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ORIGINAL ARTICLE

A new general fourth-order family of methods for finding simple roots of nonlinear equations

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Abstract In this paper, a new fourth-order family of methods free from second derivative is obtained. This new iterative family contains the King's family, which is one of the most well-known family of methods for solving nonlinear equations, and some other known methods as its particular case. To illustrate the efficiency and performance of proposed family, several numerical examples are presented. Numerical results illustrate better efficiency and performance of the presented methods in comparison with the other compared fourth-order methods. Due to that, they can be effectively used for solving nonlinear equations.

1. Introduction

In this paper, we propose an iterative methods to find a simple root a of a nonlinear equation $f(x) = 0$. i.e., $f(a) = 0$ and $f'(a) \neq 0$.

Nonlinear equations arise in a wide variety of forms in all branches of science, engineering, and technology. In recent years, a large number of root-searching algorithms and methods of solutions of different orders are available in the literature. For example, we refer the readers to Babolian and Biazar (2002), Frontini and Sormani (2003), He (1998), Abbasbandy (2006), King (1973), Chun (2007), Ostrowski (1973), Chun (2008) and the references therein.

It is well-known that the Newton's method is the most widely used (second-order) method for solving such equations, giving by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

Our approach is based on the King's fourth-order defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n) + (\alpha+2)f(y_n)}{f(x_n) + \alpha f(y_n)} \frac{f(y_n)}{f'(x_n)}, \tag{2}$$

where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{3}$$

(the notation of y_n have used throughout).

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This paper is organized as follows: In Section 2, we consider a general iterative scheme, analyze it to present a family of fourth-order methods then several known special cases of this family are listed. Section 3 is devoted to numerical comparisons between the results obtained in this work and some known iterative methods. Finally, conclusions are stated in the last section.

2. Development of method and convergence analysis

In this section to derive a fourth-order family of methods, motivated by King's family (2), we suggest and analyze the following iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{Af(x_n)^2 + Bf(y_n)f(x_n) + Cf(y_n)^2}{Df(x_n)^2 + Ef(y_n)f(x_n) + Ff(y_n)^2}\frac{f(y_n)}{f'(x_n)}, \tag{4}$$

where parameters A, B, C, D, E and F to be determined such that the iterative method defined by (4) has the order of convergence four. It can be obviously followed if the parameters are properly chosen as $A = D = 1$, $C = F = 0$, $B = E + 2$ then the iterative scheme (4), reduces to the King's family.

For the proposed family of methods (4) we have following analysis of convergence.

Theorem 1. *Let $a \in I$ be a simple root of a sufficiently differentiable function $f : I \to \Re$ on an open interval which contains x_0 as a close initial approximation to a. In the case of*

$A = B \neq 0$ and $B = E + 2A$,

the family of methods defined by (4), is of order four.

Proof. Using Taylor expansion of $f(x_n)$ about a and defining $e_n = x_n - a$, we have

$$f(x_n) = f'(a)\left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5)\right], \tag{5}$$

$$f'(x_n) = f'(a)\left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + O(e_n^4)\right], \tag{6}$$

where $c_k = f^{(k)}(a)/k!f'(a), k = 2,3,\ldots$

Substitution of (5) and (6) into (3), leads to

$$y_n = a + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (-7c_2c_3 + 4c_2^3 + 3c_4)e_n^4 + O(e_n^5). \tag{7}$$

Besides, by Taylor's expansion, we have

$$\begin{aligned} f(y_n) = f'(a)\big[&c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 \\ &+(-7c_3c_2 + 5c_2^3 + 3c_4)e_n^4 + O(e_n^5)\big]. \end{aligned} \tag{8}$$

Finally, using Eqs. (5)–(8) in (4) and some symbolic computational in Maple, we have

$$x_{n+1} = x_n - e_n + K_1e_n^2 + K_2e_n^3 + O(e_n^4),$$

where

$$K_1 = -\frac{(D - A)c_2}{D},$$

$$K_2 = \frac{2D(D - A)c_3 + (-2D^2 + 4AD - BD + AE)c_2^2}{D^2}.$$

It can be easily followed that, K_1, K_2 can be vanished, whenever

$A = D \neq 0$ and $B = E + 2A$. □

Setting the parameters of A, B, C, D, E and F which satisfying the conditions of theorem in (4) then introducing $\alpha = E/A$, $\theta = C/A$ and $\beta = F/A$ leads to the following three-parameter family of fourth-order methods

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)^2 + (2+\alpha)f(y_n)f(x_n) + \theta f(y_n)^2}{f(x_n)^2 + \alpha f(y_n)f(x_n) + \beta f(y_n)^2}\frac{f(y_n)}{f'(x_n)}, \tag{9}$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$ and $\alpha, \beta, \theta \in \Re$.

