Periodic points on the complement

Philip R. Heath \textsuperscript{a,\star}, Xuezhi Zhao \textsuperscript{b,1,2}

\textsuperscript{a} Department of Mathematics, Memorial University of Newfoundland, St. John’s, Newfoundland, Canada A1C 5S7
\textsuperscript{b} Department of Mathematics, Capital Normal University, Beijing 100037, China

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Abstract

Let $f : (X, A) \to (X, A)$ be a self-map of a pair of compact ANRs, with $X$ connected. In 1989 the second author studied fixed point theory on the complement. He defined a number $N(f; X - A)$ which is a lower bound for the number of fixed points on $X - A$ of maps $g$ that are homotopic to $f$ as a map of pairs. In this paper we generalize these ideas from fixed point theory to periodic point theory, and define two Nielsen type numbers for periodic points on the complement. We give a number of examples and follow a particular one through to show, that for this type of example one of the Nielsen type periodic numbers is given by a formula, and the other is algorithmic. We also highlight a computational simplification of the modified fundamental group approach (which we use here) over the covering space approach. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

For a self-map $f : X \to X$ of a compact connected ANR $X$, a periodic point is a point $x \in X$ such that there exists a positive integer $n$ with $f^n(x) = x$. We denote the set of all periodic points of period dividing $n$ (i.e., the fixed point set of $f^n$) by $\Phi(f^n) = \{x \in X \mid f^n(x) = x\}$. The set of periodic points of period exactly $n$ will be denoted by $P_n(f) = \{x \in \Phi(f^n) \mid x \notin \Phi(f^m) \ \forall m < n\}$. Nielsen periodic point theory was

\textsuperscript{\star} Corresponding author. Email: pheath@math.mun.ca.
\textsuperscript{1} Email: zhaoxve@sxx0.math.pku.edu.cn.
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introduced in [12] and developed in [8] and [11], and is quite a bit more subtle than merely the consideration of the Nielsen number of \( f^n \). The object is to estimate the numbers

\[
M \Phi(f^n) = \min \#(\{ \Phi(g^n) | g \sim f \}),
\]

the minimum numbers of periodic points of all periods dividing \( n \), and

\[
MP_n(f) = \min \#(\{ P_n(g) | g \sim f \}),
\]

the minimum numbers of periodic points of period exactly \( n \). Note that all the homotopies in the definitions are of \( f \), not of \( f^n \). We illustrate the considerations of this paper by an example, which we will follow through the paper and give complete information on it for the two Nielsen type numbers that we define.

**Example 1.1.** Let \( X = S^1 \times S^1 = \{(e^{i\theta}, e^{i\phi}) | 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq 2\pi\} \), and let \( A = \Delta S^1 = \{(e^{i\theta}, e^{i\phi}) | \theta = \phi \} \) be the diagonal of \( X \). We define a relative map \( f : (X, A) \to (X, A) \) by \( f(e^{i\theta}, e^{i\phi}) = (e^{2i\theta}, e^{2i\phi}) \). For \( n \) odd, the periodic points for this map on \( X \) are given by

\[
\Phi(f^n) = \left\{ (e^{i\theta}, e^{i\phi}) | \theta = \frac{2j\pi}{2^n - 1}, \phi = \frac{2^{n+1}j\pi}{2^n - 1}, j = 0, 1, \ldots, 2^n - 2 \right\},
\]

and for \( n \) even

\[
\Phi(f^n) = \left\{ (e^{i\theta}, e^{i\phi}) | \theta = \frac{2j_1\pi}{2^n - 1}, \phi = \frac{2j_2\pi}{2^n - 1}, j_1, j_2 = 0, 1, \ldots, 2^n - 2 \right\}.
\]

The name of the game, for this and other examples for a given \( n \), is to give sharp lower bounds for the numbers

\[
M \Phi(f^n; X - A) = \min \#((X - A) \cap \{ \Phi(g^n) | g \sim_A f \})
\]

(the minimum numbers of periodic points of all periods dividing \( n \) of maps \( g \) that are homotopic to \( f \) as a map of pairs, and that lie in \( X - A \)), and for

\[
MP_n(f; X - A) = \min \#((X - A) \cap \{ P_n(g) | g \sim_A f \})
\]

(the minimum numbers of periodic points of period exactly \( n \) of maps \( g \) that are homotopic to \( f \) as a map of pairs, and that lie in \( X - A \)). Note that homotopies must be relative homotopies (that is homotopies of pairs), and homotopies of \( f \) not of \( f^n \). The real challenge in this, and similar examples, is not to determine which points might or might not be combined under relative homotopies of \( f \), for this follows from existing Nielsen theory. The challenge is to determine which period point classes (or more precisely orbits) may be moved from \( X - A \) to \( A \) under such relative homotopies.

We outline a brief history of Nielsen periodic, and relative Nielsen theory. Periodic point theory was first developed by Jiang in his book ([12], 1983), and at the same time an inadequate (and unpublished) version was given by Halpern (see [3]). Let \( f : X \to X \) be a self-map of a compact connected ANR \( X \). For a fixed \( n \) there are two Nielsen type numbers for periodic points of \( f \) which in this paper are denoted by \( N\Phi_n(f) \) and \( N\Phi_n(f) \). These
numbers are lower bounds for the numbers $M\Phi_n(f)$ and $MP_n(f)$, respectively, defined above. Periodic point theory is of course a generalization of fixed point theory from the case $n = 1$. Jiang’s ideas were taken and extended in the papers [8] and [11], where a number of computational results (some of which will be generalized in this paper) were given.

