JOURNAL OF MULTIVARIATE ANALYSIS 22, 13–23 (1987)

A Class of Infinitely Divisible Multivariate Negative Binomial Distributions

R. C. GRIFFITHS

*Monash University, Clayton, Australia**

AND

R. K. MILNE

University of Western Australia, Nedlands, Australia

Communicated by P. R. Krishnaiah

A particular class of multivariate negative binomial distributions has probability generating functions of the form $|I-Q|^{\alpha}|I-QS|^{-\alpha}$, where $\alpha>0$ and $S=\mathrm{diag}(s_1,..., s_n)$. The main results of this paper concern characterizations of the infinitely divisible distributions of this class.

1. INTRODUCTION

One type of n-dimensional exponential distribution has Laplace transform

$$|I+VT|^{-1}, \tag{1}$$

where V is an $n\times n$ positive semi-definite matrix and $T=\mathrm{diag}(t_1,..., t_n)$. Such distributions arise naturally from a Wishart matrix with two degrees of freedom as the joint distribution of the diagonal elements after scaling by dividing by two.

Griffiths [4] characterizes the class of matrices V for which the distribution with Laplace transform (1) is infinitely divisible, after partial results by earlier authors. In the proof of this characterization there arises a

Received 1984; revised July 23, 1986.

AMS 1980 subject classifications: 60E05; 62H05.

Keywords and phrases: Infinite divisibility, multivariate geometric distribution, multivariate negative binomial distribution.

* Research completed while visiting the Department of Mathematics, University of Western Australia.

0047-259X/87 $3.00

class of multivariate geometric distributions whose probability generating functions (pgf's) are of the form

$$|I-Q|\;|I-QS|^{-1}, \tag{2}$$

where Q is an $n\times n$ matrix and $S=\operatorname{diag}(s_1,..., s_n)$. Clearly there is a close relationship between multivariate geometric and multivariate exponential distributions. Although this relationship is exploited in Griffiths [4], the results obtained there about multivariate geometric distributions are incomplete and not stated explicitly.

The main result of this paper, Theorem 2 in Section 3, gives a complete characterization of the $n\times n$ matrices Q for which (2) is an infinitely divisible pgf. Certain of the conditions on Q have a natural and elegant statement in graph-theoretic terms. Section 2 considers some preliminary aspects of structure, establishing a characterization of those symmetric matrices Q for which (2) is a pgf.

When the pgf (2) is infinitely divisible,

$$|I-Q|^{\alpha}|I-QS|^{-\alpha} \tag{3}$$

is a pgf for each $\alpha>0$. In this case the corresponding distributions are multivariate negative binomial. (It is convenient in this paper to admit random variables degenerate at zero as being negative binomial.)

The pgf's (2) and (3) belong to the class of pgf's of the form

$$\{H(\mathbf{s})\}^{-\gamma}, \qquad \gamma>0, \tag{4}$$

where H is a multilinear form in $s_1,..., s_n$, i.e., H is of the form

$$H(\mathbf{s})=a_0+\sum_{r=1}^{n}\sum_{\{i_1,...,i_r\}} a_{i_1\cdots i_r}s_{i_1}s_{i_2}\cdots s_{i_r},$$

the inner summation being over all r-subsets of $\{1, 2,..., n\}$. Some results about pgf's of the form (4) have been obtained by Doss [1] and Milne [7], and for the two-dimensional case by Wiid [8] and Edward and Gurland [2].

In Section 4 we specialize to the bivariate case and show that then the class of pgf's (4) is always infinitely divisible, coinciding with the infinitely divisible class (3). As a by-product it is found that negative correlation is *not* possible in bivariate pgf's of the form (3).

2. Basic Structure

We begin by establishing a simple necessary and sufficient condition for (2) to be a pgf, assuming Q is symmetric and positive semi-definite.

