)(Eg))\\ &= \sum \begin{bmatrix} n-1 \\ i \end{bmatrix} (K^i(E^{n-1}f)\, E^i g + K^i E^{n-i-1} KFE^{i+1} g)\\ &= \sum \begin{bmatrix} n-1 \\ i \end{bmatrix} [(K^i E^{n-i} f)(E^i g) + v^{n-i-1} K^{i+1} E^{n-i-1} f E^{i+1} g]\\ &= \sum \begin{bmatrix} n \\ i \end{bmatrix} K^i E^{n-i} f E^i g,\end{aligned}$$

since

$$\begin{aligned}&\begin{bmatrix} n-1 \\ i \end{bmatrix} + v^{n-i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}\\ &\quad = \frac{[nb-1]!\,[n-i] + v^{n-i}[n-1]!\,[i]}{[n-i]!\,[i]!}\\ &\quad = \frac{[n-1]!}{[n-i]!\,[i]!}\left(\frac{v^{n-i} - 1 + v^{n-i}v^i - v^{n-i}}{v-1}\right)\\ &\quad = \frac{[n]!}{[n-i]!\,[i]!} = \begin{bmatrix} n \\ i \end{bmatrix}.\end{aligned}$$

2. Deformed Derivatives in Several Dimensions

The techniques of the previous section have their analogues in higher dimensions. First it is necessary to introduce a group **K** of acceptable automorphisms. Then definition (1.4) can be extended to the case of several variables and to results obtained again which generalize those of the calculus.

Let $\mathbb{C}^{n+1}$ be denoted by coordinates $(v, z_1, \ldots, z_h) = (v, z)$. The group of *acceptable automorphisms* **K** is defined as follows.

2.1. Definition. An acceptable automorphism K is an analytic automorphism

$$K\colon \mathbb{C}^{n+1} \backslash \{(0) \times \mathbb{C}^n\} \to \mathbb{C}^{n+1}$$

such that

(i) $K(v, z) = (v, k_1(v, z) \cdots k_n(v, z))$

(ii) $K(1, z) = (1, z)$.

It is evident that the set of acceptable automorphisms form a subgroup of the group of analytic automorphisms. Examples of such a transformation are given by generalized reflections. If $(\ ,\ )\colon \mathbb{C}^n \to \mathbb{C}$ is a bilinear form, the generalized reflection with respect to an element a in $\mathbb{C}^n$ is given by

$$K_a(v, z) = (v, z - (a, z)(v-1)a).$$

For $v = -1$ this transformation coincides with the usual reflection associated with a. Note however that the set of generalized reflections does not form a subgroup. It does, however, generate a subgroup of **K**.

Given a choice of acceptable automorphism K, it remains to determine the set of deformed derivatives with respect to K. Given a complex analytic function

$$\lambda\colon \mathbb{C}^n \to \mathbb{C},$$

Definition 1.4 generalizes to the following.

2.3. Definition. $E_{K,\lambda} f(v, z) = (f(K(v, z)) - f(v, z))/(v-1)\,\lambda(z)$ for f an analytic function on $\mathbb{C}^{n+1} \backslash (\{0\} \times \mathbb{C}^n)$. Under favourable conditions on K and λ, $E_{K,\lambda} f$ is analytic and $E_{K,\lambda}$ is a deformed derivative with respect to E.

2.4. Proposition. (i) *Write $K(v, z) = (v, z) + \mu(v, z)$. If $\mu(v, z)/\lambda(z)(v-1)$ is analytic on $\mathbb{C}^{n+1} \backslash (\{0\} \times \mathbb{C}^n)$, then Ef is analytic on $\mathbb{C}^{n+1} \backslash (\{0\} \times \mathbb{C}^n)$.*

(ii) *If K, λ are as in* (i) *above, then*

$$E_{K,\lambda} fg = (E_{K,\lambda} f)\, g + (Kf)(E_{K,\lambda}\, g)$$

where $(Kf)(v, z) = f(K(v, z))$.

Proof. (i) Write $E = E_{K,\lambda}$. Clearly Ef is analytic away from the zeros of the map

$$h(v, z) = (v-1)\,\lambda(z).$$

To see that Ef is analytic at the zeros of h, it is helpful to have a particular version of Taylor's theorem.

2.5. LEMMA. *For f analytic in a neighbourhood of w_0 in $\mathbb{C}^m$, there exist analytic functions $f_i(u, w)$ on a neighbourhood of (w_0, w_0) in $\mathbb{C}^{2m}$ such that f can be written*

$$f(w) = f(u) + \partial f(u)\cdot(w-u) + \sum_{i=1}^{m} f_i(u, w)(w_i - u_i)$$

and $f_i(u, u) = 0$.

Using this lemma expand $f(K(v, z))$ writing $K(v, z) = (v, z) + \mu(v, z)$

$$f(K(v, z)) = f(v, z) + \partial f(v, z)\cdot\mu(v, z) + \sum_{i=1}^{m} f_i((v, z), K(v, z))\,\mu_i(v, z).$$

Since $\mu(v_0, z_0) = 0$ for all zeros (v_0, z_0) of h (since μ/h is analytic) δ can be chosen such that for $\|(v, z) - (v_0, z_0)\| < \delta$, $((v, z), K(v, z))$ is in the required neighbourhood of $((v_0, z_0), (v_0, z_0))$.

Now rewrite Ef as

$$\begin{aligned} Ef &= \frac{\partial f\cdot\mu + \sum f_i((v, z), K(v, z))\,\mu_i(v, z)}{h} \\ &= \partial f\cdot(\mu/h) + f_*((v, z)\,K(v, z))\cdot\mu/h. \end{aligned}$$

Thus Ef is explicitly analytic in a neighbourhood of (v_0, z_0).