Turning to relative Nielsen theory Schirmer in 1986 [13] introduced a relative Nielsen number for a self-map $f : (X, A) \to (X, A)$, of a pair of compact ANRs $(X, A)$, with $X$ connected. This was done in order to take into account the fact that such a relative map $f$ determines self-maps of both $X$ and $A$, thus possibly detecting more fixed points than the ordinary Nielsen number. The relative Nielsen number is a sort of hybrid of the corresponding ordinary Nielsen numbers of these maps, and helps explain certain apparent anomalies in the Nielsen theory of self homeomorphisms of manifolds with boundary. Schirmer’s original paper however did not give specific information about how many of the fixed points thus detected lie in $X - A$. The second author sought to address this concern in [15], where he presented his number $N(f; X - A)$, which is a lower bound for the minimum number $M(f; X - A)$ of fixed points of $f$ on the complement $X - A$, which cannot be moved to $A$ by relative homotopies of $f$.

In 1995 Schirmer’s ideas about relative theory, and the existing ideas about periodic points, were combined to give a common generalization of relative fixed point theory and periodic point theory in the paper [10]. This theory considers the periodic Nielsen theory of a map $f : (X, A) \to (X, A)$, of a pair of compact ANRs with $X$ connected. It does this by considering the interaction of the periodic point theory of the two maps on $X$ and on $A$ in a similar way to the interaction of the two maps in the fixed point case. The periodic point case is of course a good deal more complicated. As with Schirmer’s paper [13], no consideration was given in [10] to how many of the periodic points lie in the complement $X - A$.

Coming to this paper then our aim is to define Nielsen type numbers $NP_n(f; X - A)$ and $N\Phi_n(f; X - A)$ that will give reasonable estimates for the numbers $MP_n(f; X - A)$ and $M\Phi(f^n; X - A)$, respectively, and which extend relative periodic point theory in the same way that [15] extends ordinary relative Nielsen theory. Of course, $N(f^n; X - A)$ is a lower bound for $M\Phi(f^n; X - A)$. However since the former number estimates the minimum number of fixed points on $X - A$ of maps $g$ homotopic to $f^n$ (as opposed to the minimum number of fixed points of $g^n$ for $g$ homotopic to $f$), it may be a poor lower bound for $M\Phi(f^n; X - A)$ (see Examples 5.5 and 5.8). In addition in and of itself $N(f^n; X - A)$ is of little help in determining $MP_n(f; X - A)$. We will however give conditions under which $N\Phi(f^n; X - A)$ can be equated with $N(f^n; X - A)$, and also conditions under which $NP_n(f; X - A)$ can be calculated from some of the $N(f^m; X - A)$ for $m|n$ (see Corollary 5.6 and Theorem 5.7).

Our approach differs from that of [15] in that we use the modified fundamental group approach introduced in [4], and used later in [8] and [11]. This avoids having to pass to the Homology cokernels which in any case is only effective when some measure of commutativity is present. In fact in the presence of such commutativity the necessary group structure is already present in the Reidemeister classes, as seen for example in Theorem 2.1 (see also Examples 4.2, 6.4, and Lemma 6.2).
The paper is outlined as follows. Following this introduction we have a review of the necessary theory of periodic point classes and orbits, together with a brief review of fixed point theory on the complement. In Section 3 we introduce weakly common periodic point orbits and the first of the Nielsen periodic point numbers \( \mathcal{NP}_n(f; X - A) \) on the complement (presented of course in the modified fundamental group approach). In Section 4 we introduce \( \mathcal{N}\Phi_n(f; X - A) \), the other Nielsen periodic point number on the complement. Section 5 discusses the relationship of \( \mathcal{NP}_n(f; X - A) \) and \( \mathcal{N}\Phi_n(f; X - A) \) to each other, and to \( \mathcal{N}(f^m; X - A) \) for \( m \mid n \). In Section 6 we give a full exposition of the example in this introduction and also one from [9] (where the subspace is not connected), showing that for these examples there are formulae for \( \mathcal{NP}_n(f; X - A) \), and algorithmic procedures to calculate the \( \mathcal{N}\Phi_n(f; X - A) \). Finally in Section 7 we make some concluding remarks.

2. Review

In this section we give a brief review of the concepts from ordinary Nielsen periodic point theory, and fixed point theory on the complement that are needed in the study of the Nielsen periodic point numbers on the complement.

2.1. Periodic point theory

We use a modified fundamental group approach as in [4,8,11]. Throughout the paper unless otherwise stated all spaces \( X, A \), etc. will denote compact ANRs. Let \( f : X \to X \) be a self-map of \( X \), for each positive integer \( n \) we will write \( f^n \) for the \( n \)th iterate of \( f \). Let

\[
\Phi(f^n) = \{ x \in X : f^n(x) = x \}
\]

denote the fixed point set of \( f^n \). For \( x \in \Phi(f^n) \) we define the period of \( x \), denoted by \( \text{per}(x) \), to be the smallest positive integer \( m \mid n \) such that \( x \in \Phi(f^m) \).

There are two sides to the theory: the algebraic and the geometric. For the geometric side we say that \( x, y \in \Phi(f^n) \) are \( f^n \) Nielsen equivalent provided that there is a path \( c \) from \( x \) to \( y \) so that (relative end points) \( c \simeq f^n(c) \). The set of equivalence classes thus generated will be denoted by \( \Phi(f^n) \). We call these classes the set of all geometric Nielsen classes for \( f^n \). Throughout the paper we will use \( A^n, B^n, \) or \( C^n \) to denote geometric Nielsen classes of \( f^n \). When the \( n \) is omitted, then we assume \( n = 1 \). Using the standard theory of index (see, for example, [1]) we may then define an integer index \( \text{ind}(A^n) \) for each geometric class \( A^n \). The classes with non zero index are called essential Nielsen classes. For any positive integer \( m \), the Nielsen number \( \mathcal{N}(f^m) \) of \( f^m \) is then the number of essential classes of \( f^m \).