THEOREM 1. *Let Q be an $n \times n$ real symmetric positive semi-definite matrix and* $S = \operatorname{diag}(s_1, \ldots, s_n)$, where $|s_i| \leqslant 1$, $i = 1, \ldots, n$. Then

$$|I - Q|\, |I - QS|^{-1}$$

is a pgf *iff all the eigenvalues of Q lie in the interval* $[0, 1)$. *The (multivariate) marginal* pgf's *are all of the same general form, the univariate marginals being geometric distributions.*

Proof. Let $\mathbf{Y}$ have a multivariate exponential distribution with Laplace transform (1), where $V = (I - Q)^{-1} - I$. Since Q has eigenvalues in $[0, 1)$, V is positive semi-definite. Now let $\mathbf{Z}$ be such that its conditional distribution given $\mathbf{Y}$ is of independent Poisson random variables with $\mathscr{E}(\mathbf{Z} | \mathbf{Y}) = \mathbf{Y}$. Then the (unconditional) pgf of $\mathbf{Z}$ is

$$\begin{aligned} P(\mathbf{s}) &= \mathscr{E}\left(\mathscr{E}\left(\prod_1^n s_i^{Z_i} \,|\mathbf{Y}\right)\right) = \mathscr{E}\left(\prod_1^n \exp\{-Y_i(1 - s_i)\}\right) \\ &= |I + V(I - S)|^{-1} = |I - Q|\, |I - QS|^{-1}. \end{aligned} \tag{5}$$

This establishes the sufficiency of the eigenvalue condition on Q.

When the given function is a pgf, it is necessarily finite for all complex $\mathbf{s}$ such that $|s_i| \leqslant 1$, $s = 1, \ldots, n$, and hence the polynomial $|I - Qz|$ in z cannot vanish for any z less than or equal to one in modulus. This establishes the necessity, since the eigenvalues of Q must be real.

Finally, observe that any subset of variables having the given pgf, has a pgf of the same form with V replaced by its corresponding submatrix. In particular, the univariate marginal distributions have pgf's

$$[1 - v_{ii}(s_i - 1)]^{-1}, \qquad i = 1, \ldots, n. \quad \blacksquare$$

Remark 1. With V and Q related as above, either of the two equivalent forms (5) may be used. Observe that V is symmetric iff Q is symmetric, and further that V is positive semi-definite iff Q has all its eigenvalues in $[0, 1)$.

Remark 2. Clearly $Z_1, \ldots, Z_n$ are mutually independent iff V is diagonal.

Remark 3. If $\mathbf{Y}$ has the structure $Y\mathbf{v}$ where $\mathbf{v}$ is a vector of positive constants and Y has an exponential distribution with mean 1, then $\mathbf{Z}$ has pgf

$$\left[1 - \sum_1^n v_i(s_i - 1)\right]^{-1}. \tag{6}$$

This corresponds to (5) with

$$V = (\sqrt{v_i v_j}), \qquad Q = \left(1 + \sum_1^n v_i\right)^{-1} V.$$

It is the well-known form of multivariate geometric distribution. From consideration of marginals it follows that a function of the form (6) is a pgf iff $v_i \geqslant 0$, $i = 1,..., n$.

Remark 4. Random variables $Z_1,..., Z_n$ with pgf (5) are exchangeable iff

$$V = \sigma^2[(1-\rho) I + \rho J], \qquad -(n-1)^{-1} \leqslant \rho \leqslant 1, \tag{7}$$

where J is a matrix of unit entries. Thus, evaluating the determinant we deduce that the pgf of $\mathbf{Z}$ is

$$\left\{1 - \rho\sigma^2 \sum_1^n (s_i - 1)[1 - \sigma^2(1-\rho)(s_i - 1)]^{-1}\right\}^{-1}$$
$$\times \prod_1^n [1 - \sigma^2(1-\rho)(s_i - 1)]^{-1}.$$

3. Characterization of Infinite Divisibility

To facilitate an elegant statement and proof of our main characterization result, we adopt a graphic-theoretic representation of certain matrix properties. Similar graph theory was employed by Griffiths [4] in a version (Theorem 2) of his main characterization result (Theorem 1). For background and terminology not explained here, refer to Wilson [9].

