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Table Algebras and Applications to Products of Characters in Finite Groups*

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1. INTRODUCTION

We use a concept called "table algebra," introduced by two of the authors in [AB], to state and prove a general theorem which implies the following two results as corollaries:

(a) In any nonabelian finite simple group G, the product of all the distinct irreducible characters of G contains all the nontrivial irreducible characters of G as constituents.

(b) In any nonabelian finite simple group G, the product of all the distinct conjugacy classes of G contains $G - \{1\}$.

The first result is a partial solution to a conjecture stated in [ALG] and is a generalization of Theorem 1.5 in [ACH]. For the complete solution of the conjecture one would have to prove (a) with the word "nontrivial" omitted. We show in Section 3 that the conjecture holds for all finite simple groups of Lie type and for all 26 sporadic groups. Thus, using the classification theorem for nonabelian simple groups, it remains open only for the family A_n.[1] Furthermore, for simple groups of Lie type G, $\prod_{\chi \in Irr(G)} \chi = m\rho_G$, where ρ_G is the regular character and $m > 0$. The second

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[1] The conjecture for A_n has been verified recently by I. Zisser.

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result (b) is well known and is due to Brauer and Wielandt [F, p. 37]. We need to define some terms in order to state our main theorem.

DEFINITION. A *table algebra* $(A, \mathbf{B})$ is a finite dimensional, commutative, associative algebra A with identity element 1 over the complex numbers $\mathbb{C}$, and a distinguished basis $\mathbf{B} = \{b_1 = 1, b_2, ..., b_k\}$ such that the following properties hold (where (b_i, a) denotes the coefficient of b_i in $a \in A$, a written as a linear combination of $\mathbf{B}$; and where $\mathbb{R}^*$ denotes $\mathbb{R}^+ \cup \{0\}$, the set of non-negative real numbers):

(I) For all i, j, m, $b_i b_j = \sum_m \lambda_{ijm} b_m$ with λ_{ijm} a non-negative real number.

(II) There is an algebra automorphism (denoted) of A, whose order divides 2, such that $b_i \in \mathbf{B}$ implies that $\bar{b}_i \in \mathbf{B}$. (Then $\bar{i}$ is defined by $\bar{b}_i = b_{\bar{i}}$.)

(III) Hypothesis (II) holds and there is a function $g: \mathbf{B} \times \mathbf{B} \to \mathbb{R}^+$ (the positive reals) such that

$$(b_m, b_i b_j) = g(b_i, b_m) \cdot (b_i, \bar{b}_j b_m),$$

where $g(b_i, b_m)$ is independent of j, for all i, j, m.

$\mathbf{B}$ is called the *table basis* of $(A, \mathbf{B})$. The elements of $\mathbf{B}$ are called the *irreducible components* of $(A, \mathbf{B})$, and nonzero combinations of elements of $\mathbf{B}$ with coefficients in $\mathbb{R}^+$ are called *components*. If $a = \sum_{m=1}^{k} \lambda_m b_m$ is a component ($\lambda_m \in \mathbb{R}^*$) then $Irr(a) := \{b_m \mid \lambda_m \neq 0\}$ is called the set of *irreducible constituents* of a. An element $a \in A$ satisfying $a = \bar{a}$ is called a *real* element.

Proposition 2.2 of [AB] shows that if $(A, \mathbf{B})$ is a table algebra, then there exists a basis $\mathbf{B}'$, which consists of suitable positive real scalar multiples b_i' of the elements b_i of $\mathbf{B}$, such that $(A, \mathbf{B}')$ is a table algebra with $g'(b_i', b_j') = 1$ for all b_i', b_j' in $\mathbf{B}'$. Such a basis $\mathbf{B}'$ is called *normalized*. Now $Irr(b_{i_1}' b_{i_2}' \cdots b_{i_t}')$ consists of the corresponding scalar multiples of the elements of $Irr(b_{i_1} b_{i_2} \cdots b_{i_t})$, for any sequence $i_1, i_2, ..., i_t$ of indices. So in the proof of any theorem which identifies the irreducible constituents of a product of basis elements, we may assume that $\mathbf{B}$ is normalized.

Suppose that $\mathbf{B}$ is normalized. It follows from (III), as in [AB, Sect. 2], that A has a positive definite Hermitian form, with $\mathbf{B}$ as an orthonormal basis, and such that

$$(a, bc) = (a\bar{b}, c)$$

for all $a, b, c \in \mathbb{R}\mathbf{B}$.