One of the classical results of Nielsen theory is that a homotopy \( f \sim g \) induces a bijection between the essential classes of \( f \) and those of \( g \). This gives that \( \mathcal{N}(f) \leq M\Phi(f) \) with a lower bound that is sharp in many cases. For each \( m \mid n \) there is a canonical change of level function

\[
\gamma_{m,n} : \Phi(f^m) \sim \to \Phi(f^n) \sim
\]
defined as follows. If $A^m \in \Phi(f^m)\sim$, then if $x \in A^m$ we define $\gamma_{m,n}(A^m)$ to be the unique class of $\Phi(f^m)\sim$ which contains $x$. We warn the reader that the $\gamma_{m,n}$ are neither injective nor do they automatically restrict to give functions between essential classes. The map of degree $-1$ on $S^1$ with $m = 1$ and $n = 2$ illustrates these points (see Example 4.2). It is clear when $\gamma_{m,n}$ that $\gamma_{k,n} = \gamma_{m,n} \gamma_{k,m}$.

To come to the algebraic side of the theory we choose a basepoint $x_0 \in X$, along with a path $\alpha$ from $x_0$ to $f(x_0)$. Please note in what follows we shall not distinguish between a path and its path class in the fundamental groupoid of $X$. Up to canonical bijection the following constructions are independent of these choices. For each $n$ we define “coordinates” for all iterates of $f$ as follows. Define $1\omega = \omega$ and for $n > 1$ we let $n\omega = (n - 1)\omega f^n(\omega)$, so $n\omega$ is a path (class) from $x_0$ to $f^n(x_0)$. We abbreviate $(n\omega)^{-1}$ by $n\omega^{-1}$. If $x_0$ is a fixed point of $f$, we will not distinguish between $x_0$ and the constant path at $x_0$. Note that in this case $n x_0 = x_0$.

We define a homomorphism $f_n^{n\omega}: \pi_1(X, x_0) \to \pi_1(X, x_0)$ by $f_n^{n\omega}(\alpha) = n\omega f_n^\alpha(\alpha) n\omega^{-1}$. Note that we are using multiplicative notation unlike [8] and [11] which used additive notation even for the non commutative groups. Next for each $n$ we partition $\pi_1(X, x_0)$ into Reidemeister classes as follows: $\alpha \sim \beta$ if and only if there exists $\delta \in \pi_1(X, x_0)$ with $\alpha = \delta \beta f_n^{n\omega}(\delta^{-1})$. The class containing $\alpha$ will be denoted by $[\alpha]_n$, and the set of all classes by $R(f_n^{n\omega})$ (note the change from [8] and [11] where the symbol $\text{Ker}(1 - f_n^{n\omega})$ is used). We call $R(f_n^{n\omega})$ the set of (algebraic) $n$ periodic point classes of $f$. The symbol $R(f^n)$ denotes the Reidemeister number $\#R(f_n^{n\omega})$ of $f^n$. From [8] the number $R(f^n)$ is independent of the choice of $x_0$ and $\omega$. Let

$$\text{Fix}(f_n^{n\omega}) = \{\alpha \in \pi_1(X, x_0) \mid f_n^{n\omega}(\alpha) = \alpha\}.$$  

The top part of the diagram that follows is an exact sequence of based sets
Since we wish to emphasize the point that there is no need to pass to the homology we state all this as a theorem.

**Theorem 2.1** [4]. Let $f : X \to X$ be a map with $\pi_1(X,x_0)$ abelian, then the sequence

$0 \to \text{Fix}(f^{n\omega}_* ) \to \pi_1(X,x_0) \xrightarrow{1-\omega f_* } \pi_1(X,x_0) \to \mathcal{R}(f^{n\omega}_*) \to 0$

is an exact sequence of groups and homomorphisms.

The change of level function for the algebraic side of the theory is the function

$t_{m,n} : \mathcal{R}(f^{m\omega}_*) \to \mathcal{R}(f^{n\omega}_*)$ defined by

$t_{m,n}([\alpha]^m) = \left[ \alpha f^{m\omega}_*(\alpha) f^{2m\omega}_*(\alpha) \cdots f^{(n-m)m\omega}_*(\alpha) \right]^n,$

or when $\pi_1(X)$ is abelian written additively as

$t_{m,n}([\alpha]^m) = [\alpha + f^{m\omega}_*(\alpha) + f^{2m\omega}_*(\alpha) + \cdots + f^{(n-m)m\omega}_*(\alpha)].$

As in the geometric setting, we also have when $k|m$ that $t_{k,n} = t_{m,n} t_{k,m}$. The algebraic and geometric theory are related by an injective function

$\rho_n = \rho : \Phi(f^n)/\sim \to \mathcal{R}(f^{n\omega}_*)$

defined as follows. Given $x \in A^n$ we choose a path $c$ from the basepoint $x_0$ to $x$. We can then define $\rho(A^n) = [ cf^n(c^{-1})^{n\omega} ]$ This will be independent of the choice of $x$ within $A^n$ and $c$. An algebraic class $[\alpha]^n$ is said to be non-empty if it lies in the image of $\rho$. We follow the modified fundamental group approach as in [4,8,11], and assign next an index to the Reidemeister classes. The index $\text{Ind}( [\alpha]^n )$ of a class $[\alpha]^n \in \mathcal{R}(f^{n\omega}_*)$ is defined as follows

$\text{Ind}( [\alpha]^n ) = \begin{cases} \text{ind}(A^n) & \text{if } [\alpha]^n = \rho(A^n), \\ 0 & \text{otherwise}, \end{cases}$

where $\text{ind}(A^n)$ is the usual index of a fixed point class as defined, for example, in [1]. As with the geometric classes, an algebraic class is essential provided it has non zero index. It should be clear from the definition of index of a Reidemeister class that we can define the Nielsen number $N(f^n)$ of $f^n$ either as the number of essential geometric classes or as the number of essential algebraic classes of $f^n$.