Given an $n \times n$ matrix Q with the property that all off-diagonal pairs of elements satisfy

$$q_{ij} q_{ji} \geqslant 0, \qquad i \neq j,\ i, j = 1,..., n,$$

construct a simple graph, $G(Q)$, with vertex-set $\{1, 2,..., n\}$ and edge-set $\{\{i, j\}: i \neq j,\ q_{ij} + q_{ji} \neq 0,\ i, j \in \{1, 2,..., n\}\}$. Colour an edge green if $q_{ij} + q_{ji} < 0$ or red if $q_{ij} + q_{ji} > 0$. A circuit in $G(Q)$ is defined to be elementary if there are no other edges in $G(Q)$ joining pairs of vertices of the circuit, apart from the original circuit edges.

Lemma. *Let G be a simple graph whose edges are coloured red or green. Every circuit in G has an even number of green edges if and only if every elementary circuit in G has an even number of green edges.*

Proof. The necessity is clear. An induction proof on circuit length gives the sufficiency. A path $r_1 r_2 r_1$ trivially has an even number of green edges. Assume that all circuits of length less than k have an even number of green edges if all elementary circuits have this property.

If a circuit $r_1 r_2 \cdots r_k r_1$ is not elementary there exists an edge $r_l r_m$ from

G not in the circuit. Two circuits are formed with the original edges and the edge $r_l r_m$,

$$r_1 r_2 \cdots r_l r_m r_{m+1} \cdots r_k r_1, \; r_l r_{l+1} r_{l+2} \cdots r_m r_l.$$

If all circuits of length less than k have an even number of green edges, then the number of green edges in each circuit is even. Since $r_l r_m$ is included in both circuits the original circuit has an even number of green edges. This completes the induction proof of the lemma. ▮

Remark 5. Define $Q^* = Q + Q'$, where $'$ denotes transpose. Then it is clear that the statement "every circuit in $G(Q)$ has an even number of green edges" can be expressed algebraically as

$$q^*_{r_1 r_2} q^*_{r_2 r_3} \cdots q^*_{r_k r_1} \geqslant 0 \qquad \text{for all subsets } \{r_1, \ldots, r_k\} \text{ of } \{1, \ldots, n\}. \tag{8}$$

THEOREM 2. *Let Q be an $n \times n$ (real) matrix. Then*

$$P(\mathbf{s}) = |I - Q| \, |I - QS|^{-1},$$

with $S = \operatorname{diag}(s_1, \ldots, s_n)$, is an infinitely divisible (multivariate geometric) pgf *iff*:

(i) *the eigenvalues of Q are bounded (strictly) in modulus by one*;

(ii) *$q_{ii} \geqslant 0$ and $q_{ij} q_{ji} \geqslant 0$, $i \neq j$, $i, j \in \{1, 2, \ldots, n\}$; and*

(iii) *every elementary circuit in the graph $G(Q)$ has an even number of green edges.*

Proof. When (i) is satisfied, it can be shown (cf. Griffiths [4]) that the expansion

$$\ln P(\mathbf{s}) = \ln |I - Q| + \sum_{k=1}^{\infty} \operatorname{tr}\{(QS)^k\}/k \tag{9}$$

is valid and convergent for $|s_i| \leqslant 1$, $i = 1, \ldots, n$. Then $P(\mathbf{s})$ is infinitely divisible iff the coefficients, apart from the constant term, in the multiple variable power series (9) are non-negative. That is, $P(\mathbf{s})$ is infinitely divisible iff for all non-negative integers $j_1, \ldots, j_n$ (not all zero) the coefficient of $s_1^{j_1} \cdots s_n^{j_n}$ in (9) is non-negative, i.e.,

$$k^{-1} \sum q_{i_1 i_2} \cdots q_{i_{k-1} i_k} q_{i_k i_1} \geqslant 0, \tag{10}$$

where the summation is over $i_1, \ldots, i_k \in \{1, 2, \ldots, n\}$ such that $k = j_1 + \cdots + j_n$, and the number of indices $i_1, \ldots, i_k$ equal to l is j_l,

