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A Note on the Range of a Vector Measure; Application to the Theory of Optimal Control

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I. INTRODUCTION

Let χ be a compact metric space, and y^i, $i = 1, 2, \cdots, k$, be continuous vector valued functions on χ into E^n. Let μ be a nonnegative, finite regular measure on the Borel subsets of χ.[1] For each $x \in \chi$, we denote by $\mathscr{A}(x)$ the set $\{y^1(x), \cdots, y^k(x)\}$, and by co $\mathscr{A}(x)$ the convex hull of $\mathscr{A}(x)$.

With each μ measurable function z defined on χ associate the vector $v(z) = (\int_\chi z_1(x)\, d\mu, \cdots, \int_\chi z_n(x)\, d\mu)$ and denote by $\mathscr{R}_{\mathscr{A}}$ the range, in E^n, of $v(z)$ for measurable functions z such that $z(x) \in \mathscr{A}(x)$, while $\mathscr{R}_{\text{co}\mathscr{A}}$ denotes the range of $v(z)$ for measurable functions z such that $z(x) \in \text{co}\, \mathscr{A}(x)$.

It is shown that $\mathscr{R}_{\mathscr{A}} = \mathscr{R}_{\text{co}\mathscr{A}}$.

This result will be applied to obtain a uniform approximation theorem of the form found in [2], and also to obtain an extended bang-bang principle of the type given in [3], for problems in which the range set of the control vector may vary with time.

II. STUDY OF $\mathscr{R}_{\mathscr{A}}$ AND $\mathscr{R}_{\text{co}\mathscr{A}}$

It is evident that $\mathscr{R}_{\text{co}\mathscr{A}} \supset \mathscr{R}_{\mathscr{A}}$, hence it will be necessary to show that for any μ measurable function z, with $z(x) \in \text{co}\, \mathscr{A}(x)$, we can produce a μ measurable function y, with $y(x) \in \mathscr{A}(x)$, such that $v(y) = v(z)$. This will necessitate a representation of z in terms of the y^i which will make use of the following extension of a

LEMMA OF FILIPPOV [4]. *Let the vector valued function $f(x, \alpha_1, \cdots, \alpha_k)$, or more concisely $f(x, \alpha)$, be continuous on $\chi \times Q(x)$ to E^n, where for each $x \in \chi$, the set $Q(x) \subset E^k$ is closed, bounded, and upper semicontinuous with*

[1] All definitions will be consistent with those given in [1].

respect to set inclusion, in x. Let $R(x)$ denote the set $f(x, Q(x))$ and $z(x)$ be a μ measurable function such that $z(x) \in R(x)$. Then there exists a μ measurable function α such that $\alpha(x) \in Q(x)$ and $f(x, \alpha(x)) = z(x)$ for almost all $x \in \chi$.

The proof of this lemma follows word for word the proof given by Filippov, where χ was a closed interval in E^1, and the measure was Lebesgue measure. It should be noted that his proof requires: for any $\epsilon > 0$, a measurable function on χ be continuous on a closed subset of χ, with measure differing from that of χ by at most ϵ. This is assured by the assumption that μ is a nonnegative, finite regular measure. It is also required that closed sets of χ be μ measurable, a consequence of the definition of a Borel field.

LEMMA 1. *If z is any μ measurable function on χ with values in E^n, such that $z(x) \in \text{co}\, \mathscr{A}(x)$ for each $x \in \chi$, then z admits a representation of the form*

$$z(x) = \sum_{i=1}^{k} \alpha_i(x)\, y^i(x) \tag{1}$$

where the scalar valued functions α_i are μ measurable and $0 \leqslant \alpha_i(x) \leqslant 1$, $\sum_{i=1}^{k} \alpha_i(x) = 1$ for each $x \in \chi$.

PROOF: Since $z(x) \in \text{co}\, \mathscr{A}(x)$, for each x the representation (1) is certainly valid. It remains, therefore, to show that this can be accomplished with μ measurable functions α_i.