If G is a finite group then the algebra $Ch(G)$ generated by $\mathbf{B} = Irr(G)$ over $\mathbb{C}$ and the algebra $Z(\mathbb{C}[G])$ generated by the conjugacy class sums of

G are examples of table algebras. Recently, several results were proved concerning products of irreducible characters and products of conjugacy classes in finite groups which demonstrated, sometimes, an analogy between the two concepts. A full survey and bibliography about this topic can be found in [A]. The concept of table algebra led us in [AB] to form unified proofs of these analogous theorems. Table algebras are shown in [AB, Theorem 2.10] to be equivalent to a particular type of C-algebra (see [BI]). The notion of C-algebra is nearly 50 years old, but has only recently become one of interest in the study of commutative association schemes and other generalizations of the character ring and class algebra of a finite group.

Our main result is as follows:

THEOREM A. *Let* $(A, \mathbf{B})$ *be a table algebra and let* $a \in \mathbf{B}$. *Then for every positive integer* i,

$$a \in Irr\left(\prod_{b \in Irr(a\bar{a})^{i} \cup \{a\}} b\right).$$

Our first consequence is

COROLLARY 1. *Let* $(A, \mathbf{B})$ *be a table algebra and* $b_i \in \mathbf{B}$. *Assume that there exists a positive integer* n_i *such that* $\mathbf{B} = Irr(b_i^{n_i})$. *Then* $b_i \in Irr(\prod_{a \in \mathbf{B}} a)$.

To state Corollary 2 we need the following definitions from [AB]. For each of them, (A, $\mathbf{B}$) denotes a table algebra.

DEFINITION. A subset $\mathbf{D} \subseteq \mathbf{B}$ is called a *table subset* of $\mathbf{B}$ if $\mathbf{D} \neq \varnothing$ and $Irr(b_i b_j) \subseteq \mathbf{D}$ for all $b_i, b_j \in \mathbf{D}$. A subalgebra of A generated by some table subset of $\mathbf{B}$ is called a *table subalgebra* of $(A, \mathbf{B})$.

DEFINITION. $(A, \mathbf{B})$ is *simple* if the only table subsets of $\mathbf{B}$ are $\mathbf{B}$ and $\{1\}$.

DEFINITION. An element $b \in \mathbf{B}$ is called *linear* if $Irr(b^n) = \{1\}$ for some $n > 0$.

DEFINITION. $(A, \mathbf{B})$ is *abelian* if every element of $\mathbf{B}$ is linear.

As noted in [AB], the notions of "simple" and "abelian" coincide with the usual ones for the group G when $A = Ch(G)$ or $Z(\mathbb{C}G)$.

COROLLARY 2. *Let* $(A, \mathbf{B})$ *be a nonabelian simple table algebra. Then* $\mathbf{B} = Irr(\prod_{b \in \mathbf{B}} b) \cup \{1\}$.

Example (6) in [AB, Sect. 5] is that of a nonabelian simple table algebra $(A, \mathbf{B})$ with $\mathbf{B} = \{1, b, \bar{b}, c\}$ such that $Irr(b\bar{b}c) = \{b, \bar{b}, c\} = \mathbf{B} - \{1\}$. Thus the result of Corollary 2 is the best possible.

Results (a) and (b) mentioned above follow imediately from Corollary 2.

Some steps toward a general theory of table algebras and applications to finite group theory can be found in [AB]. We hope that further study of the properties of table algebras will be useful in obtaining new results about finite groups.

2. Proof of Theorem A

Let $(A, \mathbf{B})$ be a table algebra and fix $a \in \mathbf{B}$. In proving Theorem A, it suffices to assume that $a \neq 1$ and that $\mathbf{B}$ is normalized. We make these assumptions throughout this section. Thus $(b, cd) = (d, b\bar{c})$ for all components b, c, d of A.

Since $1 \in Irr(a\bar{a})$, it is clear that $Irr(a\bar{a})^{i+1} \supseteq Irr(a\bar{a})^i$ for all $i \in \mathbb{N}$ (the positive integers), and that if $Irr(a\bar{a})^{i+1} = Irr(a\bar{a})^i$ for some $i \in \mathbb{N}$ then $Irr(a\bar{a})^j = Irr(a\bar{a})^i$ for all $j \geqslant i$. Set $Irr(a\bar{a})^0 = \{1\}$ and $Irr(a\bar{a})^{-1} = \varnothing$. Note that for all $j \geqslant i \geqslant 0$, $Irr(a\bar{a})^j - Irr(a\bar{a})^i$ is stable under .

