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A Class of Matroids Derived from Saturated Chain Partitions of Partially Ordered Sets

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The simultaneously k- and $(k-1)$-saturated chain partitions of a finite partially ordered set P determine a matroid $G_k(P)$. This matroid is a gammoid. The identity on P induces a strong map from $G_k(P)$ to $G_{k+1}(P)$. This strong map has a linear representation.

1. INTRODUCTION

It was observed by Greene and Kleitman [7] that the Sperner k-families of a finite partially ordered set P determine a matroid on the set M of maximal elements of P. Greene [6] conjectured that this matroid is induced by a matroid on the partially ordered set $P-M$. We establish in this paper that this is indeed the case.

Corresponding to every positive integer k there is a matroid on P which we denote by $G_k(P)$. Using a characterization of the Greene–Kleitman geometry, $\Gamma_P^{(k)}$, given in [7], we show that $G_k(P-M)$ induces $\Gamma_P^{(k)}$. We further prove that $G_k(P)$ is a gammoid, and that the identity map on P induces a strong map from $G_k(P)$ to $G_{k+1}(P)$. We prove a theorem concerning the representation of certain strong maps as linear maps, from which it follows that the strong map from $G_k(P)$ to $G_k(P)$ always has a representation.

In Section 4 we investigate what of the structure of P is inherent in the sequence $\{G_k(P): k \geqslant 1\}$. We show that the sequence does not uniquely determine the partial order, and that the class of all matroids arising in this manner from partially ordered sets is not closed under restriction, contraction, or duality. We conclude by examining the matroids corresponding to a pair of important partially ordered sets, the lattice of subsets of an n-element set and the lattice of subspaces of an n-dimensional vector space over a finite field.

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2. Preliminaries

We bring together in this section definitions and results that will be used in the paper. The reader is referred to [1, 2, 9] for detailed expositions concerning graph theory and combinatorial geometries. All partially ordered sets are finite.

A *combinatorial pregeometry, or matroid*, G, on a finite set X is characterized by a class $\mathscr{B}(G)$ of subsets of X satisfying:

(1) If $A, B \in \mathscr{B}(G)$, then $|A| = |B|$, and

(2) for every $x \in X$, there exists $y \in B$ such that $A - \{x\} \cup \{y\} \in \mathscr{B}(G)$.

The elements of $\mathscr{B}(G)$ are the *bases* of G. A subset of X which is contained in some basis of G is an *independent set*. A set $C \subseteq X$ is a *circuit* of G if and only if C is not independent in G, but every proper subset of C is independent. For a subset Y of X, the *rank* of Y, denoted $r_G(Y)$, is the cardinality of a maximal independent subset of Y. The *rank* $r(G)$ of the matroid is $r_G(X)$. A subset F of X is called a *flat* or *closed set* provided that for every $x \notin F$, $r_G(F \cup \{x\}) > r_G(F)$. For $Y \subseteq X$, the closure of Y, denoted $\bar{Y}^G$, is the intersection of all flats of G containing Y. The set of flats of G, ordered by inclusion, form a lattice called the *geometric lattice* of G. Suppose H is a second matroid on X. The identity on X induces a *strong map* from G to H if and only if every closed set in H is also a closed set of G. The matroid F on X with $r_F(x) = |X|$ is called the *free* matroid on X. The identity on X induces a strong map from F to G, called the *closure map* of G.

The *dual* matroid G^* of G is a matroid on X having for its set of bases the complements of bases of G; i.e., $B \in \mathscr{B}(G^*)$ if and only if $X - B \in \mathscr{B}(G)$. For a set $Y \subseteq X$, the *restriction* of G to Y, $G|Y$, is characterized by the property that $Z \subset Y$ is independent in $G|Y$ if and only if Z is independent in G. The matroid $G \cdot Y = (G^*|Y)^*$ is called the *contraction* of G to Y. G is said to be *representable* over a field F provided there exists a mapping $f : X \to F^r$, $r = r(G)$, with $A \subseteq X$ independent in G if and only if $f|A$ is injective and $f(A)$ is linearly independent in F^r.

If D is a digraph and $B \subseteq V(D)$, then the class of subsets $A \subseteq V(D)$ for which there exists sets of vertex-disjoint paths linking A into B form the independent sets of a matroid, called the *strict gammoid* induced by (D, B). For vertices x and y of D, (x, y) will denote the directed edge from x to y, and $\Gamma(x)$ will represent $\{z \in V(D): (x, z) \in E(D)\}$. A theorem of Mason [8] states that for a flat F of the strict gammoid, the set $(F \cap B) \cup \{x \in F : \Gamma(x) \not\subseteq F\}$ is a basis for F. If $P = (x_1, x_2,..., x_m)$ is a path in D, then for any j, $1 \leqslant j \leqslant m$, the subpath $(x_1, x_2,..., x_j)$ is called an *initial segment* of P and the subpath $(x_j, x_{j+1},..., x_m)$ is a *terminal segment* of P.

Suppose P is a partially ordered set. A *chain partition* $\mathscr{C} = \{C_i : 1 \leqslant i \leqslant n\}$ of P is a partition for which each C_i is a linearly ordered set, or *chain*. For a positive integer k, we define a *k-family* of P to be a set which contains no chain of length $k+1$. A *Sperner* k-family is a k-family of maximum cardinality and $d_k(P)$ signifies the size of a Sperner k-family. For $k > 1$, $\Delta_k(P) = d_k(P) - d_{k-1}(P)$, with $\Delta_1(P) = d_1(P)$. We will also refer to a Sperner 1-family as a *Dilworth set*. If $\mathscr{D} = \{D_i : 1 \leqslant i \leqslant m\}$ is a chain partition of P, we define

$$\beta_k(\mathscr{D}) = \sum_{i=1}^{m} \min[k, |D_i|].$$

If A is a Sperner k-family, then

$$d_k(P) = \sum_{i=1}^{m} |A \cap D_i| \leqslant \sum_{i=1}^{m} \min[k, |D_i|] = \beta_k(\mathscr{D}).$$