Let Q be the simplex defined by $Q = \{\alpha \in E^k\colon \sum_{i=1}^{k} \alpha_i = 1,\ 0 \leqslant \alpha_i \leqslant 1\}$. Thus Q is closed and bounded. Define

$$f(x, \alpha) = \sum_{i=1}^{k} \alpha_i y^i(x) \qquad \text{for} \qquad (x, \alpha) \in \chi \times Q.$$

Then f is continuous. If z is any measurable function with $z(x)$ belonging to the set $f(x, Q)$, the extended lemma of Filippov states that z can be represented as $z(x) = f(x, \alpha(x))$ for almost all x, where α is a μ measurable function of x with $\alpha(x) \in Q$.

THEOREM 1. $\mathscr{R}_{\mathscr{A}} = \mathscr{R}_{\text{co}\mathscr{A}}$.

PROOF: Certainly $\mathscr{R}_{\text{co}\mathscr{A}} \supset \mathscr{R}_{\mathscr{A}}$. Let z be any measurable function with $z(x) \in \text{co}\, \mathscr{A}(x)$. We must produce a measurable function y, with $y(x) \in \mathscr{A}(x)$, such that $v(z) = v(y)$.

By Lemma 1, z admits the representation $z(x) = \sum_{i=1}^{k} \alpha_i(x) y^i(x)$, where the α_i are measurable and satisfy $0 \leqslant \alpha_i(x) \leqslant 1$, $\sum_{i=1}^{k} \alpha_i(x) = 1$.

For any measurable subset $E \subset \chi$ define $\mu_{ij}(E) = \int_E y_j^i(x)\, d\mu$, $i = 1, 2, \cdots, k$; $j = 1, 2, \cdots, n$, where y_j^i is the jth component of the vector y^i.

Consider the k^2n dimensional vector $\omega(x)$ defined by

$$\omega(x) = \Big(\int_\chi \alpha_1(x)\, d\mu_{11}, \cdots, \int_\chi \alpha_1(x)\, d\mu_{1n}, \int_\chi \alpha_1(x)\, d\mu_{21}, \cdots, \int_\chi \alpha_1(x)\, d\mu_{kn},$$
$$\int_\chi \alpha_2(x)\, d\mu_{11}, \cdots, \int_\chi \alpha_k(x)\, d\mu_{kn}\Big).$$

By Theorem 4 [5], there exists a measurable vector α^* with $\alpha_i^*(x) = \{^1_0$ and $\sum_{i=1}^k \alpha_i^*(x) = 1$, such that $\omega(\alpha^*) = \omega(\alpha)$. In particular,

$$\Big(\int_\chi \alpha_1^*(x)\, d\mu_{11}, \cdots, \int_\chi \alpha_1^*(x)\, d\mu_{1n}\Big) = \Big(\int_\chi \alpha_1(x)\, d\mu_{11}, \cdots, \int_\chi \alpha_1(x)\, d\mu_{1n}\Big)$$
$$\Big(\int_\chi \alpha_2^*(x)\, d\mu_{21}, \cdots, \int_\chi \alpha_2^*(x)\, d\mu_{2n}\Big) = \Big(\int_\chi \alpha_2(x)\, d\mu_{21}, \cdots, \int_\chi \alpha_2(x)\, d\mu_{2n}\Big)$$
$$\vdots$$
$$\Big(\int_\chi \alpha_k^*(x)\, d\mu_{k1}, \cdots, \int_\chi \alpha_k^*(x)\, d\mu_{kn}\Big) = \Big(\int_\chi \alpha_k(x)\, d\mu_{k1}, \cdots, \int_\chi \alpha_k(x)\, d\mu_{kn}\Big).$$

Thus

$$\int_\chi z(x)\, d\mu = \int_\chi \sum_{i=1}^k \alpha_i(x)\, y^i(x)\, d\mu = \int_\chi \sum_{i=1}^k \alpha_i^*(x)\, y^i(x)\, d\mu.$$

Let $I_i = \{x \in \chi : \alpha_i^*(x) = 1\}$, $i = 1, 2, \cdots, k$. Then each set I_i is measurable and $\bigcup_{i=1}^k I_i = \chi$, $I_i \cap I_j = \varnothing$ if $i \neq j$. Define $y(x) = y^i(x)$ for $x \in I_i$. Then y is measurable, $y(x) \in \mathscr{A}(x)$ and $v(y) = v(z)$, which completes the proof.