$l=1, 2,..., n$. To establish the theorem it will be shown that the totality of conditions (10) is equivalent to

$$q_{r_1 r_2} q_{r_2 r_3} \cdots q_{r_k r_1} \geqslant 0 \qquad \text{for all subsets } \{r_1,..., r_k\} \text{ of } \{1,..., n\}. \quad (11)$$

Under condition (ii), (8) and (11) are equivalent and, because of the lemma, equivalent to (iii). Now observe that terms in the sums on the l.h.s. of (10) can all be factored into products as in (11) and diagonal entries q_{rr}, $r \in \{1,..., n\}$, the latter being non-negative when (ii) is satisfied. Hence, the sufficiency of (i), (ii). and (iii) has been proved.

Necessity. For condition (i), necessity follows straightforwardly as in the proof of Theorem 1.

Placing $k=1$ in (10) yields $q_{ii} \geqslant 0$, $i=1,..., n$. This together with (10) for $k=2$ shows that the remaining conditions in (ii) are necessary.

It has now been shown that (i) and (ii) are necessary, and that when they are satisfied

$$(11) \Leftrightarrow (8) \Leftrightarrow (\text{iii}).$$

Hence to complete the proof of necessity, and thereby of the theorem, we have to show only that (10) implies (11) and therefore (iii). It is clearly enough to do this assuming $r_1 r_2 \cdots r_k r_1$ is an elementary circuit in $G(Q)$. In this case (10) with $j_{r_1} = \cdots = j_{r_k} = 1$ is equivalent to

$$q_{r_1 r_2} q_{r_2 r_3} \cdots q_{r_k r_1} + q_{r_1 r_k} q_{r_k r_{k-1}} \cdots q_{r_2 r_1} \geqslant 0.$$

Because (ii) is satisfied, both products on the l.h.s. are non-negative. ▮

COROLLARY 1. *Conditions* (ii) *and* (iii) *of the theorem can be replaced by*:

(ii′) $P(\mathbf{s})$ *can be expressed as*

$$|I - Q_a|\ |I - Q_a S|^{-1},$$

where $Q_a = (|q_{ij}|)$.

Proof. The sufficiency is clear from (10), and the necessity from (11). ▮

COROLLARY 2. *If Q satisfies* (i), (ii) *of Theorem* 2 *and all the off-diagonal elements of $Q^* = Q + Q'$ are non-zero, then $P(\mathbf{s})$ is an infinitely divisible* pgf *if and only if*

$$q^*_{ij} q^*_{jk} q^*_{ki} > 0, \qquad i, j, k \in \{1, 2,..., n\}.$$

Proof. In this case all the elementary circuits of $G(Q)$ are triangles.

Remark 6. When $n>2$, not all $P(\mathbf{s})$ satisfying (i) and (ii) of Theorem 2 are infinitely divisible. Consider the exchangeable distribution in Remark 4 when $\sigma^2=1$. Then

$$\begin{aligned} Q &= I-(I+V)^{-1} \\ &= (1-\rho)(2-\rho)^{-1} I + \rho(2-\rho)^{-1}\{2+(n-1)\rho\}^{-1} J. \end{aligned}$$

Hence, from Corollary 2, although $P(\mathbf{s})$ is a pgf for all $-(n-1)^{-1} \leqslant \rho \leqslant 1$, it is infinitely divisible only if $0 \leqslant \rho \leqslant 1$.