Lemma 1. *If H_1 and H_2 are two disjoint subsets of $\mathbf{B}$ with $a \in Irr(a \prod_{h \in H_i} h)$ for $i = 1, 2$, then $a \in Irr(a(\prod_{H_1 \cup H_2} h))$.*

Proof. It follows from the hypotheses that

$$a \in Irr\left(a \prod_{h \in H_1} h\right) \subseteq Irr\left[\left(a \prod_{h \in H_2} h\right) \prod_{h \in H_1} h\right] = Irr\left(a \prod_{h \in H_1 \cup H_2} h\right).$$

Lemma 2. *If $\{a\} \cup \{d\} \cup H_1 \cup H_2 \subseteq \mathbf{B}$ with $a \in Irr(ad(\prod_{H_i} h))$ for $i = 1, 2$, then there exists $c \in Irr(a\bar{a})$ such that*

$$a \in Irr\left(ac\left(\prod_{h \in H_1} \bar{h}\right)\left(\prod_{h \in H_2} h\right)\right).$$

Proof. By our assumption, $0 \neq (ad \prod_{h \in H_1} h, a) = (d, a\bar{a} \prod_{h \in H_1} \bar{h})$. So there exists $c \in Irr(a\bar{a})$ such that $0 \neq (d, c \prod_{h \in H_1} \bar{h})$. Hence $a \in Irr(ad \prod_{h \in H_2} h) \subseteq Irr[a(c\prod_{h \in H_1} \bar{h}) \prod_{h \in H_2} h]$, as desired.

Lemma 3. *If $b \in Irr(a\bar{a})^j - Irr(a\bar{a})^{j-1}$ for some $j \geqslant 1$, then there exists $c \in Irr(a\bar{a})^{j-1} - Irr(a\bar{a})^{j-2}$ such that $a \in Irr(abc)$.*

Proof. By hypothesis, $0 < (b, (a\bar{a})^j) = (a, b(a\bar{a})^{j-1}a)$. So there is $c \in Irr(a\bar{a})^{j-1}$ such that $0 < (a, bca)$ and hence $a \in Irr(abc)$. If $c \in Irr(a\bar{a})^{j-2}$

then $0 < (a, b(a\bar{a})^{j-2}a) = (b, (a\bar{a})^{j-1})$ implies that $b \in Irr(a\bar{a})^{j-1}$, a contradiction.

LEMMA 4. *Suppose that x is a component of A, $a \in Irr(a\bar{a})^t - Irr(a\bar{a})^{t-1}$ for some $t > 0$, and $a \in Irr(axa)$. Suppose also that either $a = \bar{a}$ or $x = \bar{x}$. Then there exists $c \in Irr(a\bar{a})^t - \{a\}$ with $a \in Irr(axc)$.*

Proof. Assume first that $a = \bar{a}$. So $a \in Irr(a^2)^t$ implies that there exists $u \in Irr(a^2)$ with $a \in Irr(u(a^2)^{t-1})$. Then $0 < (a, u(a^2)^{t-1}) = (u, a^{2t-1})$ yields $u \in Irr(a^{2t-1})$. Now $a \in Irr(a^2x)$ implies $Irr(a^{2t-1}) \subseteq Irr(a^{2t-2}a^2x) = Irr(a^{2t}x)$. Hence $u \in Irr(vx)$ for some $v \in a^{2t}$.

If $v = a$ then $a \in Irr(a^{2(t-1)}u) \subseteq Irr(a^{2(t-1)}ax)$. So there exists $e \in Irr(a\bar{a})^{t-1}$ (hence $e \neq a$, by hypothesis) with $a \in Irr(eax)$. The conclusion then holds, with $c = e$.

If $v \neq a$, then $a \in Irr(au)$ (as $u \in Irr(a^2)$ implies $0 < (u, a^2) = (a, au)$) and $Irr(au) \subseteq Irr(avx)$ yield $a \in Irr(avx)$ and $v \in Irr(a\bar{a})^t$. The conclusion holds with $c = v$.