Corollary. *Let $\mathscr{B}(x)$ be any set in E^n such that $\mathscr{A}(x) \subset \mathscr{B}(x)$ and* co $\mathscr{A}(x) =$ co $\mathscr{B}(x)$ *for each $x \in \chi$. If $\mathscr{R}_{\mathscr{B}}$, $\mathscr{R}_{\mathrm{co}\mathscr{B}}$ denote the ranges, in E^n, of $v(z)$ for measurable z such that $z(x) \in \mathscr{B}(x)$ and $z(x) \in$* co $\mathscr{B}(x)$ *respectively, then $\mathscr{R}_{\mathscr{A}} = \mathscr{R}_{\mathscr{B}} = \mathscr{R}_{\mathrm{co}\mathscr{B}} = \mathscr{R}_{\mathrm{co}\mathscr{A}}$.*

Proof: Since $\mathscr{A}(x) \subset \mathscr{B}(x) \subset \mathrm{co}\,\mathscr{B}(x) = \mathrm{co}\,\mathscr{A}(x)$, it follows that $\mathscr{R}_{\mathscr{A}} \subset \mathscr{R}_{\mathscr{B}} \subset \mathscr{R}_{\mathrm{co}\mathscr{B}} = \mathscr{R}_{\mathrm{co}\mathscr{A}}$. But by Theorem 1, $\mathscr{R}_{\mathscr{A}} = \mathscr{R}_{\mathrm{co}\mathscr{A}}$.

III. Applications to the Theory of Optimal Control

In this section χ will be the interval $[0, t_1] \subset E^1$ and μ Lebesgue measure. A control u will be a vector valued, measurable function defined on $[0, t_1]$ with value $u(t)$ in a given set $U(t) \subset E^r$.

We will first consider the system

$$\dot{x}(t) = A(t)\,x(t) + b(t, u(t)), \;\; x(0) = x^0 \tag{2}$$

where dot denotes differentiation with respect to t, x is an n vector, A is a measurable $n \times n$ matrix valued function of t, summable over $[0,t_1]$, while b is an n vector valued function, continuous on $[0, t_1] \times E^r$.

Assume there exist continuous controls $u^1, u^2, \cdots, u^k$ such that if $\mathscr{A}(t) = \{b(t, u^1(t)), \cdots, b(t, u^k(t))\}$ while $\mathscr{B}(t) = \{b(t, \sigma) : \sigma \in U(t)\}$, then $\operatorname{co} \mathscr{A}(t) = \operatorname{co} \mathscr{B}(t)$; $t \in [0, t_1]$.

We will show that if a state x^* can be attained in time $t^* \in [0, t_1]$ by an arbitrary control, i.e., for some control u the solution φ^u (which exists and is unique in the class of absolutely continuous functions) of (2) is such that $\varphi^u(t^*) = x^*$, then there is a control v which at each time $t \in [0, t^*]$ assumes one of the values $u^1(t), \cdots, u^k(t)$ and is such that $\varphi^v(t^*) = x^*$.

We will call a control v with range at t restricted to the values $u^1(t), \cdots, u^k(t)$ a *restricted control.*

Results of this type for an equation of the form (2) were obtained in [3].