Proof of Lemma 2.5. Suppose f is given by the power series

$$f(w) = \sum a_k(w - w_0)^k.$$

For u sufficiently close to w_0, $f(w)$ may equally well be expanded about u, thus

$$\begin{aligned} f(w) &= \sum a_k(w - u + u - w_0)^k \\ &= \sum_k a_k\left(\sum_{j<k}\binom{k}{j}(w-u)^j\,(u-w_0)^{k-j}\right). \end{aligned}$$

Writing $x=(w-u)$, $y=(u-w_0)$, the power series

$$g(x, y)=\sum_{\underline{k}} \sum_{\underline{j}<\underline{k}} \binom{\underline{k}}{\underline{j}} a_{\underline{k}} x^{\underline{j}} k^{\underline{k}-\underline{j}}$$

is convergent since $g(x, y)=f(x+y+w_0)$, and hence absolutely convergent on some neighbourhood of zero in $\mathbb{C}^{2m}$. Now write g as a sum of subseries as follows. Define

$$g_i=\sum_{\underline{k}} \sum_{\substack{j_l=0, l<i \\ j_i \geq 1}} \binom{\underline{k}}{\underline{j}} a_{\underline{k}}\, y^{\underline{k}-\underline{j}} x^{(\underline{j}-1_i)},$$

where $\underline{j}-1_i$ is the multi-index $(j_1, \ldots, j_i-1, \ldots, j_m)$; then

$$\begin{aligned} f(w)=g(x, y) &= g(0, y)+\sum_{i=1}^{m} g_i(x, y) x_i \\ &= g(0, y)+\sum_{i=1}^{m} g_i(0, y) x_i+\sum_{i=1}^{m}(g_i(x, y)-g_i(0, y)) x_i . \end{aligned}$$

But

$$\begin{aligned} g(0, y) &= f(y+w_0)=f(u), \\ g_i(0, y) &= \sum_{\underline{k}} k_i a_{\underline{k}}\, y^{\underline{k}-1_i}=\partial_i f(u). \end{aligned}$$

Defining

$$f_i(u, w)=g_i(w-u, u-w_0)-g_i(0, u-w_0)$$

completes the proof.

(ii) Now let f, g be in $C^{\omega}(\mathbb{C}^{n+1} \backslash(\{0\} \times \mathbb{C}^n))$, and write $E=E_{K, \lambda}$. Then

$$\begin{aligned} Efg(v, z) &= \frac{fgK(v, z)-fg(v, z)}{\lambda(z)(v-1)} \\ &= \frac{1}{\lambda(z)(v-1)} \cdot(f(K(v, z))\, g(K(v, z))-f(K(v, z))\, g(v, z)) \\ &\quad+\frac{1}{\lambda(z)(v-1)}(f(K(v, z))\, g(v, z)-f(v, z)\, g(v, z)) \\ &= f(K(v, z))\, Eg(v, z)+Ef(v, z)\, g(v, z), \end{aligned}$$

as desired.

Examples of acceptable automorphisms K for which associated deformed derivatives E exist are provided by the generalized reflections. In such a case, let $\lambda(z)$ be the inner product with a, $\lambda(z)=(z, a)$, so that if

$$K(v, z)=z+(v-1)(z, a)a$$

and

$$\frac{\mu(v, z)}{(v-1)\,\lambda(v, z)}=\frac{(v-1)(z, a)a}{(z, a)(v-1)}=a,$$

which is clearly analytic.

Unless otherwise specified all deformed derivatives are of the form $f(v, z)E$, where E is the deformed derivative corresponding to a generalized reflection. The following proposition extends the usual identities from calculus.

2.5. PROPOSITION. *Let $(\ ,\ )$ be a bilinear form on $\mathbb{C}^n$ under which the basis elements $e_1, ..., e_n$ are orthonormal, so that $z=\sum z_ie_i$ satisfies $(z, e_i)=z_i$. Define*

$$K_i(v, z)=(v, z+(v-1)\,z_ie_i)$$

$$K_i^{-1}(v, z)=(v, z+(v^{-1}-1)\,z_ie_i)$$

$$E_i(v, z)=\frac{K_if-f}{(v-1)z_i}$$

$$F_i(v, z)=z_i\left(\frac{K_i^{-1}f-1}{(v^{-1}-1)}\right),$$

where $K_if(v, z)=f(K_i(v, z))$ and similarly $K_i^{-1}f(v, z))$. Then the following hold.

(i) *F_i is a deformed derivative with respect to K^{-1}, and $KK^{-1}=K^{-1}K$ is the identity.*

(ii) *For any f, g in $C^\omega(\mathbb{C}^{n+1}\backslash\{0\}\times\mathbb{C}^n)$, fE_i, gF_i are deformed derivatives with respect to K_i, K_i^{-1} respectively.*

(iii) $K_iK_j=K_jK_i$.

(iv)

$$E_iz_1^{l_1}\cdots z_n^{l_n}=[l_i]\,z_1^{l_1}\cdots z_i^{l_i-1}\cdots z_n^{l_n}$$

$$F_jz_1^{l_1}\cdots z_n^{l_n}=[-l_i]\,z_1^{l_1}\cdots z_j^{l_j+1}\cdots z_n^{l_n}.$$

(v) $K_jz_i=v^{\delta_{ij}}z_iK_j$, $K_j^{-1}z_i=v^{-\delta_{ij}}z_iK_j^{-1}$.

(vi) $[E_i, E_j]=0,\ [F_i, F_j]=0.$

(vii) $z_i E_i=(K_i-1)/(v-1).$

(viii) *For* $\varepsilon=\pm 1$,

$$E_i K_j^{\varepsilon}=v^{-\delta_{ij}\varepsilon} K_j^{\varepsilon} E_i$$
$$F_i K_j^{\varepsilon}=v^{-\delta_{ij}\varepsilon} K_j^{\varepsilon} F_i.$$

(ix) $[E_i, z_j]=\delta_{ij} K_i.$

(x) $[z_i E_j, z_r E_s]=v^{\delta_{ij}}\delta_{jr} K_j z_i E_s - v^{\delta_{rs}}\delta_{is} K_i z_r E_j.$

(xi) $[E_i, F_i]=(K_i^{-1}-K_i)/(v^{-1}-1).$

Proofs. (i) It is not hard to verify that F_i is of the form required by the hypothesis of the last proposition and is thus a deformed derivative with respect to K_i^{-1}. Direct calculation verifies that $K_i^{-1}K_i$ is the identity.

(ii) This can be verified directly.

(iii) This is a consequence of the orthogonality of e_i and is demonstrated by direct calculation.

(iv) By direct calculation

$$\begin{aligned} F_i(z_1^{l_1}\cdots z_n^{l_n}) &= z_i\left(\frac{v^{-l_i}z_1^{l_1}\cdots z_n^{l_n}-z_1^{l_1}\cdots z_n^{l_n}}{(v-1)}\right)\\ &= z_i\left(\frac{v^{-l_i}-1}{v-1}\right) z_1^{l_1}\cdots z_n^{l_n}\\ &= [-l_i]\, z_1^{l_1}\cdots z_i^{l_i+1}\cdots z_n^{l_n}. \end{aligned}$$

The other identity is similar.

(v) Since K_j and K_j^{-1} are algebra maps, these identities are easily verified.