Remark 7. Suppose that $\mathbf{X}=\mathbf{X}_1 \dotplus \mathbf{X}_2$, the direct sum of $\mathbf{X}_1$ and $\mathbf{X}_2$, has pgf (5) where V is a symmetric positive semi-definite matrix. Then $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent iff $V_{12}=0$. This is necessary, since if X_i is an element of $\mathbf{X}_1$ and X_j of $\mathbf{X}_2$,

$$\operatorname{cov}(X_i, X_j) = v_{ij}^2,$$

and so independence implies $V_{12}=0$. Conversely, if $V_{12}=0$ then (in an obvious notation)

$$P(\mathbf{s}) = |I_1 - V_{11}(S_1 - I_1)|^{-1}\, |I_2 - V_{22}(S_2 - I_2)|^{-1}.$$

If $P(\mathbf{s})$ is infinitely divisible and V is not symmetric a characterization of independence is more involved. Observe that for $\mathbf{X}$ with pgf (5)

$$\operatorname{cov}(X_i, X_j) = v_{ij} v_{ji}. \tag{12}$$

Hence, it is possible that $\operatorname{cov}(X_i, X_j)=0$ whenever X_i is an element of $\mathbf{X}_1$ and X_j of $\mathbf{X}_2$ with $\mathbf{X}=\mathbf{X}_1 \dotplus \mathbf{X}_2$ but that $\mathbf{X}_1$ and $\mathbf{X}_2$ are not independent.

Define a directed graph $G_d(Q)$ with vertices $\{1, 2,..., n\}$ and edges $\{\{i, j\}: i \neq j,\ q_{ij} \neq 0,\ i, j \in \{1, 2,..., n\}\}$.

THEOREM 3. *Suppose that* $\mathbf{X}$ *has an infinitely divisible* pgf *of the form* (5) *and that the vertex set of* $G_d(Q)$ *is partitioned into* U_1 *and* U_2. *Set* $\mathbf{X}_i = (X_k: k \in U_i)$, $i=1, 2$. *Then* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are independent iff every* (*directed*) *circuit in* $G_d(Q)$ *contains vertices of either* U_1 *or* U_2, *but not both.*

Proof. This is immediate from (10) and (11) since independence is equivalent to $\ln P(\mathbf{s})$ being a sum of two functions, one of $\mathbf{s}_1$ and one of $\mathbf{s}_2$. ∎

COROLLARY. *A sufficient condition for independence is that either* $Q_{12}=0$ *or* $Q_{21}=0$.

Proof. Under the given condition it is clear that a directed circuit of $G_d(Q)$ cannot contain vertices of both U_1 and U_2. ∎

Remark 8. Observe that, in view of (12), it is not necessary that V be positive definite even when it is symmetric.

The class of pgf's obtained from Theorem 2 with symmetric Q is strictly larger than the class of infinitely divisible pgf's that can be obtained from Theorem 1. It is easy to exhibit a suitable example.

Remark 9. When $X_1, ..., X_n$ have a pgf of the form (3) it follows that $X_1 + \cdots + X_n$ has pgf

$$|I-Q|^{\alpha}\,|I-QS|^{-\alpha} = |I-V(s-1)|^{-\alpha} = \prod_{1}^{n}\,[1-\lambda_i(s-1)]^{-\alpha}$$

where $V=(I-Q)^{-1}-I$ and $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ are the eigenvalues of V. Thus $X_1 + \cdots + X_n$ will have some negative binomial distribution iff $\lambda_1 = \cdots = \lambda_k = \lambda$ and $\lambda_{k+1} = \cdots = \lambda_n = 0$ for some k, $1 \leqslant k \leqslant n$. In particular the distribution of $X_1 + \cdots + X_n$ will be negative binomial with index α iff V is of rank one.

Further, it is easy to check that the distribution of the sum of any l of $X_1, ..., X_n$ has a negative binomial distribution with index α iff V is of rank one.