Family (9) contains some known fourth-order methods as particular cases, as follows:

Case 1: For $\theta = \beta = 0$ and $\alpha \in \Re$, we obtain the King's family of methods (1).

Case 2: For $\theta = \beta = 0$, and $\alpha = -2$, we obtain a fourth-order family of methods:

$$x_{n+1} = x_n - \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)}\frac{f(x_n)}{f'(x_n)}, \tag{10}$$

which was obtained by Ostrowski (1973).

Case 3: For $\theta = 0$, $\alpha = -2$ and $\beta = 2\beta' \in \Re$, we obtain following fourth-order family of methods:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)^2}{f(x_n)^2 - 2f(x_n)f(y_n) + 2\beta' f(y_n)^2}\frac{f(y_n)}{f'(x_n)},$$

and its two members with $\beta = 1/2$ and $\beta = 1$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \left[\frac{f(x_n)}{f(x_n) - f(y_n)}\right]^2\frac{f(y_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)^2}{f(x_n)^2 - 2f(x_n)f(y_n) + 2f(y_n)^2}\frac{f(y_n)}{f'(x_n)},$$

respectively, which were introduced by Chun (2007).

Case 4: For $\alpha = -1/2$, $\beta = -1/4$ and $\theta = 3/4$, we obtain a fourth-order method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{4f(x_n)^2 + 6f(y_n)f(x_n) + 3f(y_n)^2}{4f(x_n)^2 - 2f(y_n)f(x_n) - f(y_n)^2}\frac{f(y_n)}{f'(x_n)},$$

which was introduced by Chun (2008).

Case 5: For $\alpha = \beta = \theta = 0$, we obtain a fourth-order method

$$x_{n+1} = x_n - \left[1 + \frac{f(y_n)}{f(x_n)} + 2\frac{f(y_n)^2}{f(x_n)^2}\right]\frac{f(x_n)}{f\prime(x_n)}, \tag{11}$$

which was introduced by Chun (2008).

Case 6: For $\alpha = \beta = 0$ and $\theta = 1$, we obtain a fourth-order method:

$$x_{n+1} = x_n - \left[1 + \frac{f(y_n)}{f(x_n)} + 2\frac{f(y_n)^2}{f(x_n)^2} + \frac{f(y_n)^3}{f(x_n)^3}\right]\frac{f(x_n)}{f'(x_n)}, \tag{12}$$

which was obtained by Chun (2008).

Case 7: For $\theta = \beta = 0$ and $\alpha = -1$, we obtain a fourth-order method:

$$x_{n+1} = x_n - \frac{f(x_n)^2 + f(y_n)^2}{f\prime(x_n)(f(x_n) - f(y_n))},$$

which was obtained by Kou et al. (2007).

Case 8: For $\theta = \beta = 0$ and $\alpha = -5/2, -3/2, -7/2$, respectively, we have

Table 1 Comparison of various fourth-order convergent iterative methods.

	Number of iterations						Number of function evaluations					
	NM	KM	CM1	CM2	BGM1	BGM2	NM	KM	CM1	CM2	BGM1	BGM2
$f_1, x_0 = -0.3$	55	49	25	25	8	7	110	147	75	75	24	21
$f_1, x_0 = 1$	6	4	4	4	3	3	12	12	12	12	9	9
$f_2, x_0 = 0$	5	3	3	3	2	2	10	9	9	9	6	6
$f_2, x_0 = 1$	5	3	3	3	3	3	10	9	9	9	9	9
$f_3, x_0 = -1$	6	5	5	3	3	4	12	15	15	9	12	9
$f_3, x_0 = -2$	9	5	5	6	6	5	18	15	18	18	15	15
$f_4, x_0 = 2$	6	5	5	5	4	4	12	12	15	15	12	12
$f_4, \mathrm{v}x_0 = -5$	8	5	5	5	5	5	16	15	15	15	15	15
$f_5, x_0 = 3$	7	4	4	4	3	3	14	12	12	12	12	12
$f_5, x_0 = 4$	8	5	5	5	4	4	16	15	15	15	12	12
$f_6, x_0 = 2$	9	5	6	6	5	5	18	15	18	18	15	15
$f_6, x_0 = 3.5$	11	6	7	7	6	6	22	18	21	21	18	18
$f_7, x_0 = 1$	7	4	5	5	6	4	14	12	15	15	18	12
$f_7, x_0 = 2$	6	4	4	4	3	3	12	12	12	12	9	9