So we may assume that $a \neq \bar{a}$, hence by hypothesis $x = \bar{x}$. Then $a \in Irr(axa)$ implies that

$$0 < (a, axa) = (a, \overline{axa}) = (a, ax\bar{a}).$$

Now $a \in Irr(a\bar{a})^t$ forces $\bar{a} \in Irr(a\bar{a})^t$, and we have $a \in Irr(ax\bar{a})$. So the conclusion holds, with $c = \bar{a}$.

LEMMA 5. *Suppose $b \in Irr(a\bar{a})^j - Irr(a\bar{a})^{j-1} - \{a\}$ for some $j \geqslant 1$. If any of $b = \bar{b}$, $b = \bar{a}$, or $a = \bar{a}$ holds then there exists $c \in Irr(a\bar{a})^{j-1} - \{a\}$ such that $a \in Irr(abc)$.*

Proof. By Lemma 3, there exists $d \in Irr(a\bar{a})^{j-1} - Irr(a\bar{a})^{j-2}$ such that $a \in Irr(abd)$. If there exists such $d \neq a$, then the conclusion holds with $c = d$. In particular, if $b = \bar{a}$ then $a \in Irr(a\bar{a})^j - Irr(a\bar{a})^{j-1}$ and so all such $d \neq a$.

So we may assume that either $b = \bar{b}$ or $a = \bar{a}$, that $a \in Irr(a\bar{a})^{j-1} - Irr(a\bar{a})^{j-2}$ and that $a \in Irr(aba)$. The result now follows from Lemma 4.

DEFINITION. Let $i > j$ be positive integers. An *adequate partition* (*a.p.*) $(\mathbf{S}, f)$ of $Irr(a\bar{a})^i - Irr(a\bar{a})^j - \{a\}$ is a collection $\mathbf{S}$ of disjoint subsets S whose union is $Irr(a\bar{a})^i - Irr(a\bar{a})^j - \{a\}$, and a function $f: \mathbf{S} \to Irr(a\bar{a})^j - \{a\}$ such that

(i) for all $S \in \mathbf{S}$, $a \in Irr(a \cdot f(S) \cdot \prod_{h \in S} h)$; and

(ii) for all $S \in \mathbf{S}$, $\bar{S} - \{a\} = S - \{\bar{a}\}$.

$(\mathbf{S}, f)$ is called a *fully adequate partition* (*f.a.p.*) if $(\mathbf{S}, f)$ is an a.p., and if, in addition,

(iii) if $S \neq T \in \mathbf{S}$ then $f(S) \neq f(T)$.

LEMMA 6. *Let $i > j \geqslant 1$. If $Irr(a\bar{a})^i - Irr(a\bar{a})^j - \{a\}$ has an a.p. then it has a f.a.p.*

Proof. Let $(\mathbf{S}, f)$ be an a.p. Suppose that $f(S) = f(T)$ for some $T \neq S \in \mathbf{S}$. Now $\bar{a}$ is not in at least one of S, T, so we may assume that $\bar{a} \notin S$. Then $S = \bar{S}$ by (ii). Lemma 2 and $S \cap T = \varnothing$ imply that there exists $c \in Irr(a\bar{a})$ such that

$$a \in Irr\left(ac\left(\prod_{h \in S} \bar{h}\right)\left(\prod_{h \in T} h\right)\right) = Irr\left(ac \prod_{h \in S \cup T} h\right).$$

If $c = a$, then $a, \bar{a} \in Irr(a\bar{a})$ and $S \cup T \subseteq Irr(a\bar{a})^i - Irr(a\bar{a})^j$ imply that $\bar{a} \notin S \cup T$. So $\overline{S \cup T} = S \cup T$ by (ii) in this case. Then Lemma 4 implies there exists $d \in Irr(a\bar{a}) - \{a\}$ with $a \in Irr(ad \prod_{h \in S \cup T} h)$. So we may assume that $c \neq a$ in any case.

Now form a new partition $\mathbf{T}$ of $Irr(a\bar{a})^i - Irr(a\bar{a})^j - \{a\}$ as follows: replace S and T by *one* set, $S \cup T$, and re-define $f(S \cup T) = c$. That is, choose any $c (\neq a)$ as above for the image. The other subsets in $\mathbf{T}$ and values of f are identical with those of $(\mathbf{S}, f)$. Then it is easy to see that $(\mathbf{T}, f)$ is an a.p., with $|\mathbf{T}| = |\mathbf{S}| - 1$. Since this process may be repeated if $\mathbf{T}$ does not satisfy (iii), the result holds.