4. Bivariate Geometric Distributions

Observe that in Theorem 2, when $n=2$, the third condition plays no role: (i) and (ii) alone are necessary and sufficient for $P(\mathbf{s})$, of the specified form, to be an infinitely divisible pgf. We now consider the bivariate case in a different parametrization (cf. (4)).

Theorem 4. *Let*

$$P(s_1, s_2) = (a_0 + a_1 s_1 + a_2 s_2 + a_{12} s_1 s_2)^{-1}. \tag{13}$$

Then $P(s_1, s_2)$ *is a* pgf *iff*

(i) $a_0 + a_1 + a_2 + a_{12} = 1$;

(ii) $a_0 > 0$, $a_1 \leqslant 0$, $a_2 \leqslant 0$;

and either

(iiia) $a_{12} \leqslant 0$ *or*

(iiib) $a_{12} > 0$, $a_1 a_2 \geqslant a_{12} a_0$, *and the roots of*

$$a_0+(a_1+a_2)\,s+a_{12}s^2=0 \tag{14}$$

(both real) are strictly greater than one.

Further, if $P(s_1, s_2)$ *is a* pgf *then it is infinitely divisible.*

Proof. Sufficiency of (i), (iii), *and* (iiia). This is well-known, being established first by Edwards and Gurland [2]: start from the bivariate Poisson pgf (cf. Johnson and Kotz [5, Chap. 11, Sect. 4])

$$\exp\{\lambda a_1(s_1-1)+\lambda a_2(s_2-1)+\lambda a_{12}(s_1 s_2-1)\}$$

and let λ have an exponential distribution with parameter one.

Sufficiency of (i), (ii), *and* (iiib). Clearly (i) ensures $P(1, 1)=1$. When (ii) is satisfied a 2×2 matrix Q can be found satisfying

$$q_{11}=-a_1/a_0, \qquad q_{22}=-a_2/a_0, \qquad q_{12}q_{21}=(a_1 a_2-a_{12}a_0)/a_0^2.$$

If (iiib) holds then $q_{12}q_{21}\geqslant 0$ and it is possible to choose Q to be symmetric with $q_{12}\geqslant 0$. Hence Q is positive semi-definite, since $q_{11}\geqslant 0$, $q_{22}\geqslant 0$, and $|Q|=a_{12}/a_0>0$. Further, $P(s_1, s_2)$ can be written as

$$P(s_1, s_2)=|I-Q|\;|I-QS|^{-1},$$

where $S=\operatorname{diag}(s_1, s_2)$.

Clearly the eigenvalues of Q are non-negative; they will be in $(0, 1)$ provided

$$0=|I-Qs|\;|I-Q|^{-1}=a_0+(a_1+a_2)\,s+a_{12}s^2$$

has roots in $(1, \infty)$. Since this is ensured by (iiib), Theorem 1 implies that $P(s_1, s_2)$ is a pgf. This can be deduced also from Theorem 2 which yields the further result that $P(s_1, s_2)$ is infinitely divisible.

Necessity. Clearly (i) is necessary. If (X_1, X_2) has pgf $P(s_1, s_2)$ then

$$a_0^{-1}=P(X_1=0, X_2=0),\; a_1=P(X_1=1, X_2=0),\; a_2=P(X_1=0, X_2=1)$$

and hence (ii) is necessary. If $a_{12}\leqslant 0$ the proof is completed. If $a_{12}>0$ it is necessary that $a_1 a_2\geqslant a_{12}a_0$.