If $\beta_k(\mathscr{D}) = d_k(P)$, $\mathscr{D}$ is called a *k-saturated* chain partition. Observe that if $\mathscr{D}$ is k-saturated and $D \in \mathscr{D}$ with $|D| < k$, then the chain partition obtained by replacing D with $|D|$ singleton or *trivial* chains is also k-saturated. We adopt the convention of listing only the chains of length at least k; the elements of P not contained in these chains will be taken to be trivial chains. Every chain partition is defined to be 0-saturated. A deep result of Greene and Kleitman [6] ensures that simultaneously k- and $(k-1)$-saturated chain partitions exist for every positive integer k. The expression $(k, k-1)$-*SCP* will be used to signify a simultaneously k- and $(k-1)$- saturated chain partition. For a subset A of P, the order ideal, $\mathscr{I}(A)$, generated by A consists of all x in P with $x \leqslant a$ for some a in A. The order filter, $\mathscr{F}(A)$, generated by A consists of all x in P with $x \geqslant a$ for some a in A.

The *graded multipartite* graph $\Gamma_k(P)$ is defined as follows:

$$V(\Gamma_k(P)) = P_0 \cup P_1 \cup \cdots \cup P_k, \tag{1}$$

where the sets $P_i = \{x^i : x \in P\}$ are copies of P, and

$$E(\Gamma_k(P)) = \{(x^i, y^{i+1}) : x > y \text{ in } P\}. \tag{2}$$

Note that the graph $\Gamma_1(P)$ is the bipartite graph representing the partial order. A *matching* or *linking* in $\Gamma_k(P)$ is a set of disjoint paths with initial vertices in P_0 and terminal vertices in P_k.

3. The Matroid $G_k(P)$

Let $\mathscr{C} = \{C_i : 1 \leqslant i \leqslant n\}$ be a k-saturated chain partition of P. Suppose $C_1 = \{x_1 > x_2 > \cdots > x_m\}$. Consider the class of $m-k$ chains

$\{(x_j, x_{j+1}, ..., x_{j+k}) : 1 \leqslant j \leqslant m - k\}$. Each of these chains determines a path in $\Gamma_k(P)$ and all of these paths are disjoint. For each C_i with size greater than k we perform the same construction. Since paths in $\Gamma_k(P)$ corresponding to distinct chains of $\mathscr{C}$ correspond to disjoint subsets of P, they are clearly disjoint. Hence, corresponding to the chain partiton there is a matching in $\Gamma_k(P)$ of size $\sum_{i=1}^{n} (|C_i| - k) = |P| - d_k(P)$. The following theorem of Greene asserts that this matching is maximal.

THEOREM 3.1 (Greene [5]). *The size of a maximal matching in $\Gamma_k(P)$ is $|P| - d_k(P)$.*

Let $\mathscr{Q} \cup \mathscr{R}$ be a set of disjoint paths in $\Gamma_k(P)$ satisfying:

(1) The initial vertices of paths in $\mathscr{Q}$ are elements of P_0.

(2) The initial vertices of paths in $\mathscr{R}$ are elements of P_1.

(3) The terminal vertices of paths in $\mathscr{Q} \cup \mathscr{R}$ are elements of P_k.

We say the pair $(\mathscr{Q}, \mathscr{R})$ is a *$(k, k-1)$-matching* in $\Gamma_k(P)$. The $(k, k-1)$-matching is *maximal* if and only if for every $(k, k-1)$-matching $(\mathscr{S}, \mathscr{E})$, $|\mathscr{Q}| \geqslant |\mathscr{S}|$ and $|\mathscr{Q} \cup \mathscr{R}| \geqslant |\mathscr{S} \cup \mathscr{E}|$.

Now suppose further that $\mathscr{C}$ is a $(k, k-1)$-*SCP* of P. Let $\mathscr{Q}$ be the maximal matching in $\Gamma_k(P)$ corresponding to $\mathscr{C}$ as described at the beginning of this section. For each i, $1 \leqslant i \leqslant n$, let D_i be the k greatest elements of C_i, say $D_i = \{y_{i1} > y_{i2} > \cdots > y_{ik}\}$. Let R_i, $1 \leqslant i \leqslant n$, be the path with initial vertex y_{i1}^1 in P_1 and terminal vertex y_{ik}^k in P_k determined by the points of D_i. If we take $\mathscr{R}$ to be the set $\{R_i : 1 \leqslant i \leqslant n\}$, the pair $(\mathscr{Q}, \mathscr{R})$ is easily seen to be a $(k, k-1)$-matching in $\Gamma_k(P)$.

THEOREM 3.2. *The matching $(\mathscr{Q}, \mathscr{R})$ is a maximal $(k, k-1)$-matching.*

Proof. Let $(\mathscr{S}, \mathscr{E})$ be a $(k, k-1)$-matching in $\Gamma_k(P)$. Then $\mathscr{Q}$ and $\mathscr{S}$ are matchings in $\Gamma_k(P)$. Since $\mathscr{C}$ is k-saturated, $|\mathscr{Q}| \geqslant |\mathscr{S}|$. For each path W in $\mathscr{Q} \cup \mathscr{S}$, let W^T be the truncated path obtained by deleting the initial vertex of W. Observe the correspondence between the set of paths $\{W^T : W \in \mathscr{Q}\} \cup \mathscr{R} = \mathscr{U}$ and the matching $\mathscr{M}$ determined by the chains of in $\Gamma_{k-1}(P)$; $(x_1^1, x_2^2, ..., x_k^k) \in \mathscr{U}$ if and only if $(x_1^0, x_2^1, ..., x_k^{k-1}) \in \mathscr{M}$. Similarly the set of paths $\{W^T : W \in \mathscr{S}\} \cup \mathscr{E} = \mathscr{T}$ corresponds to a second matching in $\Gamma_{k-1}(P)$. As $\mathscr{C}$ is also $(k-1)$-saturated, $|\mathscr{Q} \cup \mathscr{R}| = |\mathscr{U}| \geqslant |\mathscr{T}| = |\mathscr{S} \cup \mathscr{E}|$. ∎

There is a partial converse to this proposition, for which we need the following lemma.