LEMMA 7. *For any $i \geqslant 2$, $Irr(a\bar{a})^i - Irr(a\bar{a})^{i-1} - \{a\}$ has a f.a.p.*

Proof. We may assume that $Irr(a\bar{a})^i - Irr(a\bar{a})^{i-1} - \{a\} \neq \varnothing$, as otherwise we can define $\mathbf{S} = \{\varnothing\}$ and $f(\varnothing) = 1$.

For each $b \neq \bar{b} \in Irr(a\bar{a})^i - Irr(a\bar{a})^{i-1} - \{a\}$, with $b \neq \bar{a}$, define $S_b = S_{\bar{b}} = \{b, \bar{b}\}$ and define $f(S_b) = 1$. For each $b \in Irr(a\bar{a})^i - Irr(a\bar{a})^{i-1} - \{a\}$ such that either $b = \bar{b}$ or $b = \bar{a}$, Lemma 5 implies that there exists $c \in Irr(a\bar{a})^{i-1} - \{a\}$ such that $a \in Irr(abc)$. Define $S_b = \{b\}$ and $f(S_b) = c$.

Now let $\mathbf{S} = \{S_b \mid b \in Irr(a\bar{a})^i - Irr(a\bar{a})^{i-1} - \{a\}\}$, and let $f: \mathbf{S} \to Irr(a\bar{a})^{i-1} - \{a\}$ be as in the paragraph above. Then $(\mathbf{S}, f)$ is clearly an a.p. for $Irr(a\bar{a})^i - Irr(a\bar{a})^{i-1} - \{a\}$. Lemma 7 now follows from Lemma 6.

LEMMA 8. *Suppose that $i > j \geqslant 2$. If there is an a.p. for $Irr(a\bar{a})^i - Irr(a\bar{a})^j - \{a\}$ then there is an a.p. for $Irr(a\bar{a})^i - Irr(a\bar{a})^{j-1} - \{a\}$.*

Proof. By Lemma 6, $Irr(a\bar{a})^i - Irr(a\bar{a})^j - \{a\}$ has a f.a.p. $(\mathbf{S}, f)$. We proceed to construct an a.p. $(\mathbf{T}, g)$ for $Irr(a\bar{a})^i - Irr(a\bar{a})^{j-1} - \{a\}$. Let $b \in Irr(a\bar{a})^j - Irr(a\bar{a})^{j-1} - \{a\}$. We define a set T_b so that $b \in T_b$ (T_b will be an element of $\mathbf{T}$), and a value $g(T_b)$, as follows:

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(1) *Suppose that* $b = f(S)$ *for some (necessarily unique)* $S \in \mathbf{S}$, *and that either* $b = \bar{b}$ or $b = \bar{a}$. Define $T_b = S \cup \{b\}$ and $g(T_b) = 1$. Note that $a \in Irr(ab \prod_{h \in S} h)$, since $(\mathbf{S}, f)$ satisfies (i).

(2) *Suppose that* $b \neq f(S)$ *for all* $S \in \mathbf{S}$, *and that either* $b = \bar{b}$ or $b = \bar{a}$. Lemma 5 implies that there exists $c \in Irr(a\bar{a})^{i-1} - \{a\}$ such that $a \in Irr(abc)$. Define $T_b = \{b\}$ and $g(T_b) = c$.

(3) *Suppose that* $b \neq \bar{b}$, $b \neq \bar{a}$, *and that neither* b *nor* $\bar{b}$ *is in* $f(\mathbf{S})$. Define $T_b = T_{\bar{b}} = \{b, \bar{b}\}$ and $g(T_b) = 1$.

(4) *Suppose that* $b \neq \bar{b}$, $b \neq \bar{a}$, *and that* $\bar{b} = f(S)$ *for some* $S \in \mathbf{S}$ *but* $b \notin f(\mathbf{S})$.

If $a = \bar{a}$ then Lemma 5 implies that there exists $c \in Irr(a\bar{a})^{i-1} - \{a\}$ with $a \in Irr(abc)$. Then $\{b, c\} \cap (\{\bar{b}\} \cup S) = \varnothing$, $a \in Irr(a\bar{b} \prod_{h \in S} h)$, and Lemma 1 imply that $a \in Irr(acb\bar{b}(\prod_{h \in S} h))$. Then define $T_b = T_{\bar{b}} = S \cup \{b, \bar{b}\}$ and $g(T_b) = c$.