To prove the preceding assertion, let $X(t)$ denote a fundamental solution of the homogeneous equation $\dot{x} = A(t)x$. Then for any control u, the solution of (2) is given by

$$\varphi^u(t) = X(t)\,x^0 + X(t) \int_0^t X^{-1}(\tau)\, b(\tau, u(\tau))\, d\tau, \; t \in [0, t_1].$$

Let u be a control such that $\varphi^u(t^*) = x^*$. It will suffice to show there exists a restricted control v such that

$$\int_0^{t^*} X^{-1}(\tau)\, b(\tau, u(\tau))\, d\tau = \int_0^{t^*} X^{-1}(\tau)\, b(\tau, v(\tau))\, d\tau.$$

Define

$$\mathscr{A}^*(t) = \{X^{-1}(t)\, b(t, u^1(t)), \cdots, X^{-1}(t)\, b(t, u^k(t))\}$$

$$\mathscr{B}^*(t) = \{X^{-1}(t)\, b(t, \sigma) : \sigma \in U(t)\}.$$

Since $X^{-1}(t)$ is a linear operator, and $\operatorname{co} \mathscr{A}(t) = \operatorname{co} \mathscr{B}(t)$, it follows that $\operatorname{co} \mathscr{A}^*(t) = \operatorname{co} \mathscr{B}^*(t)$ for each $t \in [0, t^*]$. The corollary to Theorem 1 yields $\mathscr{R}_{\mathscr{A}^*} = \mathscr{R}_{\operatorname{co}\mathscr{B}^*}$. Thus there is a measurable function y such that $y(t) \in \mathscr{A}^*(t)$ and $\int_0^{t^*} y(\tau)\, d\tau = \int_0^{t^*} X^{-1}(\tau) b(\tau, u(\tau))\, d\tau$. By Filippov's lemma y can be realized in the form $y(t) = X^{-1}(t) b(t, v(t))$ for almost all t, where the measurable function v is such that $v(t) \in \{u^1(t), \cdots, u^k(t)\}$. This completes the argument.

We next consider the system

$$\dot{x}(t) = f(t, x(t), u(t)), \; x(0) = x^0. \tag{3}$$

It will be assumed that each component f_i of the n vector valued function f is bounded by M in absolute value, continuous in all arguments, and satisfies a uniform Lipschitz condition of the form

$$| f_i(t, x, u) - f_i(t, \bar{x}, u) | \leqslant K \| x - \bar{x} \|,$$

where

$$\| x - \bar{x} \| = \sum_{i=1}^{n} | x_i - \bar{x}_i |.$$

Again, for any control u, a unique solution of (3) exists in the class of absolutely continuous functions.

Assume $u^1, u^2, \cdots, u^k$ are k continuous controls such that if $\mathscr{A}(t, x) = \{f(t, x, u^1(t)), \cdots, f(t, x, u^k(t))\}$ while $\mathscr{B}(t, x) = \{f(t, x, \sigma) : \sigma \in U(t)\}$, then $\text{co}\,\mathscr{A}(t, x) = \text{co}\,\mathscr{B}(t, x)$, for all $(t, x) \in [0, t_1] \times E^n$.

We will show that if given any $\epsilon > 0$ and an arbitrary control u, there exists a restricted control v such that $\| \varphi^u(t) - \varphi^v(t) \| < \epsilon$ for all $0 \leqslant t \leqslant t_1$.

Let φ^u be the trajectory to be approximated and, for the moment, let v be an arbitrary control. Then

$$\begin{aligned} \varphi^u(t) - \varphi^v(t) &= \int_0^t [f(\tau, \varphi^u(\tau), u(\tau)) - f(\tau, \varphi^v(\tau), v(\tau))]\, d\tau \\ &= \int_0^t [f(\tau, \varphi^u(\tau), u(\tau)) - f(\tau, \varphi^u(\tau), v(\tau))]\, d\tau \\ &\quad + \int_0^t [f(\tau, \varphi^u(\tau), v(\tau)) - f(\tau, \varphi^v(t), v(\tau))]\, d\tau. \end{aligned}$$