(vi) Write

$$G_i f(v,z)=K_i f(v,z)-f(v,z)$$
$$h_i(v,z)=\frac{1}{(v-1)z_i}.$$

Now compute, for $i\neq j$,

$$\begin{aligned} E_iE_j f &= E_i(h_j G_j f)\\ &= (E_i h_j)\, G_j f + (K_i h_j)\, E_i G_j f. \end{aligned}$$

But $K_i h_j = v^{-\delta_{ij}} h_j$, so that $E_i h_j = \delta_{ij}(1 - v^{-1}) h_i h_j = 0$. Thus

$$\begin{aligned} E_i E_j f &= h_j h_i G_i G_j f \\ &= h_j h_i (K_i K_j f - K_i f - K_j f + f). \end{aligned}$$

Since $K_i K_j = K_j K_i$ (iii), it is now clear that the commutator will vanish. The analogous identity $[F_i, F_j] = 0$ is established in the same way.

(vii) This is immediate from the definition.

(viii) Compare

$$\begin{aligned} E_i K_j^{\varepsilon} f &= h_i (K_i K_j^{\varepsilon} f - K_j^{\varepsilon} f) \\ &= h_i K^{\varepsilon} (K_i f - f), \\ K_j^{\varepsilon} E_i f &= K_j^{\varepsilon} h_i K_j^{\varepsilon} (K_i f - f). \end{aligned}$$

The result follows since $K_j^{\varepsilon} h_i = v^{-\delta_{ij}\varepsilon} h_i$. Similarly, write

$$F_i f = -v h_i (K^{-1} f - f).$$

Then

$$\begin{aligned} F_i K_j^{\varepsilon} &= -v h_i K_j^{\varepsilon} (K_i^{-1} f - f) \\ K_j^{\varepsilon} F_i &= -v^{1-\delta_{ij}\varepsilon} h_i K_j^{\varepsilon} (K_i^{-1} f - f). \end{aligned}$$

(ix) Compute directly

$$\begin{aligned} [E_i, z_j] &= E_i z_j f - z_j E_i f \\ &= \delta_{ij} f + v^{\delta_{ij}} z_j E_i f - z_j E_i f \\ &= \delta_{ij} f + \delta_{ij}(v - 1) z_j E_i f \\ &= \delta_{ij} f + \delta_{ij}(K_i - 1) f \end{aligned}$$

by vii. Thus $[E_i, z_j] = \delta_{ij} K_i$ as desired.

(x) Compute

$$\begin{aligned} z_i E_j z_r E_s f &= z_i \delta_{jr} E_s f + z_i k_j z_r E_j E_s f \\ &= \delta_{jr} z_i E_s f + v^{\delta_{jr}} z_i z_r E_j E_s f. \end{aligned}$$

The commutator then becomes

$$\begin{aligned} [z_i E_j, z_r E_s] f &= \delta_{jr} z_i E_s f + v^{\delta_{jr}} z_i z_r E_j E_s f - \delta_{is} z_r E_j f \\ &\quad - v^{\delta_{is}} z_r z_i E_s E_j f - z_i z_r E_j E_s f + z_i z_r E_j E_s f \\ &= \delta_{jr}(z_i E_s f + (v - 1) z_i z_r E_j E_s f) \\ &\quad - \delta_{is}(z_r E_j f + (v - 1) z_r z_i E_s E_j f) \end{aligned}$$

using the fact that $E_s E_j = E_j E_s$. Using identity vii, this simplifies to

$$\begin{aligned}[z_i E_j, z_r E_s] f &= \delta_{jr}(z_i E_s f + z_i(K_j - 1) E_s f) \\ &\quad - \delta_{is}(z_r E_j f + z_r(K_i - 1) E_j f) \\ &= \delta_{jr} v^{\delta_{ij}} K_j z_i E_s f - \delta_{is} v^{\delta_{rs}} z_r E_j f\end{aligned}$$

as desired.

(xi) Compute directly suppressing the subscript i,

$$\begin{aligned}EFf &= E\left(\frac{z_i K^{-1} f - z_i f}{(v^{-1} - 1)}\right) \\ &= \frac{-v}{(v-1)} E(z_i K^{-1} f - z_i f) \\ &= \frac{-v}{(v-1)^2} (vf - vKf - K^{-1} f + f) \\ FEf &= F\left(\frac{Kf - f}{(v-1) z_i}\right) \\ &= z_i \frac{1}{(v-1)} \frac{1}{(v^{-1} - 1)} \left(\frac{f - K^{-1} f}{v^{-2} z_i} - \frac{Kf - f}{z_i}\right) \\ &= \frac{-v}{(v-1)^2} (vf - vK^{-1} - Kf + f).\end{aligned}$$

The commutator is then

$$\begin{aligned}[E, F] f &= -\frac{v}{(v-1)^2} (-vKf - K^{-1} f + vK^{-1} f + Kf) \\ &= \frac{v}{(v-1)} (Kf - K^{-1} f) \\ &= \left(\frac{K^{-1} - K}{v^{-1} - 1}\right) f.\end{aligned}$$

3. Measuring Coalgebras

Measuring coalgebras supply the means of generating Hopf algebras from a set of deformed derivatives. The idea is that given algebras B_1, B_2 there exists a coalgebra P and a linear map $P \to \mathrm{Hom}(B_1 B_2)$ such that multiplication in B_1 and B_2 is compatible with the comultiplication in P.

The coalgebra P is highly functorial, and, in the case $B_1 = B_2$, P has a natural bialgebra structure. All the Hopf algebras discussed in this paper will be subalgebras of such bialgebras.

3.1. Definitions. Let B_1, B_2 be algebras over $A = \mathbb{C}[v, v^{-1}]$. Let C be a coalgebra (over A) and suppose $\psi: C \to \mathrm{Hom}(B_1, B_2)$ is a linear map such that

(i) $\psi(c)(ab) = \sum_{(c)} (\psi c_{(1)}(a))(\psi c_{(2)}(b))$

(ii) $\psi(c)(1) = \varepsilon(c)$,

where $\Delta c = \sum_{(c)} c_{(1)} \otimes c_{(2)}$ and ε is the counit in C.

Thus comultiplication in C determines a "product rule" for elements in ψC. Such a pair (C, ψ) is called a measuring coalgebra (with respect to B_1, B_2). A measuring coalgebra (P, π) is a *universal* measuring coalgebra if, for any other measuring coalgebra (C, ψ) there exists a unique coalgebra map $\rho: C \to P$ which makes the following diagram commute.

$$\begin{tikzcd} P \arrow[r, "\pi"] & \mathrm{Hom}(B_1, B_2) \\ C \arrow[u, "\rho"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

The following theorem summarizes results about measuring coalgebras.

3.2. Theorem. (i) *The universal measuring coalgebra $P = P(B_1, B_2)$ exists and is unique.*

(ii) *$P(B, B)$ is a bialgebra.*

Proof. (i) This was proved in Sweedler [7] in the case in which A is a field. The validity holds over general rings on categorical grounds.