**Example 2.2.** Let $X = S^1$, we define a self-map $f$ of $X$ by $f(e^{i\theta}) = e^{i\theta}$. Then $\Phi(f^2) = \{ e^{i\pi/4} | j = 0, 1, \ldots, 7 \}$, and each point is in its own Nielsen class. We let $x_0$ be the point $e^{i0}$, and $\omega$ the constant path at $x_0$. Then $\pi_1(X,x_0) \cong \mathbb{Z}$, and $1 - f^{2\omega}_*$ is multiplication by $-8$, so $\mathcal{R}(f^{2\omega}_*) \cong \mathbb{Z}_8$. Let the generator $[1]^2$ be represented by the path $t \rightarrow e^{-it}$, then $\rho = \rho_2([e^{i\pi/4}]) = [j]^2$. To see this consider $j = 2$, for example, and let $c$ be the shortest path from $x_0$ to $e^{2\pi i/4}$. Then $cf^2(c^{-1})$ essentially wraps itself twice around $S^1$ in the clockwise direction, and so is represented by $[2]^2$. We also observe for all $j$ that $\text{Ind}( [j]^2 ) \neq 0$, and so all these classes are essential (see Theorem 2.3). For $n = 1$ we have $\mathcal{R}(f^n_*) \cong \mathbb{Z}_2$, and $t_{1,2}$ is multiplication by $1 + 3 = 4$, so that $[0]^2$, and $[4]^2$ are reducible (defined precisely below).
We remark that \( \text{Ind}(\alpha^n) = \text{ind}(\phi(\alpha \circ \tilde{f}^n)) \) in the covering space approach, where \( \tilde{f}^n \) is the unique lift of \( f^n \) determined by the path \( n(\omega) \). The point of the definition of the index of a Reidemeister class is of course, that the index of \( [\alpha]^n \) is the same as the index of the geometric class (empty or not) that determines it. Since the periodic point classes are in one to one correspondence with the coordinates we are in this way able to deal with possibly empty classes without bringing in the machinery of covering spaces (see [4,8] or [11] for more details). In particular, we give the familiar result of Jiang in this setting (Theorem 2.3 below). Recall that a Jiang space is a space for which the induced map \( x_0 : \pi_1(X) \to \pi_1(X, x_0) \) is surjective. In the theorem \( L(f) \) denotes the Lefschetz number of \( f \).

**Theorem 2.3** (Jiang). If \( X \) is a Jiang space, and \( f : X \to X \) a map then \( N(f^n) = 0 \) if \( L(f^n) = 0 \), and \( N(f^n) = R(f^n) \) if \( L(f^n) \neq 0 \).

The algebraic and the geometric change of level functions are related by the following equation. For any \( A^m \) we have (see, for example, [8])

\[
\rho(\gamma_{m,n}(A^n)) = \iota_{m,n}(\rho(A^m)).
\]

For both algebraic and geometric classes at level \( n \) (denoted, respectively by \( [\alpha]^n \) and \( A^n \)) we have the notions of reducibility, depth, and length. We say that a class \( [\alpha]^n \) (respectively \( A^n \)) is reducible provided there exists an \( m \) and a class \( [\beta]^m \) (respectively \( B^m \)) for which \( \iota_{m,n}(\beta^m) = [\alpha]^n \) (respectively \( \gamma_{m,n}(B^m) = A^n \)). The depth \( d([\alpha]^n) \) (or \( d(A^n) \)) is the smallest integer \( d\) such that the class \( [\alpha]^n \) (respectively \( A^n \)) reduces to level \( d \). The class \( [\alpha]^n \) (respectively \( A^n \)) is irreducible provided it has depth \( n \). The map \( f \) induces an index preserving bijection on the algebraic or geometric classes at level \( n \) in such a way that \( f_u^{n,\omega} \) (respectively \( f^n \)) is the identity. This allows for a natural breaking up of the of the algebraic (respectively geometric) classes into orbits in such a way that depth and hence reducibility (or not) are properties of orbits. The orbit of \( [\alpha]^n \) \( (A^n) \) is denoted by \( [\{\alpha\}^n] \) \( ([A^n]) \), respectively. The length \( l(\{\alpha\}^n) \) \( (l(A^n)) \) of an orbit is the smallest integer \( l\) such that \( f_u^{l\omega}([\alpha]^{n}) = [\alpha]^n \) \( (f_u^{l}A^n = A^n) \). From the division algorithm if \( f_u^{l\omega}([\alpha]^{n}) = [\alpha]^n \) \( (f_u^{l}A^n = A^n) \), then \( l|q \). From [8, Corollary 2 of Proposition 1.14] we have that algebraic and geometric lengths coincide, and that the algebraic depth of a non-empty class is less than or equal to its geometric depth, i.e.,

\[
d(A^n) \leq d(\rho(A^n)). \tag{2}
\]

The fundamental lemma, which underlies all the Nielsen type numbers for periodic points, is the following:

**Lemma 2.4.** Let \( [\alpha]^n \in R(f_u^{n,\omega}) \) be an essential \( n \)-periodic point class. Then the orbit \( [\{\alpha\}^n] \) \( ([A^n]) \) of \( [\alpha]^n \) contains at least \( d([\alpha]^n) \) periodic points of \( f \).

It is from this lemma that we see the necessity of working with orbits rather than classes. In particular it is possible for a single essential class to contain \( n \) periodic points. Ordinary
Nielsen theory would detect only a single point in this situation (see Example 5.5). The following definition which is the first of the Nielsen type numbers for periodic points, takes into account then the fact that if there is one periodic point of least period \( n \) then are at least \( n \) of them. For each positive integer \( n \) we define

\[
NP_n(f) = n\#(IEO_n),
\]

where \( IEO_n \) is the set of irreducible essential orbits of \( f \) at level \( n \).

For the other periodic point number \( N\Phi_n(f) \), we need a definition. Recall that a set \( S \) of algebraic orbits is said to be a set of representatives for a set \( T \) of algebraic orbits if every element of \( T \) is reducible to some element of \( S \). The height \( h(S) \) of such a set \( S \) is the sum of the depths of all its members.

**Definition 2.5.** \( N\Phi_n(f) = \min\{h(S) : S \text{ a set of representatives of the set of essential algebraic orbits of } f \text{ at levels } n \text{ or less} \} \).