LEMMA 3.3. *Suppose $(\mathscr{Q}, \mathscr{R})$ is a maximal $(k, k-1)$-matching in $\Gamma_k(P)$, $\mathscr{Q} = Q_i$: $1 \leqslant i \leqslant m$, and that there is a maximal y in P belonging to every*

Sperner k-family, with no initial vertex of a path in $\mathscr{R}$ corresponding to y. Then there is a maximal $(k, k-1)$-matching $(\mathscr{S}, \mathscr{R})$ with no initial vertex of a path in $\mathscr{S}$ corresponding to y.

Proof. Consider the strict gammoid G induced by $(\Gamma_k(P), P_k)$, and let F be the flat of G spanned by P_0. Set $B = \{x \in F: \Gamma(x) \not\subseteq F\} \cup (F \cap P_k)$. This set is a basis for F, and no vertex of a path in $\mathscr{R}$ can be contained in F. Furthermore, every path Q_i meets B in a single vertex, say $\{b_i\} = B \cap Q_i$. Let Q_i' be the terminal segment of Q_i with initial vertex b_i. Since $d_k(P - \{y\}) = d_k(P) - 1$, there is a maximal matching in $\Gamma_k(P)$ which misses y_0, say $\mathscr{T} = \{T_i: 1 \leqslant i \leqslant m\}$. Every path in $\mathscr{T}$ also meets B in a single vertex, so we assume $\mathscr{T}$ is indexed so that b_i is on T_i. Let T_i' be the initial segment of T_i with terminal vertex b_i. Clearly each T_i' is contained in F. Hence $(\mathscr{S}, \mathscr{R})$ is a maximal $(k, k-1)$-matching in $\Gamma_k(P)$, where $\mathscr{S}$ is defined to be the set of paths obtained by joining each T_i to Q_i' at b_i. ∎

THEOREM 3.4. *Let $(\mathscr{Q}, \mathscr{R})$ be a maximal $(k, k-1)$-matching in $\Gamma_k(P)$. The initial vertices of paths in $\mathscr{R}$ correspond to the tops of chains of length at least k in some $(k, k-1)$-SCP of P.*

Proof. Proof is by induction on $|P|$. The theorem is clearly true for all partially ordered sets of cardinality at most $k+1$. Say $|P| = n$ and assume the statement is true for every partially ordered set which fewer than n elements.

Case 1. For some maximal element b, $d_k(P - \{b\}) = d_k(P)$.

Under these circumstances, Greene and Kleitman [6] have shown that $d_{k-1}(P - \{b\}) = d_{k-1}(P)$. Hence b^0 and b^1 must be initial vertices of paths in $\mathscr{Q}$ and $\mathscr{R}$, respectively, say b^0 is on Q and b^1 is on R. The matching $(\mathscr{Q} - \{Q\}, \mathscr{R} - \{R\} \cup \{Q^T\})$ is therefore maximal in $\Gamma_k(P - \{b\})$. By the inductive hypothesis there is a $(k, k-1) - SCP$ of $P - \{b\}$ whose set of tops corresponds to the initial vertices of $\mathscr{R} - \{R\} \cup \{Q^T\}$. If c^1 is the initial vertex of Q^T, the element b can be adjoined to the chain with top c to form a $(k, k-1) - SCP$ of P.

Case 2. For every maximal element x, $d_k(P - \{x\}) = d_k(P) - 1$.

Let $\mathscr{C}$ be a $(k, k-1)$-*SCP* of P with $(\mathscr{S}, \mathscr{T})$ the corresponding $(k, k-1)$-matching in $\Gamma_k(P)$. Let A_1 and B_1 be, respectively, the initial vertices of paths in $\mathscr{R}$ and $\mathscr{T}$. If $B \neq A$, then there exists a b in $B - A$. Were b not maximal in P, there would be a c maximal in P with $c > b$. If C is the chain with top b and S is any Sperner k-family of P, then $|C \cap S| = k$. But by hypothesis c belongs to S, and $(C \cap S) \cup \{c\}$ would be a chain of length $k+1$ in S. So b must be maximal in P. Therefore no vertex corresponding to b is on any path of $\mathscr{R}$, and by the previous lemma we may assume that no vertex of a path in $\mathscr{Q}$ corresponds to b. Hence $(\mathscr{Q}, \mathscr{R})$ is a maximal $(k, k-1)$-matching in

$\Gamma_k(P - \{b\})$; so by the inductive hypothesis there is a $(k, k-1)$-*SCP* $\mathscr{D}$ of $P - \{b\}$ with A as its set of tops. Note that

$$d_{k-1}(P - \{b\}) = (n-1) - |\mathscr{Q} \cup \mathscr{R}| = d_{k-1}(P) - 1.$$

Therefore b also belongs to every Sperner $(k-1)$-family of P and $\mathscr{D}$ is a $(k, k-1)$-*SCP* of P. ∎

Let G be the restriction to $P_0 \cup P_1$ of the strict gammoid induced by $(\Gamma_k(P), P_k)$ and consider $G \cdot P_1$, the contraction of G to P_1. A set X_1 is a basis of $G \cdot P_1$ if and only if there exists a set $Y_0 \subseteq P_0$ such that

(1) $X_1 \cup Y_0$ is a basis for G, and

(2) Y_0 is a basis for $G|P_0$; i.e., Y_0 is the set of initial vertices in a maximal linking of P_0 to P_k.

This is equivalent to the condition that there exists a maximal $(k, k-1)$-matching $(\mathscr{Q}, \mathscr{R})$ in $\Gamma_k(P)$ with X_1 as the set of initial vertices of $\mathscr{R}$. This observation together with Theorem 3.4 establishes the following result.