Define a map $f(C_1, \psi) \to (C_2, \psi_2)$ of measuring coalgebras to be a coalgebra map $f: C_1 \to C_2$ for which the diagram

$$\begin{tikzcd} C_1 \arrow[r, "\psi_1"] \arrow[d, "f"'] & \mathrm{Hom}_A(B_1, B_2) \\ C_2 \arrow[ur, "\psi_2"'] & \end{tikzcd}$$

commutes. Let $\mathcal{M}(B_1, B_2)$ be the category of measuring coalgebras. The following lemma holds.

3.3. LEMMA. (i) *Coproducts exist in* $\mathcal{M}(B_1, B_2)$.

(ii) *Coequalizers exist in* $\mathcal{M}(B_1, B_2)$.

Given this lemma, the universal measuring coalgebra can be expressed in terms of the following coequalizer. Let $\{C_\lambda, \psi_\lambda\}_{\lambda = \lambda}$ be a set of measuring coalgebras, with one coalgebra from each isomorphism class of finite-dimensional measuring coalgebras, and let $\{f_{\mu,\nu}: (C_\mu, \psi_\mu) \to (C_\nu, \psi_\nu)\}$ be the set of morphisms between the C_λ. The universal measuring coalgebra will be the coequalizer of the diagram

$$\coprod_{\{f_{\mu,\nu}\}} (C_\mu, \psi_\mu) \underset{j}{\overset{i}{\rightrightarrows}} \coprod_{\Lambda} (C_\lambda, \psi_\lambda) \to P,$$

where i is given by the inclusion $(C_\mu, \psi_\mu) \to \coprod_\Lambda (C_\lambda, \psi_\lambda)$ and j is given by the maps

$$(C_\mu, \psi_\mu) \xrightarrow{f_{\mu,\nu}} (C_\nu, \psi_\nu) \to \coprod_{\Lambda} (C_\lambda, \psi_\lambda).$$

The universal properties of P can be readily verified. Since any coalgebra is the sum of its finite-dimensional subcoalgebras any measuring coalgebra $(C_\lambda, \psi_\lambda)$ includes in the coproduct $\coprod_\Lambda (C_\lambda, \psi_\lambda)$ and thus maps to P. The uniqueness of that map is guaranteed by the coequalizer property. I am indebted to Thomas Schmitt, who pointed out the need for restricting Λ to the set of isomorphism classes of finite-dimensional coalgebras.

Proof of Lemma 3.3. (i) Given any set $\{(C_\lambda, \psi_\lambda)\}_{\lambda \in \Lambda}$ form the coproduct $\coprod_\Lambda C_\lambda$ as A-modules. Then check that the maps

$$\Delta = \coprod \Delta_\lambda : \coprod_\Lambda C_\lambda \to \left(\coprod_\Lambda C_\lambda\right) \otimes \left(\coprod_\Lambda C_\lambda\right)$$

$$\varepsilon = \coprod \varepsilon_\lambda : \coprod_\Lambda C_\lambda \to A$$

$$\psi = \coprod \psi_\lambda : \coprod_\Lambda C_\lambda \to \mathrm{Hom}(B_1, B_2)$$

give $\coprod_\Lambda C_\lambda$ the structure of a measuring coalgebra.

(ii) Consider a pair of maps of measuring coalgebras

$$(C_1, \psi_1) \underset{g}{\overset{f}{\rightrightarrows}} (C_2, \psi_2),$$

and define

$$J = \{f(c) - g(c) : c \in C_1\}.$$

Let $C_3 = C_2/J$. Initially C_3 is only an A module, but it is not hard to verify that J is a coideal, since

$$\begin{aligned}\Delta(f(c)-g(c)) &= \sum_{(c)} f(c_{(1)})\otimes f(c_{(2)}) - g(c_{(1)})\otimes g(c_{(2)})\\ &\quad + \sum_{(c)} -f(c_{(1)})\otimes g(c_{(2)}) + f(c_{(1)})\otimes g(c_{(2)})\\ &= \sum_{(c)} f(c_{(1)})\otimes (f(c_{(2)}) - g(c_{(2)}))\\ &\quad + \sum_{(c)} (f(c_{(1)}) - g(c_{(1)}))\otimes g(c_{(2)}).\end{aligned}$$

Moreover, the map $\psi_2\colon C_2 \to \mathrm{Hom}(B_1, B_2)$ is zero on J since

$$\psi_2(f(c)-g(c)) = \psi_2 f(c) - \psi_2 g(c) = \psi_1(c) - \psi_1(c).$$

Thus (C_3, ψ_2) is a measuring coalgebra. That (C_3, ψ_2) has the property required of coequalizers holds since C_3 is evidently a coequalizer in the category of A-modules.

(iii) Composition provides a map

$$P(B, B)\otimes P(B, B) \xrightarrow{\hat{\mu}} \mathrm{Hom}(B, B)$$

which is clearly A-linear. That $\hat{\mu}$ measures can be verified directly, so that there is a unique coalgebra map

$$\mu\colon P(B, B)\otimes P(B, B) \to P(B, B).$$

Since μ is a coalgebra map, $P(B, B)$ is a bialgebra.

Examples of bialgebras can thus be constructed by finding subcoalgebras C of P and considering the algebra they generate. The following theorem gives conditions under which the resulting bialgebra is a Hopf algebra.

3.4. THEOREM. *Let (C, π) be a measuring coalgebra for B.*

(i) *The algebra H generated by C is a bialgebra.*

(ii) *Suppose s is a map of A-modules*

$$s\colon C \to H$$

such that

$$\sum s(c_{(1)})\, c_{(2)} = \sum c_{(1)} s(c_{(2)}) = \varepsilon(c).$$

Then s extends to an antipode

$$s\colon H \to H,$$

making H into a Hopf algebra.

Proof. (i) Since the multiplication μ is a coalgebra map, for c_1, c_2 in C

$$\Delta(c_1 c_2) = (\Delta c_1)(\Delta c_2) \subset C \cdot C \otimes C \cdot C.$$

Hence $\Delta H \subset H \otimes H$.

(ii) Suppose $s\colon C \to H$ has the properties stated. The program is to extend s to H via the following steps.

1. Endow the tensor algebra TC with the structure of a bialgebra such that the map $p\colon TC \to H$ is a bialgebra map.

2. Extend s to an algebra anti-automorphism.

$$s\colon TC \to H$$

and observe that the kernel J of s is an ideal and a coideal on which ε_{TC} vanishes.

3. Show that J is the kernel of the map $p\colon TC \to H$. Thus

$$s\colon TC/J = H \to H$$

is the antipode for H.