Since \( NP_n(f) \) and \( N\Phi_n(f) \) are homotopy invariants we have

**Theorem 2.6.** \( NP_n(f) \leq MP_n(f) \) and \( N\Phi_n(f) \leq M\Phi_n(f) \).

It is easy to confuse the set \( MP_n(f) = \min\{\#(\Phi(g^n)) : g \sim f \} \) with the set \( \min\{\#(\Phi(g^n)) : g^n \sim f^n \} \). In fact \( N\Phi_n(f) \) is not a lower bound for this latter set. Take, for example, the map \( f \) of degree \(-1\) on the circle \( S^1 \). Then with \( id \) denoting the identity map, \( f^2 \sim id^2 \), but \( f \not\sim id \). Thus \( Mp(f) = 2 \), while \( MP_2(id) = 0 \). The point is that the homotopy invariance is with respect to homotopies of \( f \) and not with respect to homotopies of higher iterates.

### 2.2. Fixed point theory on the complement

Actually this subsection is not quite review, since we need to convert the theory in [15] to the modified fundamental group approach (found, for example, in [8], and [11]). In the paper [2] the details of the modified fundamental group approach is given for coincidence theory on the complement. Since certain aspects of fixed point theory can be regarded as a special case of coincidence theory (put \( g = 1_X \), etc.), many of the details of the results in this section can be found in [2]. Let \( f : (X, A) \rightarrow (X, A) \) be a map of a pair of compact ANRs, with \( X \) connected, and let \( \hat{A} = \bigcup_k A_k \) be the disjoint union of those components \( A_k \) of \( A \) which are mapped by \( f \) into themselves (this is in fact finite since \( A \) is compact). We shall write \( f_k : A_k \rightarrow A_k \) for the restriction of \( f \) to \( A_k \). We choose points \( x_0 \in X \), and a path \( \omega : x_0 \rightarrow f(x_0) \) in \( X \) as in the last section. In addition for each component \( A_k \) of \( \hat{A} \) we choose base points \( a_k \in A_k \), and paths \( u_k : x_0 \rightarrow a_k \) in \( X \), and \( \omega_k : a_k \rightarrow f_k(a_k) \) in \( A_k \). Thus for each \( k \) there is a Reidemeister set \( \mathcal{R}(f^{\omega_k}_{k\*}) \). We define a function

\[
\hat{v}_k : \mathcal{R}(f^{\omega_{k\*}}_{k\*}) \rightarrow \mathcal{R}(f^\omega_{k\*})
\]

on a class \([\beta]\) to be \( \hat{v}_k([\beta]) = [v_k(\beta)] \) where \( v_k : \pi_1(A_k, a_k) \rightarrow \pi_1(X, x_0) \) is defined by \( v_k(\beta) = u_k \beta \omega_k f(u_k^{-1}) \omega^{-1} \). Note that since we identify paths and classes, we also identify
the path \( \beta \) in \( A_k \) on the right hand side of the definition with its image \( i_k(\beta) \) in \( X \), where \( i_k : A_k \to X \) is the inclusion. For \( \tilde{A} \) connected we will write \( v_A \) rather than \( v_1 \). The details of the proof of the following lemma can be found in [2, 3.2].

**Lemma 2.7.** The designation \( [\alpha] \to [v_k(\alpha)] \) is a well defined function \( \tilde{v}_k : \mathcal{R}(f^{(\alpha)}_k) \to \mathcal{R}(f^{(\mu)}_k) \). Moreover the following diagram is commutative

\[
\begin{array}{ccc}
\Phi(f_k)/\sim & \overset{i_k}{\longrightarrow} & \mathcal{R}(f^{(\alpha)}_k) \\
\downarrow & & \downarrow \tilde{v}_k \\
\Phi(f)/\sim & \overset{\rho}{\longrightarrow} & \mathcal{R}(f^{(\mu)}_k)
\end{array}
\]

As we have seen when \( \pi_1(X) \) and \( \pi_1(A) \) are abelian then there are group structures on \( \mathcal{R}(f^{(\alpha)}_k) \) and \( \mathcal{R}(f^{(\mu)}_k) \). However even in such cases \( \tilde{v}_k \) is not in general a homomorphism (see, for example, the proof of Proposition 6.5). However if in addition \( x_0 = a_k \) is a fixed point, and if \( u_k, \omega_k \) and \( \omega \) are all constant then it is.

**Definition 2.8.** A class \([\alpha] \in \mathcal{R}(f^{(\alpha)}_k)\) is said to be a weakly common fixed point class of \( f \) and \( f_A \) (or of \( f \) and \( f_k \)) if there is a \( k \) and a \([\beta] \in \mathcal{R}(f^{(\alpha)}_k)\) such that \([\alpha] = \tilde{v}_k([\beta])\). If \([\alpha] \) is essential, it is called an essential weakly common fixed point class of \( f \) and \( f_A \), if in addition \([\beta] \) is essential we call it an essential common fixed point class of \( f \) and \( f_A \).

The number of essential weakly common fixed point classes of \( f \) is denoted by \( E(f; f_A) \), and the number of essential common fixed point classes of \( f \) is denoted by \( N(f; f_A) \) (see [13]).

Note that \( 0 \leq N(f; f_A) \leq E(f; f_A) \), and that each inequality may be strict (see [15, p. 191]). If \( \rho(A^n) = [\beta] \) is a (weakly) common fixed point class by abuse of notation we sometimes refer to \( A^n \) as a (weakly) common fixed point class. Note that \([\beta] \) in Definition 2.8 need not be in the \( \rho \) image of \( \Phi(f_k)/\sim \). Thus the class \([\beta] \) may be empty. The next result is Lemma 2.3 of [15].

**Lemma 2.9.** A fixed point \( x \in \Phi(f) \) belongs to a weakly common fixed point class if and only if there is a path \( \alpha : (I, 0, 1) \to (X, x, A) \) from \( x \) to \( A \) and a homotopy \( H : \alpha \simeq f(\alpha) : (I, 0, 1) \to (X, x, A) \).

To emphasize the point we observe that \( H : I \times I \to X \) in Lemma 2.9 has \( H(0, t) = x \) for all \( t \in I \), but \( H(1, t) \) is only required to lie in \( A \).