Using the assumed Lipschitz condition we obtain

$$\begin{aligned} \| \varphi^u(t) - \varphi^v(t) \| \leqslant & \left\| \int_0^t [f(\tau, \varphi^u(\tau), u(\tau)) - f(\tau, \varphi^u(\tau), v(\tau))\, d\tau \right\| \\ & + nK \int_0^t \| \varphi^u(\tau) - \varphi^v(\tau) \|\, d\tau. \end{aligned} \tag{4}$$

It can now be seen that the result would be an immediate consequence of the Gronwall inequality if we could show the existence of a restricted v which makes the term $\| \int_0^t [f(\tau, \varphi^u(\tau), u(\tau)) - f(\tau, \varphi^u(\tau), v(\tau))]\, d\tau \|$ arbitrarily small, uniformly for $t \in [0, t_1]$.

Subdivide the interval $[0, t_1]$ into m equal subintervals each of length $\delta = t_1/m$. Let I_j denote the jth subinterval. By the corollary to Theorem 1, for each j there exists a measurable function y^j defined on I_j with values $y^j(t) \in \mathscr{A}(t, \varphi^u(t))$ such that

$$\int_{I_j} f(\tau, \varphi^u(\tau), u(\tau))\, d\tau = \int_{I_j} y^j(\tau)\, d\tau. \tag{5}$$

By Filippov's lemma y^j can be realized in the form

$$\dot{y}^j(t) = f(t, \varphi^u(t), v^j(t)) \tag{6}$$

for almost all t in I_j, where v^j is a restricted control on I_j. Define $v(t) = v^j(t)$ for $t \in I_j$, $j = 1, 2, \cdots, m$.

For any $t \in [0, t_1]$ let ν be an integer such that $\nu\delta < t \leqslant (\nu + 1)\delta$. Using (5) and (6)

$$\left\| \int_0^t [f(\tau, \varphi^u(\tau), u(\tau)) - f(\tau, \varphi^u(\tau), v(\tau)) \, d\tau] \right\|$$

$$= \left\| \int_{\nu\delta}^t [f(\tau, \varphi^u(\tau), u(\tau)) - f(\tau, \varphi^u(\tau), v(\tau))] \, d\tau \right\| \leqslant 2nt_1M/m,$$

M being the bound on the components of the vector f. If given any $\epsilon_1 > 0$, one can choose $m > 2nt_1M/\epsilon_1$, thereby obtaining

$$\left\| \int_0^t [f(\tau, \varphi^u(\tau), u(\tau)) - f(\tau, \varphi^u(\tau), v(\tau))] \, d\tau \right\| < \epsilon_1, \; t \in [0, t_1].$$

Using this bound in (4), the Gronwall inequality yields $\| \varphi^u(t) - \varphi^v(t) \| \leqslant \epsilon_1 e^{nKt_1}$ for all $t \in [0, t_1]$, which completes the argument.

Other results concerning uniform approximation theorems can be found in [2].

References

1. Dunford, N., and Schwartz, J. T., "Linear Operators." Interscience, New York, 1958.
2. Warga, J., Relaxed Variational Problems, *J. Math. Anal. Appl.* **4**, 111-128 (1962).
3. Neustadt, L. W., The existence of optimal controls in the absence of convexity conditions. Aerospace Rept. A-62-1732.1-16. (To appear in *J. Math. Anal. Appl.*)
4. Filippov, A. F., On certain questions in the theory of optimal control. Transl. *J. Soc. Ind. Appl. Math.* Series on Control, 1, No. 1 (1962).
5. Dvoretzky, A., Wald, A., and Wolfowitz, J., Relations among certain ranges of vector measures. *Pacific J. Math.* 1 59-74 (1951).
6. Blackwell, D., The range of certain vector integrals. *Proc. Am. Math. Soc.* No. 3, 390-395 (1951).
7. Eggleston, H. C., "Convexity," Cambridge Tracts in Math. and Math. Physics No. 47. Cambridge Univ. Press, 1958.
8. LaSalle, J. P., The time optimal control problem. "Contributions to the Theory of Nonlinear Oscillations," Vol. V, pp. 1-24. Princeton Univ. Press, Princeton, New Jersey, 1960.