Suppose that $a \neq \bar{a}$. Lemma 3 says that there exists $c \in Irr(a\bar{a})^{i-1}$ with $a \in Irr(abc)$. If some such $c \neq a$ then again, $a \in Irr(a\bar{b} \prod_{h \in S} h)$ and Lemma 1 imply that $a \in Irr(acb\bar{b}(\prod_{h \in S} h))$. Again, define $T_b = T_{\bar{b}} = S \cup \{b, \bar{b}\}$ and $g(T_b) = c$. Otherwise, we have that $\bar{a} \neq a \in Irr(a\bar{a})^{i-1}$ and $a \in Irr(aba)$. So $0 < (a, aba) = (a, a\bar{b}\bar{a})$ implies that $a \in Irr(a\bar{b}\bar{a})$. Also, $S = \bar{S}$ (as $\bar{a} \notin S$) and $a \in Irr(a\bar{b}(\prod_{h \in S} h))$ yield that

$$0 < \left(a, a\bar{b}\left(\prod_{h \in S} h\right)\right) = \left(a, ab\left(\prod_{h \in \bar{S}} h\right)\right) = \left(a, ab\left(\prod_{h \in S} h\right)\right),$$

and hence that $a \in Irr(ab(\prod_{h \in S} h))$. Since $\{\bar{b}, \bar{a}\} \cap (\{b\} \cup S) = \varnothing$, Lemma 1 implies that $a \in Irr(ab\bar{b}\bar{a}(\prod_{h \in S} h))$. Then define $T_b = T_{\bar{b}} = S \cup \{b, \bar{b}\}$ and $g(T_b) = \bar{a}$.

(5) *Suppose that* $b \neq \bar{b}$ *and that* $b = f(S_1)$, $\bar{b} = f(S_2)$ for some $S_1, S_2 \in \mathbf{S}$. Then $a \in Irr(ab(\prod_{h \in S_1} h)) \cap Irr(a\bar{b}(\prod_{h \in S_2} h))$ and $(\{b\} \cup S_1) \cap (\{\bar{b}\} \cup S_2) = \varnothing$. So Lemma 1 yields $a \in Irr(ab\bar{b}(\prod_{h \in S_1} h)(\prod_{h \in S_2} h))$. Define $T_b = T_{\bar{b}} = \{b, \bar{b}\} \cup S_1 \cup S_2$ and $g(T_b) = 1$.

Now define $\mathbf{T}$, a collection of subsets of $Irr(a\bar{a})^i$, by

$$\mathbf{T} := \{T_b \mid b \in Irr(a\bar{a})^i - Irr(a\bar{a})^{i-1} - \{a\}\}$$
$$\cup \{S \mid S \in \mathbf{S} \text{ and } f(S) \in Irr(a\bar{a})^{i-1}\},$$

where T_b and $g(T_b)$ are as defined in (1)–(5). Let $g(S) = f(S)$ if $f(S) \in Irr(a\bar{a})^{i-1}$. It is now easy to check that $(\mathbf{T}, g)$ is an a.p. of $Irr(a\bar{a})^i - Irr(a\bar{a})^{i-1} - \{a\}$. The result follows from Lemma 6.

Proof of Theorem A. Lemmas 7 and 8 imply that there is a f.a.p. $(\mathbf{S}, f)$ for $Irr(a\bar{a})^i - Irr(a\bar{a}) - \{a\}$. (If $i = 1$, we may take $\mathbf{S} = \{\varnothing\}$ and $f(\varnothing) = 1$.)

So for each $S \in \mathbf{S}$, $f(S) \in Irr(a\bar{a}) - \{a\}$ and $a \in Irr(a \cdot f(S) \cdot (\prod_{h \in S} h))$. Now the sets $S \cup \{f(S)\}$ and $\{b\}$, as S runs over $\mathbf{S}$ and b runs over $Irr(a\bar{a}) - f(\mathbf{S}) - \{a\}$, are all disjoint from one another. For each $b \in Irr(a\bar{a}) - f(\mathbf{S}) - \{a\}$, $0 < (a\bar{a}, b) = (a, ab)$ implies that $a \in Irr(ab)$. So Lemma 1 yields that

$$a \in Irr\left[a\left(\prod_{S \in \mathbf{S}} f(S)\left(\prod_{h \in S} h\right)\right)\left(\prod_{Irr(a\bar{a}) - f(\mathbf{S}) - \{a\}} b\right)\right] = Irr\left(\prod_{h \in Irr(a\bar{a})^{i} \cup \{a\}} h\right).$$

This completes the proof of Theorem A.