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(x_n)}\frac{f(y_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f(x_n) + f(y_n)}{2f(x_n) - 3f(x_n)}\frac{f(y_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f(x_n) - 3f(y_n)}{2f(x_n) - 7f(x_n)}\frac{f(y_n)}{f'(x_n)},$$

which were introduced by Chun and Ham (2008) (they are actually the special cases of the King's family).

Furthermore, family (9) introduces some new cases, as other its particular cases. For example:

In the Case of $\theta = 4$, $\alpha = 2$ and $\beta = 1$, we obtain the following fourth-order method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \left[\frac{f(x_n) + 2f(y_n)}{f(x_n) + f(y_n)}\right]^2 \frac{f(y_n)}{f'(x_n)}. \tag{13}$$

In the Case of $\alpha = -2$, $\beta = 1$ and $\theta = -1$, we obtain the following fourth-order method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n) + f(y_n)}{f(x_n) - f(x_n)}\frac{f(y_n)}{f'(x_n)}. \tag{14}$$

Obviously, the number of function evaluations per iteration required in the methods defined by (4) is three. We consider the definition of efficiency index (Grau and Diaz, 2006) as $\sqrt[r]{p}$, where p is the order of the method and r is the number of function evaluations per iteration required by the method. We have that the family of methods defined by (4) has the efficiency index equal to $\sqrt[3]{4} \approx 1.5874$, which is much better than the $\sqrt{2} \approx 1.4142$ of Newton's method.

3. Numerical examples

In this section, some numerical test of some various root-finding methods as well as our new methods and Newton's method are presented. Compared methods were Newton's method (1) (NM), King' method with $\beta = 3$ (KM), Chun' method (11) (CM1) and another Chun' method (12) (CM2) with the new presented methods by Eq. (13) (BGM1) and Eq. (14) (BGM2), introduced in this contribution. All computations were done using MAPLE with 128 digit floating point arithmetics (Digits = 128). Displayed in Table 1 are the number of iterations and functional evaluations required such that $|f(x_n)| < 10^{-15}$. The following functions (which are taken from Ostrowski, 1973; Chun, 2008; Kou et al., 2007), are used for the comparison and we display the approximate zeros x_* found, up to the 28th decimal place.

$f_1(x) = x^3 + 4x^2 - 10, x_* = 1.3652300134140968457608068290,$

$f_2(x) = x^2 - e^x - 3x + 2,$

$x_* = 0.2575302854398607604553673049$

$f_3(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5,$

$x_* = -1.2076478271309189270094167584,$

$f_4(x) = \sin(x)e^x + \ln(x^2 + 1), x_* = 0,$

$f_5(x) = (x - 1)^3 - 2, x_* = 2.2599210498948731647672106073,$

$f_6(x) = (x + 2)e^x - 1,$

$x_* = -0.4428544010023885831413280000,$

$f_7(x) = \sin^2(x) - x^2 + 1,$

$x_* = 1.4044916482153412260350868178.$

The results presented in Table 1 show that for the functions we tested, the new methods introduced in this contribution need reduce the number of iterations and needed functional evaluations show that this family can be competitive to the known fourth-order methods and Newton's method and converge more quickly than the other compared methods.

4. Conclusion

In this paper, we have constructed a new general fourth-order iterative family of methods for solving nonlinear equations with three parameters. This proposed iterative family contains King's family and several well-known methods as special case. This family has the advantage that one has more freedom than King' family to choose the parameters. It is noteworthy that the presented methods show better performance and faster convergence than the King's method and some recent its variants.

Further researches are required to find the optimal values of parameters to achieve faster convergence can be considered as the next studies in this field.

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