THEOREM 3.5. *Corresponding to every partially ordered set P and every positive integer k there is a matroid $G_k(P)$ whose bases are the tops of chains of length at least k of the simultaneously k- and $(k-1)$-saturated chain partitions of P. This matroid is in fact a gammoid, being the contraction of a restriction of a strict gammoid.*

The rank of $G_k(P)$ can be determined using Theorem 3.1:

$$r(G_k(P)) = (|P| - d_{k-1}(P)) - (|P| - d_k(P)) = \Delta_k(P).$$

THEOREM 3.6. *If $\mathscr{C}$ is a k-saturated chain partition of P, then there exists a $(k, k-1)$-SCP $\mathscr{D}$ such that the tops of $\mathscr{C}$ are contained in the set of tops of $\mathscr{D}$. Thus subsets of the tops of k-saturated chain partitions are independent sets in $G_k(P)$.*

Proof. Let $(\mathscr{Q}, \mathscr{R})$ be the $(k, k-1)$-matching in $\Gamma_k(P)$ determined by $\mathscr{C}$. Since $\mathscr{C}$ is k-saturated, $\mathscr{Q}$ is a maximal matching. The initial vertices of $\mathscr{Q} \cup \mathscr{R}$ are an independent set in the strict gammoid induced by $(\Gamma_k(P), P_k)$. Therefore we can find a maximal $(k, k-1)$-matching $(\mathscr{S}, \mathscr{E})$ with the initial vertices of $\mathscr{R}$ contained in the set of initial vertices of $\mathscr{E}$. Then by Theorem 3.4, there is a $(k, k-1)$-*SCP* $\mathscr{D}$ with set of tops containing the set of tops of $\mathscr{C}$. ∎

We wish to show that if P' is the set of maximal elements of P, the matroid $G_k(P - P')$ induces the corresponding Greene–Kleitman geometry

$\Gamma_P^{(k)}$. The Greene–Kleitman geometry is defined by means of a rank function r_k. If $P^* = P - P'$, then for $X \subset P'$

$$r_k(X) = |X| + d_k(P^*) - d_k(P^* \cup X).$$

THEOREM 3.7. *$G_k(P^*)$ induces $\Gamma_P^{(k)}$.*

Proof. Greene and Kleitman [6] have shown that a subset X of P' is independent in $\Gamma_P^{(k)}$ if and only if there exists a k-saturated chain partition of P^* for which X can be matched into the tops of chains of length at least k. By Theorem 3.6, this is equivalent to the condition that there exists a $(k, k-1)$-SCP of P^* having the same property. ∎

LEMMA 3.8. *Let Γ be a digraph on $V(\Gamma) = X$ and let B be a subset of X with $b \in B$. Say $B' = B - \{b\}$. Let G and G' be the strict gammoids induced by (Γ, B) and (Γ, B'), respectively. Then the identity map on X extends to a strong map from G to G'.*

Proof. Let F be a flat of G' with $A = \{x \in F: \Gamma(x) \not\subseteq \mathrm{F}\}$. By Mason's theorem, $C = A \cup (B' \cap F)$ is a basis for F in G'. Let $x \in X - F$. Then there exists a G'-independent linking of $C \cup \{x\}$. This linking is also G-independent. Hence $x \notin \bar{F}^G$. Therefore F is a closed set in G. ∎

A strong map $f: G \to H$ is *elementary* if and only if $r(H) = r(G) - 1$. Suppose G and H are matroids on X and that the identity on X induces an elementary strong map from G to H. Let $\mathscr{F}$ be the set of flats F of H for which $r_H(F) < r_G(F)$. Then $\mathscr{F}$ is an order filter in the lattice of flats of G. If $\mathscr{F}$ is a principal order filter, that is, $\mathscr{F}$ is generated by a single flat, then the map is called *principal.*

THEOREM 3.9. *Let Γ, G, and G' be as in Lemma 3.8. Then the identity on X induces a principal map.*

Proof. Let $F = \overline{\{b\} \cup \Gamma(b)}^{G'}$. We will show that F generates the order filter of flats F' of G' for which $r_G(F') > r_{G'}(F')$. Let $A = \{x \in F: \Gamma(x) \not\subseteq F\}$. Since $\Gamma(b) \subseteq F$, $b \notin A$. Therefore,

$$r_G(F) = |A \cup (F \cap B)| > |A \cup (F \cap B')| = r_{G'}(F').$$

Suppose that F' is a flat of G' with $r_G(F') > r_{G'}(F')$. Let $A' = \{x \in F': \Gamma(x) \not\subseteq F'\}$. Now $r_G(F') = |A' \cup (F' \cap B)|$ and $r_{G'}(F') = |A' \cup (F' \cap B')|$. Hence $\{b\} \cup \Gamma(b)$ is contained in F', and $F \subseteq F'$. ∎

We obtain as a corollary to Theorem 3.9 a result due to Ingleton and Piff [7].

COROLLARY 3.10. *Strict gammoids are duals of transversal matroids.*

Proof. Dowling and Kelly [3] have characterized the duals of transversal matroids (cotransversal matroids) as those matroids whose closure map admits a principal factorization. A strict gammoid is induced by a pair (Γ, B) while the corresponding matroid on $V(\Gamma)$ induced by $(\Gamma, V(\Gamma))$ is free. ∎

THEOREM 3.11. *For $k > 1$, the identity on P induces a strong map from $G_{k-1}(P)$ to $G_k(P)$.*

Proof. Let H and H' be the strict gammoids induced by $(\Gamma_k(P), P_{k-1} \cup P_k)$ and $(\Gamma_k(P), P_k)$, respectively. By repeated use of Lemma 3.8 we have that the identity on $V(\Gamma)$ induces a strong map from H to H'. Note, however, that H is a direct sum of two matroids, the strict gammoid H'' induced by $(\Gamma_{k-1}(P), P_{k-1})$ and the free matroid on P_k. The restriction of this map to $P_0 \cup P_1$ is a strong map from $G'' = H'' | (P_0 \cup P_1)$ to $G' = H' | (P_0 \cup P_1)$. But then the identity on P_1 induces a strong map from $G'' \cdot P_1 = G_{k-1}(P)$ to $G' \cdot P_1 = G_k(P)$. ∎

The following corollary is a theorem of Greene and Kleitman [6] concerning the structure of the Sperner k-families of P.