3. Proofs of the Corollaries and Further Remarks

Proof of Corollary 1. Let $b = b_i$, $n = n_i$. By assumption, $\mathbf{B} = Irr(b^n)$. So $Irr(b\bar{b})^n = Irr[Irr(b^n)\, Irr(\bar{b}^n)] = \mathbf{B}$. Thus Corollary 1 is an immediate consequence of Theorem A.

The following example illustrates that Corollary 1 is the best possible.

Example. Let $\mathbf{B} = \{\chi_1, \chi_2, \chi_3\}$ where χ_1, χ_2, χ_3 are the irreducible characters of S_3 with the table of values

χ_1:	1	1	1
χ_2:	1	-1	1
χ_3:	2	0	-1.

Then χ_3 is a real nonlinear character with $(\chi_3)^2 = \chi_1 + \chi_2 + \chi_3$. Thus $Irr(\chi_3^2) = \mathbf{B}$. Now $Irr(\prod_{\chi_i \in Irr(\chi_3)^2 \cup \{\chi_3\}} \chi_i) = Irr(\chi_1\chi_2\chi_3) = \{\chi_3\}$.

Proof of Corollary 2. Since $(A, \mathbf{B})$ is a nonabelian simple table algebra, then by Proposition 4.2 of [AB], for every $b_i \neq 1$ in $\mathbf{B}$, there exists n_i such that $\mathbf{B} = Irr(b_i^{n_i})$. So Corollary 2 follows from Corollary 1.

Remark. Let G be a finite group. Recall that $\chi \in Irr(G)$ is of *p-defect zero* if $\chi(g) = 0$ for all p-singular elements g in G[I, pp. 133–134].

Consider the following properties (which may or may not hold for an arbitrary finite group G):

(1) G has an irreducible character of p-defect zero for every $p \mid |G|$.

(2) For each $g \neq 1$ in G, there exists $\chi \in Irr(G)$ with $\chi(g) = 0$.

(3) $(\prod_{\chi \in Irr(G)} \chi) = m\rho_G$, where ρ_G is the regular character and $m > 0$.

(4) $1_G \in Irr(\prod_{\chi \in Irr(G)} \chi)$.

It is easy to see that $(1) \Rightarrow (2) \Leftrightarrow (3) \Rightarrow (4)$. Michler and Willems have shown that (1) holds for all finite simple groups of Lie type [M, W]. Using the Atlas [At], one sees that (2) holds for all sporadic groups except M_{22} and M_{24}, where (4) holds anyway. So by the classification theorem for nonabelian simple groups, the conjecture mentioned in our Introduction remains open only for A_n. Little is known about (1) for A_n, except that A_n has characters of 2-defect zero for only certain values of n, and has characters of 5-, 7-, and 11-defect zero for all $n \geqslant 5$ by Atkins and Olsson [M].

References

[A] Z. Arad, On products of conjugacy classes and irreducible characters in finite groups, *in* "Proc. Sympos. Pure Math.," Vol. 47 (Arcata, 1986), Part 2, pp. 3–9, Amer. Math. Soc., Providence, RI, 1987.

[AB] Z. Arad and H. Blau, On table algebras and applications to finite group theory, **138** (1991), 137–185.

[ACH] Z. Arad, D. Chillag, and M. Herzog, Powers of characters of finite groups, *J. Algebra* **103** (1986), 241–255.

[ALG] Z. Arad and H. Lipman-Gutweter, On products of characters in finite groups, *Houston J. Math.* **15** (1989), 305–326.

[At] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, "Atlas of Finite Groups," Oxford Univ. Press (Clarendon), Oxford, 1985.

[BI] E. Bannai and T. Ito, "Algebraic Combinatorics. I. Association Schemes," Benjamin/Cummings, Menlo Park, 1984.

[F] W. Feit, "Characters of Finite Groups," Benjamin, New York, 1967.

[I] I. M. Isaacs, "Character Theory of Finite Groups," Academic Press, New York, 1976.

[M] G. O. Michler, Modular representation theory and the classification of finite simple groups, *in* "Proc. Sympos. Pure Math.," Vol. 47 (Arcata, 1986), Part 1, pp. 223–232, Amer. Math. Soc., Providence, RI, 1987.

[W] W. Willems, Blocks of defect zero in finite simple groups of Lie type, *J. Algebra* **113** (1988), 511–522.