**Corollary 2.10.** The definition of weakly common fixed point class is independent of the choices of \( x_0, a_k, u_k, \omega \) and \( \omega_k \).

**Corollary 2.11.** A fixed point class of \( f \) containing a fixed point on \( A \) is a weakly common fixed point class.
Definition 2.12. The number of essential fixed point classes of \( f : X \to X \), which are not weakly common fixed point classes is called the Nielsen number of \( f \) on the complement \( X - A \). It is denoted by \( N(f; X - A) \).

In other words, \( N(f; X - A) = N(f) - E(f; f_A) \).

Theorem 2.13. \( N(f; X - A) \) is a homotopy invariant.

Theorem 2.14. Any map that is relatively homotopic to a map \( f : (X, A) \to (X, A) \) has at least \( N(f; X - A) \) fixed points on \( X - A \).

The following computational theorem is the modified fundamental group version of one in [15] which is given in terms of homology cokernels. As we will see for Jiang spaces (needed in the hypotheses of the theorem) there is no need to pass to the homology cokernels, since by Theorem 2.1 the abelian group structure is already present at the level of the Reidemeister sets. Furthermore the cohomology approach can introduce unnecessary complications (see [15, 4.10]). The easy proof is omitted.

Theorem 2.15. Let \( f : (X, A) \to (X, A) \) be a map in which \( X \) is a Jiang space, then \( N(f, g) = 0 \) if \( L(f) = 0 \), and if \( L(f) \neq 0 \) then

\[
N(f; X - A) = \# \left( \mathcal{R}(f_A) - \bigcup_{k=1}^{\ell} \mathcal{V}_k(\mathcal{R}(f_A^n)) \right).
\]

3. Weakly common periodic point orbits

Although the theory of periodic points on the complement requires us to know some relative periodic point theory, a full working knowledge is not necessary. We develop here sufficient background to enable the reader to follow what is going on while at the same time introducing the new concepts needed in order to define our complement periodic numbers.

As in Section 2.2 we work in the context of a map \( f : (X, A) \to (X, A) \) of a pair of compact ANRs, where \( X \) is path-connected, but \( A \) need not be (its compactness implies that the number of its path-components is finite). We extend the notation from Section 2.2. If \( i \) is the inclusion \( i : A \to X \), then \( i \) is a morphism from \( f_A : A \to A \) to \( f : X \to X \), that is \( i \circ f = f \circ i \).

Recall the following definition from [9]

Definition 3.1. Let \( A \) be a compact (not necessarily path-connected) ANR and let \( f_A : A \to A \) be a self-map of \( A \). A set consisting of \( r \geq 1 \) distinct components \( \{A_1, A_2, \ldots, A_r\} \) of \( A \) is called an \( f_A \)-cycle in \( A \) (or simply a cycle) if \( f_A(A_j) \subseteq A_{j+1} \) for \( j = 1, \ldots, r - 1 \) and \( f_A(A_r) \subseteq A_1 \). We call \( r \) the length of the cycle, and denote the cycle by \([A_j]\).
Note that since the $A_k$ are distinct, $r$ is the smallest integer such that $f^r_A(A_1) \subseteq A_1$. In fact this is true for any other $A_k$ in the cycle. Thus for any $A_k$ in $[A_j]$, we define $c(k)$ to be the cycle length of $A_k$. Clearly if $A_k$ and $A_j$ belong to the same cycle, then $c(k) = c(j)$.

It is clear that cycles are disjoint in $A$, but their union need not be the whole of $A$ as a component need not belong to a cycle. (See [9] for a wider discussion of this point.)

Now let $f_A : A \to A$ be a map and $n$ be a given positive integer. Then (extending the definition from Section 3) we let $\tilde{A}_n$ be the disjoint union of those components $A_j$ of $A$ that have cycle lengths dividing $n$. Let $[A_j]$ be an $f_A$-cycle of length $c(j)n$, and let $A_j \in [A_j]$. We denote the restriction of $f^m$ to $A_j$ by $f^m_j$. Suppose that $f_j : A_j \to A_k$, then there are commutative diagrams

$$
\begin{array}{ccc}
A_j & \xrightarrow{f_j} & A_k \\
\downarrow i_j & & \downarrow i_k \\
X & \xrightarrow{f} & X
\end{array}
$$

and

$$
\begin{array}{ccc}
A_k & \xrightarrow{(f^m_j)^n} & A_k \\
\downarrow i_k & & \downarrow i_k \\
X & \xrightarrow{f^{mc(j)}} & X
\end{array}
$$

of path-connected spaces, where $m$ is any positive integer.

In what follows we will assume that $x_0$ and $\omega : x_0 \to f(x_0)$ are given, and for each $A_j \in \tilde{A}_n$ a base point $a_j$ and a path $\omega_j : a_j \to f^c(j)(a_j)$ in $A_j$. Since we identify paths and classes, we will also identify $f_j(\omega_j)$, etc., with $f(\omega_j)$, etc.

**Definition 3.2.** Let $[\alpha] \in R(f^n_A)$ be an $n$-periodic point class of $f$. Then $[\alpha]$ is a weakly common $n$-periodic point class of $f$ and $f_A$ if it is a weakly common fixed point class of $f^n$ and $(f^{c(k)n/c(k)})$ for some component $A_k$ in $\tilde{A}_n$. It is an essential weakly common $n$-periodic point class of $f$ and $f_A$ if it is a weakly common $n$-periodic point class which is itself essential, and it is an irreducible weakly common $n$-periodic point class of $f$ and $f_A$ if it is weakly common and irreducible.

From the general theory outlined in Section 2.2 the definition is independent of the choices of basepoint, etc. In order to develop our theory for periodic points we need:

**Proposition 3.3.** Let $[\alpha]^n \in R(f^n_A)$ be an $n$-periodic point class of $f$. Then if $[\alpha]^n$ is a weakly common $n$-periodic point class of $f$ and $f_A$ then so is $f^n([\alpha]^n)$, that is the property of being weakly common or not, is a property of the whole orbit $\{[\alpha]^n\} = \{f^n([\alpha]^n), f_{x_0}^{c_{x_0}}([\alpha]^n), \ldots, f_{x_0}^{c_{x_0}^{(k)}}([\alpha]^n)\}$.