To show $a_1 a_2\geqslant a_{12}a_0$ first note that regression of X_1 on X_2 is linear. The coefficient of s_1-1 in $P(s_1, s_2)$ is

$$\mathscr{E}(X_1 s_2^{X_2})=-[1+(a_2+a_{12})(s_2-1)]^{-2}[a_1+a_{12}+a_{12}(s_2-1)],$$

and the coefficient of s_2^x in this expression, divided by $P(X_2=x)$ is

$$\mathscr{E}(X_1|X_2=x)=(a_1a_2-a_{12}a_0)\,cx+b, \tag{15}$$

where $c>0$ and both sides of (15) must be non-negative for all $x=0, 1,\ldots.$

Finally, observe that (ii), $a_{12}>0$, and $a_1a_2 \geqslant a_{12}a_0$ being satisfied, it follows from the proof of sufficiency above that the roots of (14) are strictly greater than one. ∎

It seems useful to restate the result of this theorem in the form of the following corollary.

COROLLARY 1. *Any* pgf *of the form* (13) *is infinitely divisible, and the class of all such* pgf's *coincides with the class of all* (*infinitely divisible*) pgf's *of the form* (4) *with* $n=2$.

COROLLARY 2. *Negative correlation is not possible in* pgf's *of the form* (13), *or in* pgf's *of the form* (4) *with* $n=2$.

Proof. The result follows since, for the pgf (13), the covariance is $a_1a_2-a_{12}a_0$ and this quantity is always non-negative. ∎

Remark 10. If (13) is to be a non-degenerate bivariate pgf, we must have at least one of the inequalities $a_{12}\neq 0$, $a_1a_2\neq 0$ satisfied. Then each marginal distribution will be geometric.

Remark 11. The two-dimensional pgf $|I-V(S-I)|^{-\alpha}$ can be expressed as

$$\begin{aligned}|I-V(S-I)|^{-\alpha}&=(1-\mu)^{\alpha}(1-\theta_1(s_1-1))^{-\alpha}(1-\theta_2(s_2-1))^{-\alpha}\\&\quad\times\{1-\mu(1-\theta_1(s_1-1))^{-1}(1-\theta_2(s_2-1))^{-1}\}^{-\alpha},\end{aligned} \tag{16}$$

where $\mu=v_{12}v_{21}v_{11}^{-1}v_{22}^{-1}$, $\theta_1=v_{22}^{-1}|V|$, $\theta_2=v_{11}^{-1}|V|$. If (X, Y) has pgf (16), then

$$\begin{aligned}&P(X=x,\ Y=y)\\&\quad=\sum_{k=0}^{\infty}P(k;\alpha,\mu)\,P(x;\alpha+k,\theta_1)\,P(y;\alpha+k,\theta_2),\quad x, y=0, 1,\ldots,\end{aligned} \tag{17}$$

where $a_{(x)}=a(a+1)\cdots(a+x-1)$ and

$$P(x;\beta,\theta)=\beta_{(x)}(x!)^{-1}(1+\theta)^{-(\alpha+x)}\theta^x.$$

The distribution (17) is a mixture of independent negative binomial distributions, mixed by another negative binomial distribution. The conditional pgf of X given $Y = y$ is

$$\begin{aligned}\{1-\lambda(s_1-1)\}^{-(\alpha+y)}\{1-\phi(s_1-1)\}^{y}\\ =\{1-\lambda(s_1-1)\}^{-(\alpha+y)}\{(1-\eta)+\eta(1-\lambda(s_1-1))\}^{y},\end{aligned} \tag{18}$$

where $\lambda = (v_{11} + |V|)(1 + v_{22})^{-1}$, $\phi = |V|\, v_{22}^{-1}$, $\eta = \phi\lambda^{-1}$. The distribution corresponding to (18) is

$$g(x\,|\,y) = \sum_{r=0}^{y} \binom{y}{r} (1-\eta)^r \eta^{y-r} P(x;\alpha + r, \lambda), \qquad x = 0, 1.... \tag{19}$$

The conditions of Theorems 4 imply that $\lambda \geqslant 0$ $0 \leqslant \eta \leqslant 1$ when $\phi \geqslant 0$, so then (19) is a mixture. If $\phi < 0$, then $-1 \leqslant \phi < 0$, and (18) represents a convolution of negative binomial and binomial distributions.

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