COROLLARY 3.12. $\Delta_k(P) \leqslant \Delta_{k-1}(P)$.

Proof. As was noted earlier, $r(G_k(P)) = \Delta_k(P)$. ∎

THEOREM 3.13. *Suppose G and G' are matroids on X with the identity inducing a principal map from G to G'. Suppose further that G has a representation $\sigma: X \to K^r$ over the field K. If the order of K is sufficiently large, there is a linear map $L: K^r \to K^{r-1}$ such that $L \circ \sigma$ is a representation of G'.*

Proof. Let A be the flat of G which generates the order filter of flats F with $r_{G'}(F) < r_G(F)$ and let $\sigma: G \to K^r$ be a representation of G. For K sufficiently large, we can find a vector $\mathbf{v}$ in the linear span of $\sigma(A)$ which not contained in the linear span of $\sigma(F)$ for any flat F of G properly contained in A. Let $L: K^r \to K^{r-1}$ be the canonical linear transformation having the linear span of $\mathbf{v}$ for its kernel. We show that $L(\sigma(G))$ is a representation of G'.

For any rank $(r-1)$ flat F of G not containing A, the G-flat $A \cap F$ has rank $r_G(A) - 1$. Therefore the linear span of $\sigma(A \cap F)$ is the intersection of the linear spans of $\sigma(A)$ and $\sigma(F)$. Hence $\mathbf{v}$ is not in the span of any set $\sigma(F)$, F a flat of G not containing A.

First, if Y is an independent set in G', it is also independent in G, and $\sigma(Y)$ is linearly independent. The flat A is not contained in $\bar{Y}^G$ since $\bar{Y}^G = \bar{Y}^{G'}$. Therefore $\{\mathbf{v}\} \cup \sigma(Y)$ is linearly independent. Since L has nullity one, $L(\sigma(Y))$ must be linearly independent.

Next assume that Y is a dependent set in G'. If Y is dependent in G, then certainly $L(\sigma(Y))$ is linearly dependent. So suppose Y is independent in G. Then $A \subseteq \bar{Y}^G$ and v is contained in the linear span of $\sigma(Y)$. Hence $L(\sigma(Y))$ is linearly dependent. ∎

THEOREM 3.14. *The strong map from $G_k(P)$ to $G_{k+1}(P)$ is representable as a linear map between representations of $G_k(P)$ and $G_{k+1}(P)$.*

Proof. Let G_1 and G_2 be, respectively, the strict gammoids induced by $P_k \cup P_{k+1}$ and P_{k+1} on $\Gamma_{k+1}(P)$. Repeated applications of Theorem 3.13 to the principal maps from the free matroid on $\Gamma_{k+1}(P)$ to G_1 and from G_1 to G_2 yield a representation σ of G_1 and a linear representation L of the strong map from G_1 to G_2.

Let B be the linear span of $\sigma(P_0)$ and let Π be the canonical linear transformation on $\sigma(G_1)$ with kernel B. It is easily seen that $\Pi \circ \sigma$ is a representation of the contraction $G_1 \cdot (V(\Gamma) - P_0)$ and that the map L applied to the restriction of $\Pi \circ \sigma$ to P_1 is a linear representation of the strong map from $G_k(P)$ to $G_{k+1}(P)$. ∎

4. STRUCTURAL PROPERTIES OF P INHERENT IN $G_k(P)$: $k \geqslant 1$.

THEOREM 4.1. $d_k(P) = \sum_{i=1}^{k} r(G_i(P))$.

Proof. By Theorem 3.1, we have that $r(G_i(P)) = \Delta_i(P)$. $1 \leqslant i \leqslant k$. ∎

THEOREM 4.2. *The height of P is the maximal k for which $r(G_k(P)) > 0$.*

Proof. Let h be the height of P. Then a maximal chain in P determines a path in $\Gamma_h(P)$ from P_1 to P_h. Since there is no path of length $h+1$ in $\Gamma_h(P)$, the initial vertex of this path corresponds to a point independent in $G_h(P)$. Clearly $r(G_{h+1}(P)) = 0$.

Recall that $\mathscr{F}(A)$ and $\mathscr{I}(A)$ are, respectively, the order-filter and order ideal generated by $A \subseteq P$. In [6] a partial order is defined on the class of Sperner k-families of an arbitrary partially ordered set, and under this ordering the Sperner k-families are shown to form a distributive lattice. In P there is a Sperner k-family A such that for every Sperner k-family B of P, $B \subseteq \mathscr{I}(A)$. We call A the *unit* Sperner k-family of P. Let $A' = \max[A]$.

LEMMA 4.3. *If $x \notin \mathscr{F}(A')$, then $x \in \mathscr{I}(A')$.*

Proof. Suppose there exists an $x \notin \mathscr{F}(A') \cup \mathscr{I}(A')$. Then $\{x\} \cup A$ is a k-family of P, contradicting the maximality of A. ∎

THEOREM 4.4. *If $x \notin \mathscr{F}(A')$, then x is a loop in $G_k(P)$.*

Proof. Suppose the theorem is false. Then there exists a $(k, k-1)$-*SCP* $\mathscr{C}$ with $x \notin \mathscr{F}(A')$ the top of a nontrivial chain $C \in \mathscr{C}$. By Lemma 4.3 there is an $a \in A'$ with $a > x$. But then $\{a\} \cup (C \cap A)$ is a chain of length $k+1$ in A. ∎

For $k = 1$, we have a stronger result.

THEOREM 4.5. *x is a loop of $G_1(P)$ if and only if $x \notin \mathscr{F}(A)$.*

Proof. The proof is by induction on $|P|$. We assume the theorem is true for every partially ordered set Q with $|Q| < |P|$. Let $x \in \mathscr{F}(A)$ and choose $y \leqslant x$ with y in A.

Case 1. $d_1(P - \{y\}) = d_1(P) - 1$.