The proposition of course is the algebraic counterpart of the geometric fact that if $x \in \Phi(f^n)$ lies in $X - A$, then the whole orbit $\{x, f(x), \ldots, f^{n-1}(x)\}$ must lie in $X - A$ since $f^n(x) = x$ does.

**Proof.** The statement of the proposition assumes we have already chosen a basepoint $x_0$ for $X$ and path $\omega : x_0 \to f(x_0)$. By hypothesis there exists an $A_k \in \tilde{A}_n$ such that $[\alpha]^n$ is a weakly common fixed point class of $f^n$ and $(f^{c_{x_0}^{(k)}})^{n/c(k)}$. For simplicity we let $m = n/c(k)$, and assume that $f_k : A_k \to A_{k+1}$. Next we choose a base point $a_k \in A_k$,
and paths $u_k : x_0 \to a_k$ in $X$, and $\omega_k : a_k \to (f_k^{(k)})^m(a_k) = f^n(a_k)$ in $A_k$. We choose the base point $a_{k+1} \in A_{k+1}$ to be $f(a_k)$, and the paths $u_{k+1} : x_0 \to a_{k+1}$ and $\omega_{k+1}$ in $X$ to be $\omega f(u_k)$ and $f(\omega_k)$, respectively. Since $[\alpha]^n$ is weakly common there is by definition a $\beta \in [\alpha]^n$ in $\mathcal{P}(\mathcal{R}(f_k^{(k)})^{\omega_k})$ such that $v_k(\beta) = u_k \beta \omega_k f^n(u_k^{-1})(n\omega)^{-1} \sim \alpha$. That is, there is a $\delta \in \pi_1(X, x_0)$ such that $u_k \beta \omega_k f^n(u_k^{-1})n\omega^{-1} = \delta f^n(\delta^{-1})n\omega^{-1}$. In what follows we will use the observation that $\omega f(n\omega) = n \omega f^n(\omega)$. Now

$$v_{k+1}(f_k(\beta)) = u_{k+1}f_k(\beta)\omega_{k+1}f^n(u_k^{-1})n\omega^{-1}$$

$$= \omega f(u_k)f(\omega_k)f^n(f(u_k^{-1})f^n(\omega^{-1}))n\omega^{-1}$$

$$= \omega f(u_k)f_k(\beta)f(\omega_k)f^n(\omega^{-1})f(n\omega^{-1})\omega^{-1}$$

$$= \omega f(u_k)f_k(\beta)f(\omega_k)f^n(u_k^{-1})n\omega^{-1}$$

$$= \omega f(\delta)n\omega f^n(\delta^{-1})n\omega^{-1}$$

$$= \omega f(\delta)n\omega f^n(\delta^{-1})n\omega^{-1}$$

$$= \omega f(\delta)n\omega f^n(\delta^{-1})n\omega^{-1}$$

$$= \omega f(\delta)n\omega f^n(\delta^{-1})n\omega^{-1}$$

$$= \omega f(\delta)n\omega f^n(\delta^{-1})n\omega^{-1}$$

$$\sim f^n(\alpha).$$

Thus $v_{k+1}(f_k([\alpha]^n)) = f^n([\alpha]^n)$, and $f^n([\alpha]^n)$ is a weakly common periodic point class as required. $\square$

Thus the property of being weakly common or not is a property of the whole orbit. We define

$$EP_n(f, f_A)$$

to be $n$ times the number of essential weakly common irreducible essential orbits of $f^n$ and $f_A^n$.

**Definition 3.4.** Let $f : (X, A) \to (X, A)$ be a map of a pair of compact ANRs. The relative Nielsen type number of period $n$ on the complement is

$$NP_n(f ; X - A) = NP_n(f) - EP_n(f, f_A).$$

**Theorem 3.5** (Lower bound property). If $(X, A)$ is a pair of compact ANRs, then every map $f : (X, A) \to (X, A)$ has at least $NP_n(f ; X - A)$ periodic points of least period $n$ on $X - A$.

**Proof.** From the fundamental Lemma 2.4 each essential irreducible orbit detects $n$ periodic points of $f$ on $X$. Thus the $NP_n(f ; X - A)/n$ essential irreducible orbits that are not weakly common essential orbits of $f^n$ and $f_A^n$ detect $NP_n(f ; X - A)$ periodic points of least period $n$ on $X$. If any of these periodic points were to lie in $A$ they would be
weakly common by Corollary 2.11. Thus there are \( NP_n(f; X - A) \) periodic points of least period \( n \) on \( X - A \) as required. \( \square \)

The proofs of the next three propositions are standard, and so are omitted.

**Proposition 3.6** (Homotopy invariance). *If the maps \( f, g : (X, A) \to (X, A) \) are homotopic, then \( NP_n(f; X - A) = NP_n(g; X - A) \).*

**Proposition 3.7** (Commutativity). *If \( f : (X, A) \to (Y, B) \) and \( g : (Y, B) \to (X, A) \), then \( NP_n(g \circ f; X - A) = NP_n(f \circ g; Y - B) \).*

**Proposition 3.8** (Homotopy type invariance). *If \( f : (X, A) \to (X, A) \) and \( g : (Y, B) \to (Y, B) \) are maps of the same (pairwise) homotopy type, then \( NP_n(f; X - A) = NP_n(g; Y - B) \).*

We close this section with a preliminary calculation on Example 1.1. Before we do this we recall the concept of \( n \)-torality from [8].

**Definition 3.9.** A map \( f : X \to X \) is said to be \( n \)-toral if (i) for every \( m|n \), and every \( [\alpha]^m \in R(f^n_{\alpha}) \), \( d([\alpha]^m) = d([\alpha]^m) \) that is the depths and length of orbits coincide, and (ii) for every \( m|n \), \( m = \) injective.