To put a coalgebra structure on TC observe that

$$C \to C \otimes C \hookrightarrow TC \otimes TC$$

is an A linear map, which extends uniquely to an algebra homomorphism

$$TC \to TC \otimes TC.$$

Since H is generated as an algebra by C there are unique algebra homomorphisms

$$p\colon TC \to H,$$

$$TC \to H \otimes H.$$

By uniqueness of the last map, the following square must commute

$$\begin{array}{ccc} TC & \longrightarrow & TC \otimes TC \\ \downarrow & & \downarrow \\ HJ & \longrightarrow & H \otimes H \end{array}$$

so that the map $TC \to H$ is a map of bialgebras.

Now let H^{op} be H with the opposite multiplication. The universal property of tensor algebras guarantees that s extends to an algebra homomorphism

$$s: TC \to H^{\mathrm{op}}.$$

But this is exactly an anti-automorphism s: $TYC \to H$. Moreover, s has the antipodal property for all w in TC,

$$\sum_{(w)} s(w)_1 \, \bar{w}_{(2)} = \varepsilon(w),$$

where $\bar{w}_{(2)}$ indicates the image of $w_{(2)}$ in H. This holds since, for c, c' in C,

$$\begin{aligned} \sum_{cc'} s(cc'_{(1)})(cc')_{(2)} &= \sum_{(c),(c')} s(c'_{(1)})\, s(c_{(1)})\, c_{(2)}c'_{(2)} \\ &= \sum s(c'_{(1)})\, \varepsilon(c)\, c'_{(2)} \\ &= \varepsilon(c)\, \varepsilon(c') = \varepsilon(cc'). \end{aligned}$$

Clearly $\ker s = J$ is an ideal. Less obviously it is a coideal: explicitly,

$$\Delta sw = \sum s(w_{(2)}) \otimes s(w_{(1)}) \tag{$*$}$$

so that if $sw = 0$, $\sum s(w_{(2)}) \otimes s(w_{(1)}) = 0$, or $\Delta w \in TC \otimes J + J \otimes TC$. To see this, define maps

$$\begin{aligned} &K, P, N \in \mathrm{Hom}_A(TC, H \otimes H) \\ &K(w) = \sum_{(w)} \bar{w}_{(1)} \otimes \bar{w}_{(2)} = \Delta \bar{w} \\ &P(w) = \sum_{(w)} s(w_{(2)}) \otimes s(w_{(1)}) \\ &N(w) = \Delta sw. \end{aligned}$$

The identity $(*)$ is the statement that $P(w) = N(w)$. The program (due to Sweedler, p. 74, Proposition 4.0.1.(4)) is to show that $K * P = N * K = 1$

where $*$ is the convolution in $\operatorname{Hom}(TC, H\otimes H)$, and $1 = u\varepsilon$ is the unit for the algebra $\operatorname{Hom}(TC, H\otimes H)$. Then $N \mathscr{V} K * P = P = N$ as desired. To establish this

$$\begin{aligned} K * P(w) &= \sum_{(w)} K(w_{(1)})\, P(w_{(2)}) \\ &= \sum (\bar{w}_{(1)} \otimes \bar{w}_{(2)})(sw_{(4)} \otimes sw_{(3)}) \\ &= \sum \bar{w}_{(1)} s(w_{(4)}) \otimes \bar{w}_{(2)} sw_{(3)} \\ &= \sum \bar{w}_{(1)} s(w_{(3)})\, \varepsilon(\bar{w}_{(2)}) \\ &= \sum \bar{w}_{(1)} s(w_{(2)}) \\ &= \varepsilon(\bar{w}) \end{aligned}$$

as desired, and

$$\begin{aligned} N * K(w) &= \sum_{(w)} (\Delta s(w_{(1)}))(\Delta \bar{w}_{(2)}) \\ &= \sum \Delta(s(w_{(1)})\, \bar{w}_{(2)}) \\ &= \Delta\varepsilon(w) = \varepsilon(w) 1 \qquad \text{in} \quad H\otimes H. \end{aligned}$$

This establishes that J is an ideal and a coideal, so that $s\colon TC/J \to H$ is injective. Moreover,

$$\varepsilon_{TC} J = 0$$

since for w in J

$$\begin{aligned} \varepsilon_{TC} w &= \varepsilon_H u_H \varepsilon_{TC} w \\ &= \varepsilon_H \sum \bar{w}_{(1)} sw_{(2)} \\ &= \sum \varepsilon_H \bar{w}_{(1)} \varepsilon_H s(w_{(2)}) \\ &= \varepsilon_H s\left(\sum \varepsilon_{TC}(w_{(1)})\, w_{(2)} \right) \\ &= \varepsilon_H s(w) = 0. \end{aligned}$$

It remains to show that the kernel K of the projection $p: TC \to H$ is exactly K. Observe that $p * s = s * p$ is the unit in $\mathrm{Hom}(TC, H)$. Thus $s * p * s = s$, so that for w in J

$$\Delta^2 w \varepsilon J \otimes TC \otimes TC + TC \otimes K \otimes TC + TC \otimes TC \otimes J.$$

But, since J is a coideal

$$\Delta^2 w \varepsilon J \otimes TC \otimes TC + TC \otimes J \otimes TC + TC \otimes TC \otimes J.$$

Thus

$$\Delta^2 w \varepsilon J \otimes TC \otimes TC + TC \otimes (J \cap K) \otimes TC + TC \otimes TC \otimes J$$

so that applying $\varepsilon \otimes 1 \otimes \varepsilon$ to $\Delta^2 w$,

$$w = \varepsilon \otimes 1 \otimes \varepsilon \Delta^2 w \in J \cap K$$

since $\varepsilon J = 0$. Thus $J \subset K$.

Similarly, using the identity $p * s * p = p$, $K \subset J$. This establishes

$$s: TC/J = TC/K = H \to H$$

as an antipode.

The classic example is the construction of a Hopf algebra from a set of vector fields.

3.5. THEOREM. *Let S be a vector space of derivations of a $\mathbb{C}$-algebra $\bar{B}$. Then $S \oplus \mathbb{C}1$ can be given the structure of a coalgebra such that the map*

$$\psi: S \oplus \mathbb{C}1 \to \mathrm{Hom}_{\mathbb{C}}(\bar{B}, \bar{B}),$$

which includes S and sends 1 to the identity measures. Moreover, the bialgebra H generated by $S \oplus \mathbb{C}1$ in $P(\bar{B}, \bar{B})$ is isomorphic to the universal enveloping algebra of the Lie algebra of derivations generated by S.