Let $\mathscr{C}$ be a minimal chain partition of $P - \{y\}$. If $x = y$, then $\mathscr{C} \cup \{x\}$ is a minimal chain partition of P. Otherwise, let C_x be the chain containing x. Then the chain partition obtained by deleting x from C_x and adjoining the new chain $\{x, y\}$ is a minimal chain partition of P with x as the top of $\{x, y\}$.

Case 2. $d_1(P - \{y\}) = d_1(P)$.

Let B be the unit Dilworth set in $P - \{y\}$ and let $\mathscr{C}$ be a minimal chain partition of P. Since B is a Dilworth set of P, every chain of $\mathscr{C}$ contains a point of B. Say C is the chain containing y, and let $\{z\} = C \cap B$. By the inductive hypothesis, since z is in $\mathscr{F}(B)$, z is not a loop in $G_1(P - \{y\})$. Therefore there is a minimal chain partition $\mathscr{C}'$ of $P - \{y\}$ with z as the top of, say, C'. Then the chain partition of P obtained by deleting x from the chain of $\mathscr{C}'$ containing it and adjoining $\{x, y\}$ to C' is a minimal chain partition of P, with x as the top of a chain. ∎

The assumption $k = 1$ in the previous theorem is necessary. If P contains an element x not comparable to any other element of P, then for $k \geqslant 2$, x is in A', hence $\mathscr{F}(A')$, and is clearly a loop in $G_k(P)$.

THEOREM 4.6. *Every maximal element of P which is not contained in A is an isthmus in $G_k(P)$.*

Proof. That every element of $P - A$ is contained in a nontrivial chain in any k-saturated chain partition of P is an immediate consequence of the definition of k-saturation. The maximal elements of P are certainly the tops of the chains containing them.

LEMMA 4.7. *Let $\mathscr{C} = \{C_1, C_2, \ldots, C_n\}$ be an m-saturated chain partition of P, with x as the top of C_1. Let $x < y$ in P. Then y is on a nontrivial chain of $\mathscr{C}$, say C_2, and the chain partition $\mathscr{C}' = \mathscr{C} - \{C_1, C_2\} \cup \{C_1 \cup \{y\}, C_2 \cup \{y\}\}$ is m-saturated.*

Proof. Let S be a Sperner m-family of P. Since $|S \cap C_1| = m$, y cannot belong to S. Therefore, y is on a nontrivial chain of $\mathscr{C}$, say C_2, and $|S \cap C_2 - \{y\}| = m$. Hence $\beta_m(\mathscr{C}) = \beta_m(\mathscr{C}')$, and $\mathscr{C}'$ is m-saturated. ∎

THEOREM 4.8. *If x is an isthmus in $G_k(P)$, then x is maximal in P.*

Proof. Suppose x is not maximal in P, say $x < y$. Let $\mathscr{C}$ be any $(k, k-1)$-SCP of P. If x is not the top of a chain of $\mathscr{C}$, we are finished. If x is the top of a chain, we can apply Lemma 4.7 with $m = k$ and $m = k-1$ to obtain a $(k, k-1)$-SCP with x on a chain having y as its top. ∎

COROLLARY 4.9. *The maximal elements of P are precisely the isthmuses of $G_1(P)$.*

Proof. Certainly every maximal element must be the top of the chain containing it in any minimal chain partition; hence every maximal element is an isthmus. By Theorem 4.8 these are the only isthmuses. ∎

Let us denote by $\mathscr{G}_k$ those matroids G for which there exists a partially ordered set P with $G = G_k(P)$. Define $\mathscr{G}_0 = \bigcup_{k>0} \mathscr{G}_k$ and let $\mathscr{G}$ represent the class of all gammoids. We have already seen that $\mathscr{G}_0 \subseteq \mathscr{G}$. The following theorem establishes that this containment is proper.

THEOREM 4.10. *For every partially set P with $|P| \geqslant 2$, $G_k(P)$ is not connected.*

Proof. For $k = 1$, the maximal elements of P are isthmuses of $G_1(P)$. So suppose $k \geqslant 2$ and let A be the unit Sperner k-family of P with $A' = \max|A|$. Then $A - A' \subseteq \mathscr{Q}(A') - A'$, so that every element of $A - A'$ is a loop in $G_k(P)$. Now $|A - A'| \geqslant d_k(P) - d_1(P) = \sum_{i=2}^{k} \Delta_i(P) \geqslant \Delta_k(P)$. So either $A - A' \neq 0$ or $r(G_k(P)) = \Delta_k(P) = 0$, in which case every element of P is a loop. ∎

THEOREM 4.11. *If $G \in \mathscr{G}$, then there exists a partially ordered set P for which $G_2(P) = G \oplus H$, where H is a preboolean matroid consisting only of loops and isthmuses.*

Proof. Let G be a gammoid on a set X. Ingleton and Piff [7] have shown that there exists a transversal matroid T and a set $Y \subset X$ for which $G = T \cdot (X - Y)$, $|Y| = r(T) - r(G)$. Let $R \subseteq X \times Z$ be a presentation of T and let Y' be a copy of Y with $S\colon Y' \to Y$ an injection. Let P be the partial order on $X \cup Z \cup Y'$ defined by the transitive and reflexive closure of $R \cup S$. Clearly P has height 3. A maximal $(2, 1)$-matching in $\Gamma_2(P)$ determines a maximal matching in R in which each vertex of Y is covered. Hence every basis in $G_2(P)$ is a basis of G. On the other hand, let B be a basis in G. Let

M be a corresponding maximal matching in R which covers $Y \cup B$. Then $M \cup S$ is a chain partition of P which determines a maximal (2, 1)-matching in $\Gamma_2(P)$. Therefore B is also a basis in $G_2(P)$. ∎

Although it is not true in general that $\mathscr{G}_k \subseteq \mathscr{G}_{k+1}$, the next theorem does establish that every member of $\mathscr{G}_k$ can be embedded as a direct summand of a memer of $\mathscr{G}_{k+1}$.

THEOREM 4.12. *Let P be a partially ordered set and let $G = G_k(P)$ for some positive integer k. Then there exists a partially ordered set P' for which $G_{k+1}(P') = G \oplus \mathscr{B}_{0,r}$ represents the preboolean matroid consisting of r loops, $r = r(G) = \Delta_k(P)$.*

Proof. Since $G_{k+1}(P)$ is a strong map image of $G_k(P)$, if $\Delta_k(P) = \Delta_{k+1}(P)$, then $G_k(P) = G_{k+1}(P)$. So we will assume that $\Delta_{k+1}(P) < \Delta_k(P)$. Let Q be a set of r elements disjoint from P. Define a partial order on $P' = P \cup Q$ by setting

$$x \leqslant_{P'} y \qquad \text{if and only if} \quad (1)\ x, y \in P, x \leqslant_P y;$$

$$(2)\ x \in Q, y \in P;$$

$$(3)\ x = y \in Q.$$

That is, the set Q is adjoined to P as an antichain with each element of Q being less than every element of P. The following sequence of lemmas establishes that P' satisfies the requirements of the proposition.

LEMMA 4.13. *Let j be a positive integer and let A be a Sperner j-family of P'. Then A does not cut Q; i.e., either $A \cap Q = \varnothing$ or $Q \subseteq A$.*

Proof. Suppose $A \cap Q \neq \varnothing$, say $x \in A - Q$. Let $A' = \min[A]$, and let $y \in A'$. Then $x \nless y$, so $y \in Q$; i.e., $A' \subseteq Q$. The set $A - A'$ is a $(j-1)$-family of P', so $A \subseteq (A - A') \cup Q$ is a j-family of P'. The maximality of A implies $A = (A - A') \cup Q$; that is, $Q \subseteq A$. ∎

LEMMA 4.14. $d_k(P') = d_k(P)$.

Proof. Clearly $d_k(P') \geqslant d_k(P)$. Let B be any Sperner k-family of P'. If $B \subseteq P$ we are done, so we assume that $B \cap Q \neq \varnothing$. By Lemma 4.14, $Q \subseteq B$. Since $B - Q$ is a $(k-1)$-family of P, $d_k(P') = |B| = |B - Q| + |Q| \leqslant d_{k-1}(P) + \Delta_k(P) = d_k(P)$. Therefore $d_k(P') = d_k(P)$. ∎

LEMMA 4.15. *A set $A \subseteq P'$ is a Sperner $(k+1)$-family of P' if and only if $Q \subseteq A$ and $A - Q$ is a Sperner k-family of P.*

Proof. Let B be a Sperner k-family of P. Then $A = B \cup Q$ is a $(k+1)$-family of P' with

$$|A| = d_k(P) + \Delta_k(P) > d_k(P) + \Delta_{k+1}(P) = d_{k+1}(P).$$

Therefore, no Sperner $(k+1)$-family of P' is contained in P. Using Lemma 4.13, we can conclude that every Sperner $(k+1)$-family A of P' contains Q. Since $A - Q$ must be a k-family P', we obtain from the maximality of A that $|A - Q| = d_k(P)$. ∎

LEMMA 4.16. *Let $\mathscr{C}$ be a $(k+1, k)$-SCP of P' and let $\mathscr{C}'$ be the restriction of $\mathscr{C}$ to P. Then $\mathscr{C}'$ is a $(k, k-1)$-SCP of P.*

Proof. Let B be a Sperner k-family of P. Since $B \cup Q$ is a Sperner $(k+1)$-family of P', for each $C \in \mathscr{C}$ we have that $|(B \cup Q) \cap C| = k + 1$. Since Q is an antichain, $|Q \cap C| \leqslant 1$. Therefore $|B \cap C| \geqslant k$, from which we can conclude that $|B \cap C| = k$ and $|Q \cap C| = 1$. Hence every chain of $\mathscr{C}'$ meets every Sperner k-family in k points, so $\mathscr{C}'$ is k-saturated.

Next we let B be a Sperner $(k-1)$-family of P. Since $B \cup Q$ is a Sperner k-family of P', the above argument also establishes that $\mathscr{C}'$ is $(k-1)$-saturated. ∎

Let $\mathscr{C}$ be an arbitrary $(k, k-1)$-SCP of P. We know that $\mathscr{C}$ contains $\Delta_k(P)$ chains of length at least k, say $C_1, C_2, \dots, C_n$. Let $Q = \{q_1, q_2, \dots, q_n\}$ and define $\mathscr{C}'$ to be the chain partition of P' with nontrivial chains $\{C_i \cup \{q_i\}: 1 \leqslant i \leqslant n\}$.

LEMMA 4.17. *The chain partition $\mathscr{C}'$ of P' as defined above is simultaneously $(k+1, k)$-saturated.*

Proof. We calculate

$$\begin{aligned}\beta_{k+1}(\mathscr{C}') &= \sum_{C' \in \mathscr{C}'} \min[k+1, |C'|] \\ &= \Big(\sum_{C \in \mathscr{C}} \min[k, |C|]\Big) + n \\ &= d_k(P) + \Delta_k(P) = d_{k+1}(P').\end{aligned}$$

Therefore, $\mathscr{C}'$ is $(k+1)$-saturated. Similarly,

$$\begin{aligned}\beta_k(\mathscr{C}') &= \sum_{C' \in \mathscr{C}'} \min[k, |C'|] \\ &= \sum_{C \in \mathscr{C}} \min[k, |C|] \\ &= d_k(P) = d_k(P').\end{aligned}$$

Hence, $\mathscr{C}'$ is also k-saturated.

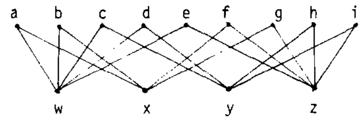

FIGURE 1

By Lemmas 4.17 and 4.18 it is evident that the bases of $G_k(P)$ and $G_{k+1}(P')$ are identical. Thus the elements of $Q = P' - P$ are loops in $G_{k+1}(P')$. Therefore G_{k+1} can be factored as the direct sum of th trivial matroid, $\mathscr{B}_{0,r}$, on Q and $G = G_k(P)$. ∎

We conclude this section with examples which show that $\mathscr{G}_0$ is not closed under restriction, contraction, duality, and that the sequence $\{G_k(P): k > 0\}$ does not uniquely determine the partial order P.