Proof. Define a coalgebra structure on $S \oplus \mathbb{C}1$ by setting

$$\Delta 1 = 1 \otimes 1, \qquad \Delta X = X \otimes 1 + 1 \otimes X$$

$$\varepsilon 1 = 1, \qquad \varepsilon X = 0$$

for X in S. Evidently ψ measures, and $S \oplus \mathbb{C}1$ generates a bialgebra H.

Let L be the subspace of H generated by the commutators of elements of S. Observe that for X, Y in S, $[X, Y]$ is again a primitive element of H, and thus maps to a derivation of $\bar{B}$. By the universal property of enveloping algebras there is a unique algebra homomorphism θ which makes the diagram

$$\begin{tikzcd} L \arrow[r] & U(L) \arrow[d, "\theta"] \\ & H \end{tikzcd}$$

commute. Since θ maps onto the generating set $S \oplus \mathbb{C}1$ of H, θ must be surjective. Since $U(L)$, H are bialgebras, and θ restricted to $L \oplus \mathbb{C}1 \subset U(L)$ is a coalgebra map, θ is a bialgebra map. In particular, θ is a coalgebra map. Using Corollary 11.02.2 of Sweedler (p. 218), since $\ker \theta \cap L = 0$, θ itself must be injective.

Thus the measuring coalgebra provides a method of recovering the universal enveloping algebras. But their great virtue in this application is that they apply to generating subcoalgebras which may be more general than sets of derivations. In particular, sets of deformed derivatives can equally be used to generated Hopf algebras. These are the Hopf algebras which are reminiscent of quantum groups. The last theorem of this section describes the sense in which the algebras generated by deformed derivatives are deformations of universal enveloping algebras.

Consider $A = \mathbb{C}[v, v^{-1}]$. This is an augmented algebra with augmentation

$$\lambda: A \to \mathbb{C} \qquad \lambda(v) = 1,$$

where $\ker \lambda$ is the ideal J generated by $(v-1)$. Any A module C can be projected onto a $\mathbb{C}$-module

$$\lambda: C \to \bar{C} = C/JC.$$

Thus we have maps

$$\mathrm{Hom}_A(B, B) \to \mathrm{Hom}_{\mathbb{C}}(\bar{B}, \bar{B})$$
$$C \to \bar{C}.$$

for any A coalgebra C which makes the diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathrm{Hom}_A(B, B) \arrow[d] \\ \bar{C} \arrow[r] & \mathrm{Hom}_{\mathbb{C}}(\bar{B}, \bar{B}) \end{tikzcd}$$

commute. Moreover, $\bar{C}$ is a $\mathbb{C}$ coalgebra and the map $\bar{\pi}$ measures. By the universal property there is a unique map from $\bar{C}$ to the universal measuring coalgebra for $(\bar{B}, \bar{B})$.

3.6. THEOREM. *Let B be an algebra over $A = \mathbb{C}[v, v^{-1}]$ with $\bar{B} = B/JB$. Let C be a measuring coalgebra for (B, B) and suppose that the image of $\bar{C}$ in* $\mathrm{Hom}(\bar{B}, \bar{B})$ *maps to $S \oplus \mathbb{C}1$, where S is a set of derivations of $\bar{B}$, as in Theorem* 3.5. *Let H be the bialgebra generated by C.*

(i) *There is a map of bialgebras over $\mathbb{C}$,*

$$\tau\colon \bar{H} \to U(L).$$

(ii) *If the map $\bar{C} \to S \oplus \mathbb{C}1$ is surjective (injective), so is τ.*

Proof. (i) Let K denote the subalgebra of $P(\bar{B}, \bar{B})$ generated by $\bar{C}$. Evidently K is a subalgebra of $U(L)$.

Multiplication $H \otimes H \to H$ determines a multiplication $\bar{H} \otimes \bar{H} \to \bar{H}$. By uniqueness of the universal measuring map, the diagram

$$\begin{tikzcd}
\bar{H} \otimes \bar{H} \arrow[r] \arrow[d] & \bar{H} \arrow[d] \\
P(\bar{B}, \bar{B}) \otimes P(\bar{B}, \bar{B}) \arrow[r] & P(\bar{B}, \bar{B})
\end{tikzcd}$$

must commute.

Moreover, the square

$$\begin{tikzcd}
H \otimes_A H \arrow[r] \arrow[d] & H \arrow[d] \\
\bar{H} \otimes \bar{H} \arrow[r] & \bar{H}
\end{tikzcd}$$

also commutes. Then since C generates H, $\bar{C}$ generates $\bar{H}$, and $\bar{H} = K$. This map τ is then the map

$$\tau\colon \bar{H} \to K \subset P(\bar{B}, \bar{B}).$$

Since $\bar{C} \to S \oplus \mathbb{C}1$, the image of τ is contained in $U(L)$.

(ii) Evidently, if $\bar{C} \to S \oplus \mathbb{C}1$ is surjective, $\tau\bar{H} = U(L)$. If $\bar{C}$ includes as a subcoalgebra of $S \oplus \mathbb{C}1$, the image of $\bar{C}$ will be of the form $S' \oplus \mathbb{C}1$, where S' is a subspace of S. Let L' be the Lie subalgebra of L generated by L'. Then by 3.5 $\bar{H} = U(L')$ and the inclusion $L' \to L$ induces an inclusion $U(L') \to U(L)$.

4. Examples

4.1. *The Algebra of Deformed Derivatives of* $C^\omega(\mathbb{C}^n)$

In this case let B be the algebra $C^\omega(\mathbb{C}^{n+1}\backslash(\{0\}\times\mathbb{C}^n))$. This is evidently a module over $A=\mathbb{C}[v, v^{-1}]$, under the action

$$fg(v, z_1, ..., z_n)=f(v)\, g(v, z_1, ..., z_n)$$

for f in $\mathbb{C}[v, v^{-1}]$, g in B.

Let $\mathbf{K}$ be the group of acceptable automorphisms as defined in 2.1, and for K in $\mathbf{K}$, let $\mathbf{E}_K$ be the set of deformed derivatives with respect to K. Set

$$\mathbf{E}=\oplus_{K\in\mathbf{K}}\mathbf{E}_K$$

Note that $\mathbf{E}_K$, and hence $\mathbf{E}$, are A-modules.

The A-module $\mathbf{E}\oplus AK$ has an obvious A-coalgebra structure

$$\Delta K=K\otimes K \qquad \varepsilon(K)=1$$

$$\Delta E=E\otimes 1+K\otimes E, \qquad \varepsilon(E)=0 \text{ for } E \text{ in } \mathbf{E}_K$$

Moreover, the obvious map

$$p\colon \mathbf{E}\oplus A\mathbf{K}\to \mathrm{Hom}_A(B, B)$$

measures, and by Theorem 3.6, $\mathbf{E}\oplus AK$ generates a bialgebra H. The map

$$s\colon \mathbf{E}\oplus A\mathbf{K}\to H$$

$$sK=K^{-1}, \qquad sE=-K^{-1}E$$

provides a suitable antipode, so that H becomes a Hopf algebra.

The image of $\mathbf{E}\oplus A\mathbf{K}$ in $\mathrm{Hom}(\bar{B}, \bar{B})$ can be determined by setting $v=1$ throughout. By definition, alle elements K reduce to the identity. Any element of $\mathbf{E}$ reduces to a derivation of $\bar{B}=C^\omega(\mathbb{C}^n)$. Moreover, since any element of Der $\bar{B}$ can be written as a sum

$$\bar{v}=\sum_{i=1}^{n} f_i\frac{\partial}{\partial z_i}$$

the element

$$v=\sum_{i=1}^{n}\hat{f}_iE_i$$

where $\hat{f}_i(v, z_1, ..., z_n) = f_i(z_1, ..., z_n)$ is an element of $\mathbf{E}$ which reduces to $\bar{v}$ in Der $\bar{B}$.

Thus H is a deformation of $U(\text{Der } C^{\omega}(\mathbb{C}^n))$.

A sub-Hopf algebra U of H which generalizes $U(gl(n))$ is given by setting

$$\mathbf{K}' = \{K_i, 1\}$$

and letting $\mathbf{E}'$ be the A module generated by $\{z_j E_i\}$ where K_i, E_i are as defined in 2.5. The subcoalgebra $\mathbf{E}' \oplus A\mathbf{K}'$ of $\mathbf{E} \oplus AK$ generates a Hopf algebra which is a deformation of $gl(n)$.

4.2. *Vector Fields on a Circle and* $sl(2)$

In this case let $B = \mathbb{C}[v, v^{-1}, z, z^{-1}]$. Let $\mathbf{K}$ be the set of acceptable automorphisms K^j determined by

$$K^j z^n = v^{nj} z^n.$$

Note that the deformed derivative $E: A[z] \to A[z]$ described in the example following Definition 1.1 extends to a deformed derivative

$$E: A[z, z^{-1}] \to A[z, z^{-1}], \quad Ez^m = [m]\, z^{m-1}$$

for *all* m in $\mathbb{Z}$. Let $\mathbf{E}$ be the set of deformed derivatives

$$\mathbf{E} = \{z^{m+1} E\}.$$

Note that as elements of $\text{Hom}_A(B, B)$,

$$zE = \frac{K-1}{v-1}$$

(Proposition 1.5(i)). The comultiplication

$$\Delta z^{m+1} E = z^{m+1} E \otimes 1 + K \otimes z^{m+1}$$
$$\Delta K^i = K^i \otimes K^i$$

and augmentation

$$\varepsilon z^{m+1} E = 0, \qquad \varepsilon K^i = 1$$

provide $\mathbf{E} \oplus A\mathbf{K}$ with a coalgebra structure for which $\mathbf{E} \oplus A\mathbf{K}$ becomes a measuring coalgebra under the obvious map $\mathbf{E} \oplus A\mathbf{K} \to \text{Hom}_A(B, B)$.

Again let H denote the bialgebra generated by the image of $\mathbf{E} \oplus A\mathbf{K}$ in the measuring coalgebra. The map

$$s\colon \mathbf{E} \oplus A\mathbf{K} \to H$$
$$s(z^{m+1}E) = -K^{-1}(z^{m+1}E)$$
$$s(K^i) = K^{-i}$$

generates an antipode, giving H the structure of a Hopf algebra.

Once again it is not hard to compute the image of $\mathbf{E} \oplus A\mathbf{K}$ in $\mathrm{Hom}(\bar{B}, \bar{B})$. Again all acceptable automorphisms reduce to the identity, and the deformed derivatives reduce to the ordinary derivations $z^{m+1}(\partial/\partial z)$ of $\bar{B} = \mathbb{C}[z, z^{-1}]$. Thus H is a deformation of $U(V)$ where V is the Lie algebra of vector fields on the circle.

The Hopf algebra H contains the sub-Hopf algebra H' generated by the A-module spanned by $\{E, zE, z^2E, K^i\}$. This is evidently a subcoalgebra of $\mathbf{E} \oplus A\mathbf{K}$, which reduces to the Lie algebra of $sl(2)$ acting on $\mathbb{C}[z, z^{-2}]$ via $\partial/\partial z$, $z(\partial/\partial z)$, and $z^2(\partial/\partial z)$. Thus H' is a deformation of $U(sl(2))$.

4.3. *Lie Algebras of Analytic Vector Fields*

By exponentiating a vector field X it is possible to construct a deformed derivative corresponding to X. In this way the algebra of deformed derivatives corresponding to a set of fector fields on $\mathbb{R}^n$ can be constructed.

In this case consider $A = \mathbb{C}[v, v^{-1}]$ to be functions of a real variable, and let B be analytic functions on a neighborhood $D \times U$ of (1,0) in $\mathbb{R}^{n+1} \setminus \{0\} \times \mathbb{R}^n$. Clearly B is an A module. Suppose that

$$S = \{X_1, ..., X_n\}$$

is a set of vector fields such that each X_i exponentiates to a one-parameter family of transformations

$$\exp(v-1)X_l = K_l(v) : \mathbb{R}^n \to \mathbb{R}^n.$$

The maps $K_l(v)$ can be regarded as algebra homomorphisms

$$K_l\colon B \to B.$$

It is not hard to verify that

$$E_l\colon B \to B,\ E_l f(v, x_1, ..., x_n) = \frac{K_l f(v, x_1, ..., x_n) - f(v, x_1, ..., x_n)}{(r-1)}$$

is a deformed derivative with respect to K_l.

Now use these elements to form a measuring coalgebra. Let $\mathbf{K}$ be the complex vector space spanned by the K_l. Write $A\mathbf{K}$ for $A \otimes \mathbf{K}$. The A module $\mathbf{E} \oplus A\mathbf{K}$ can be given the structure of a measuring coalgebra over A by setting

$$\Delta \mathbf{E} \oplus A\mathbf{K} \to (\mathbf{E} \oplus A\mathbf{K}) \otimes (\mathbf{E} \oplus A\mathbf{K})$$

$$\Delta K_l = K_l \otimes K_l, \qquad \Delta E_l = E_l \otimes 1 + K_l \otimes E_l$$

$$\varepsilon K_l = 1, \qquad \varepsilon E_l = 0$$

The measuring map is the obvious inclusion of $\mathbf{E} \oplus A\mathbf{K}$ in $\mathrm{Hom}_A(B, B)$. As before,

$$s \colon \mathbf{E} \oplus A\mathbf{K} \to H$$

$$s(K_l) = K_l^{-1}$$

$$s(E_l) = -K_l^{-1} E_l$$

again gives H the structure of a Hopf algebra. Since for $v = 1$, $K_l = 1$ and $E_l = X_l$, the Hopf algebra H is a deformation of the Lie algebra of vector fields generated by S.