EXAMPLE. Let P be the partially ordered set whose graph is given in Fig. 1. $G_2(P)$ is a rank four transversal matroid with only the elements of $\{w, x, y, z\}$ being loops. Therefore the matroid $G' = G_2(P) - \{w\} = G_2(P)/\{w\}$ is also a rank four transversal matroid, since the class of transversal matroids is closed under restriction. Since G' contains only three loops, G' cannot be a member of $\mathscr{G}_k$ for $k > 1$. Also, the matroid $G'/\{x, y, z\} = G/\{w, x, y, z\}$ is not a cotransversal matroid. Since the class of cotransversal matroids is closed under contraction, G' cannot be cotransversal. Therefore $G' \notin \mathscr{G}_1$, and hence $G' \notin \mathscr{G}_0$.

EXAMPLE. Let P be the partially ordered set with graph shown in Fig. 2. $G_1(P)$ has rank three, contains no loops, and has the unique isthmus x. Thus $G = G_1(P)^*$ has rank three, no isthmuses and x is its unique loop. Since G contains no isthmuss, $G \notin \mathscr{G}_1$. But by Theorem 4.11 $G \notin \mathscr{G}_k$ for $k > 1$. So $G \notin \mathscr{G}_0$. (Figure 3)

$G_k(P_1) = G_k(P_2)$ for every positive integer k. However, there is no order-preserving map from P_1 into P_2.

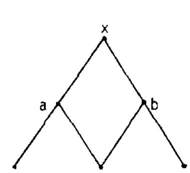

FIGURE 2

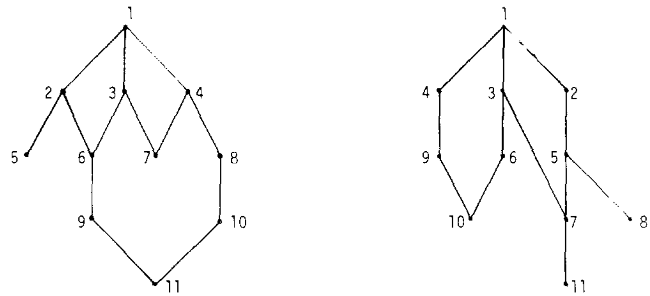

FIGURE 3

5. THE MATROIDS $G_k(\mathscr{B}_n)$ AND $G_k(V_n(q))$

In this section $\mathscr{B}_n$ and $V_n(q)$ will denote, respectively, the lattice of subsets of an n-element set and the lattice of subspaces of an n-dimensional vector space over $GF(q)$. A well-known result of Erdös [4] states that for every $k \leqslant n$, $d_k(\mathscr{B}_n)$ is the sum of the k largest binomial coefficients and that the only Sperner k-families are the obvious ones. A corresponding result holds for $V_n(q)$; $d_k(V_n(q))$ is the sum of the k largest Gaussian coefficients and, depending on the parity of k and n, there are either one or two Sperner k-families.

LEMMA 5.1. *Let $\mathscr{C}$ be a k-saturated chain partition of a partially ordered set P with unit Sperner k-family S. Let $A = \max[S]$. The restriction of $\mathscr{C}$ to $Q = \mathscr{F}(A)$ is a minimal chain partition of Q.*

Proof. Let $\mathscr{C}'$ be the restriction of $\mathscr{C}$ to Q. Since every element of $Q - A$ is on a nontrivial chain of $\mathscr{C}'$ containing a point of A,

$$\beta_1(\mathscr{C}') = |\mathscr{C}'| = |A| \leqslant d_1(Q). \quad \blacksquare$$

For both $\mathscr{B}_n$ and $V_n(q)$, the set of maximal elements in the unit Sperner k-family contains $\Delta_k(P)$ elements. We say that a partially ordered set is k-*principal* provided that the set of maximal elements in the unit Sperner k-family of P has cardinality $\Delta_k(P)$.

THEOREM 5.2. *Let P be a k-principal partially ordered set with unit Sperner k-family S and $A = \max[S]$. Then $G_k(P)$ is a principal matroid. Indeed, the loopless part of $G_k(P)$ is $G_1(Q)$ where $Q = \mathscr{F}(A)$.*

Proof. By Lemma 5.1, every basis of $G_k(P)$ is a basis of $G_1(Q)$. So we assume that B is a basis of $G_1(Q)$ corresponding to a minimal chain partition $\mathscr{C} = \{C_i : i \leqslant r\}$. Let $\mathscr{D} = \{D_i : i \leqslant r\}$ be a $(k, k-1)$-SCP of P with

$\mathscr{D}' = \{D_i' : i \leqslant r\}$ being its restriction to $\mathscr{I}(A)$. We assume $\mathscr{C}, \mathscr{D}, \mathscr{D}'$, and A are indexed so that for $i \leqslant r$, $C_i \cap A = D_i \cap A = D_i' \cap A = \{a_i\}$. For each $i \leqslant r$, $S \cap D_i \subset D_i'$, with $|S \cap D_i| = k$. Hence $\beta_j(\mathscr{D}) = \beta_j(\mathscr{D}')$ for $j = k, k-1$. Consider the chain partition $\mathscr{C}' = \{C_1 \cup D_i' : i \leqslant r\}$. Since we are extending chains of length at least k, we have that $\beta_j(\mathscr{C}') = \beta_j(\mathscr{D}')$ for $j = k, k-1$. Therefore $\mathscr{C}'$ is a $(k, k-1)$-*SCP* of P with set of tops B; i.e., B is a basis of $G_k(P)$. ∎

COROLLARY 5.3. *Each term of the sequences* $G_k(\mathscr{B}_n) : k \geqslant 1\}$ *and* $\{G_k(V_n(q)) : k \geqslant 1\}$ *is a principal matroid.*

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