4.4. *Difference Operators on Lie Groups*

This time let G be a Lie group with Lie algebra L, and let

$$B = C^\infty((1 - \varepsilon, 1 + \varepsilon) \times G)$$

for some $\varepsilon > 0$. Let $A = \mathbb{R}[v, v^{-1}]$. Again, B is an A-module. Suppose

$$S = \{X_1, ..., X_n\}$$

is a basis for L such that

$$g_l = \exp(v - 1) X_l$$

defines a map

$$g_l \colon (1 - \varepsilon, 1 + \varepsilon) \to G.$$

The maps g_l can be interpreted as "acceptable" automorphisms

$$K_l \colon (1 - \varepsilon, 1 + \varepsilon) \times G \to (1 - \varepsilon, 1 + \varepsilon) \times G$$

$$K_l(v, g) = (v, g_j^{-1} g)$$

if Definition 2.1 is generated in an obvious way replacing $\mathbb{C}^n$ by arbitrary (real) manifolds. Similarly,

$$E_l \colon C^\infty((1-\varepsilon, 1+\varepsilon)\times G) \to C^\infty(1-\varepsilon, 1+\varepsilon)\times G)$$

$$E_l f(v, g) = \frac{f(K_l(v, g)) - f(v, g)}{(v-1)}$$

evidently defines a deformed derivative with respect to K_l. Now setting $\mathbf{K} = \{K_l\}$ and letting $\mathbf{E}$ be the free A module generated by the set $\{E_l\}$, $\mathbf{E} \oplus A\mathbf{K}$ is once more a coalgebra with the familiar coproducts. Once again $\mathbf{E} \oplus A\mathbf{K}$ generates a Hopf algebra generalizing the enveloping algebra $U(L)$.

4.5. *Remark.* Throughout this paper all deformed derivatives E have had reproducts of the form

$$\Delta E = E \otimes 1 + K \otimes E$$

and have been represented by difference operators of the form

$$E = f \cdot \left(\frac{K-1}{v-1}\right).$$

There is no sacred reason for decreeing that the above comultiplication is the coproduct for deformations of ordinary derivations. Difference operators of the form

$$E' = f \cdot \left(\frac{K - K^{-1}}{v - v^{-1}}\right)$$

have also been used [6] and have coproducts given by

$$\Delta E' = E' \otimes K + K^{-1} \otimes E'.$$

5. Comments and Conclusions

The program described above has three features which should be stressed in conclusion.

5.1. *Quantum Group-like Objects and the Space of Maps*

My initial application of the universal measuring coalgebra was to the problem of handling the differential information concerning the space of smooth maps between two manifolds X and Y [1]. This theory goes as follows.

Suppose X is a point. Then the universal measuring coalgebra $P(C^\infty(Y), C^\infty(X))$ is the dual coalgebra $C^\infty(Y)^0$. For this coalgebra there is a decomposition theorem

$$C^\infty(Y)^0 = \bigoplus_{y\in Y} T_y$$

$$T_y = \bigcup_{k=0}^{\infty} T_y^k,$$

where T_y^k is canonically isomorphic to the dual of the kth jet bundle of $C^\infty(Y)$ at y.

This geometric interpretation led to the use of $P(C^\infty(Y), C^\infty(X))$ to supply jet bundle information about the space of smooth maps from X to Y. Explicitly, if $C(C^\infty(Y), C^\infty(X))$ denotes the cocommutative part of $P(C^\infty(Y), C^\infty(X))$ then

$$C(C^\infty(Y), C^\infty(X)) = \bigoplus_{\substack{\sigma: X\to Y\\ \sigma \text{ smooth}}} T_\sigma$$

$$T_\sigma = \bigcup_{k=1}^{\infty} T_\sigma^k.$$

The subcoalgebras T_σ^k could then be interpreted as the fibre at σ of the dual jet bundle for the space of smooth maps.

Now, for $Y = X = (1-\varepsilon, 1+\varepsilon)\times M$ for some manifold M, the non commutative part of the measuring coalgebra appears to play a significant role. Allowing the interpretation of $P(C^\infty(Y), C^\infty(X))$ as a generalized space of smooth maps, the quantum group-like Hopf algebras described here may be interpreted as generalized automorphisms.

5.2. *Example* 4.4. *and Group Coalgebras*

While the quantum group-like generators E_l appear to behave like derivations, by enlarging the function ring A by inverting $v-1$ the Hopf algebra generated in 4.4 can be represented as a sub-Hopf algebra of a group Hopf algebra.

Let $\mathscr{G}$ be the group

$$\mathscr{G} = \{h: (1-\varepsilon, 1+\varepsilon) \to G\}.$$

Let $\check{A}$ be the algebra A with $(v-1)$ inverted. Form the group algebra $\check{A}\mathscr{G}$. Then the map

$$\check{A}\mathscr{G} \xrightarrow{\psi} \operatorname{Hom}_{\check{A}}(\check{A}\otimes B, \check{A}\otimes B)$$

$$\psi h f(v, g) = f(v, h^{-1}(v)\, g)$$

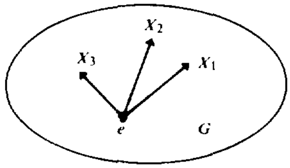
is a measuring map. Moreover, the map

$$\check{A}\otimes \mathrm{E}\oplus \check{A}K \to \mathrm{Hom}_{\check{A}}(\check{A}\otimes B, \check{A}\otimes B)$$

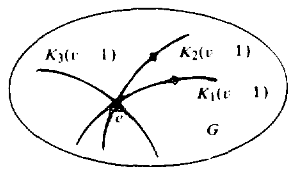
has its image in $\check{A}\mathcal{G}$.

5.3. *The Interpretation of Quantum Group-like Objects*

The last observation is particularly satisfying in the light of the popular philosophy of quantum theory, in which the parameter $(v-1)$ is not allowed to vanish but "stops" at Planck's constant. Example 4.4 indicates how Lie algebra theory, where the limit is taken as $v \to 1$, may be replaced by "quantum Lie theory" where the parameter v is recorded.



The classical Lie algebra picture



The quantum Lie picture

The virtue of the universal measuring coalgebra is that it provides a context in which the Lie algebra theory and the quantum Lie theory may be handled in the same manner.

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