A minor modification of the proof of [8, 3.5] yields:

**Proposition 3.10.** *If \( f : X \to X \) is a map, and if \( \pi_1(X) \) is abelian, then \( f \) is \( n \)-toral if \( \text{Fix} f^n = 0 \). In this case \( NP_n(f; X - A) \) is equal to the number of non weakly common irreducible essential classes in \( R(f^n_{\alpha}) \).*

**Example 3.11.** Let \( f : (X, A) \to (X, A) \) be the map defined in Example 1.1, then \( X = S^1 \times S^1 \), and \( A = \Delta S^1 \) is the diagonal of \( X \). We set the base point to be \( x_0 = (1, 1) = (e^{it}, e^{it}) \), and the path \( \omega \) to be the constant path at \( x_0 \) in both \( X \) and \( A \), then \( \pi_1(A, x_0) = \mathbb{Z}, \) and \( \pi_1(X, x_0) = \mathbb{Z} \times \mathbb{Z} \). In this preliminary calculation we calculate only \( NP_A(f; X - A) \). Note that \( A \equiv S^1 \) and that with this identification \( f_A \) is a map of degree 2. Thus from Theorem 2.1 applied to the map \( f^n_A \) (of degree \( 2^n \)) we obtain the exact sequence of groups and homomorphisms

\[
\text{Fix}(f^n_{A^*}) \to \pi_1(A, x_0) \xrightarrow{1 - f^n_A} \pi_1(A, x_0) \xrightarrow{f^n_{A^*}} \mathcal{R}(f^n_{A^*}).
\]

So on the subspace \( A \) we have that \( \mathcal{R}(f^n_{A^*}) = \mathbb{Z}_{2^n - 1} \), and Fix \( f^n_{A^*} = 0 \) for all \( n \). Note from Theorem 2.3 that \( N(f^n_{A^*}) = 2^n - 1 \).

Coming to \( X \), note that \( t_{2,4} \circ t_{1,2} = t_{1,4} \), so in order to consider irreducibility at level 4, we need only consider those classes that come up from level 2. Let \( \zeta \) and \( \xi \) be the loops given by \( \zeta = (e^{2\pi i t}, 1)_{0 \leq t \leq 1} \) and \( \xi = (1, e^{2\pi i t})_{0 \leq t \leq 1} \), then \( \zeta \) and \( \xi \) are generators for \( \pi_1(X) \). If \( n = 2k \) is even, then \( f^n_A(\zeta) = 2^n \zeta \) and \( f^n_A(\xi) = 2^n \xi \), so from
Theorem 2.1, applied this time to $f^n : X \to X$, we see that $R(f^{m \omega}) = Z_{2n-1} \times Z_{2n-1}$, and that $\text{Fix} f^{m \omega} = 0$. Thus from Proposition 3.10 $f$ is $n$-toral, and we may compute $NP_4(f; X - A)$ as the number of non-weakly common irreducible essential classes at level 4. Consider the commutative diagram (see also Proposition 4.1 and [8, 1.8])

\[
\begin{array}{ccc}
\pi_1(X) & \xrightarrow{1-f_2^{2\omega}} & \pi_1(X) \\
\pi_1(X) & \xrightarrow{1-f_4^{4\omega}} & \pi_1(X)
\end{array}
\]

where by the choice of base points the vertical functions $(\tilde{v}_1, \tilde{v}_2)$ are simply the induced homomorphisms namely the diagonal homomorphisms. In fact for $n$ even, the sequences for $f$ can be thought of as the product of two copies of the sequences for $f_A$. In particular, since with respect to $f_A \iota_{2,4} : Z_3 \to Z_15$ is multiplication by $1 + 2^2$, then $\iota_{2,4} : Z_3 \times Z_3 \to Z_{15} \times Z_{15}$ for $f$ takes $[(a, b)]^2$ to $[(5a, 5b)]^2$. Note that diagonal elements are taken to diagonal elements, and non-diagonal elements are taken to non-diagonal elements.

From the diagram then it is easy to see that the weakly common classes are on the diagonals, and since both $\iota_{2,4}$ are injective, $X$ is a Jiang space, and $L(f^2) \neq 0$, then from Theorem 2.3 there are $(15 \times 15 - 15 \times 15 = 3 \times 3 - 3)$ irreducible essential non-weakly common classes at level 4. Thus $NP_4(f; X - A) = 204$.

We remark that in this particular example we can see from the geometry that the number $NP_4(f; X - A)$ is precisely the number of irreducible essential periodic points of period 4 on the complement. We also remark however that by the count of actual points in this way it is not possible (as it is from the Nielsen theory) to deduce that this is the minimum number of such points, nor that none of them can be moved to $A$ by relative homotopies of $f$.

4. The Nielsen type number for the $n$th iterate on the complement

The main technical tool that is required for the extension of the second periodic point number to the complement is the following proposition

**Proposition 4.1.** Let $[\alpha]^m \in R(f^{m \omega})$ be an $m$-periodic point class of $f$. If $[\alpha]^m$ is a weakly common $m$-periodic point class of $f$ and $f_A$ and if $m|n$ then $\iota_{m,n}([\alpha]^m)$ is also weakly common.
Proof. In what follows we show that if \( A \) in the left hand diagram is path connected then the right hand diagram is commutative.

\[
\begin{array}{cccc}
A & \xrightarrow{f_A} & A & \xrightarrow{\mathcal{R}(f_{A^2})} \mathcal{R}(f_{A^2}) \\
\downarrow f & & \downarrow f_A & \downarrow \mathcal{R}(f_A) \\
X & \xrightarrow{f} & X & \xrightarrow{\mathcal{R}(f_{X^2})} \mathcal{R}(f_{X^2})
\end{array}
\]

where \( \mathcal{R} \) and \( \mathcal{R}(f_{A^2}) \) are defined as in Section 2.2 using \( \omega : x_0 \to f(x_0) \) in \( A \) and \( u : x_0 \to a_0 \) in \( X \) respectively. Now we apply this to the second diagram in (3) to obtain the required result.

Now

\[
\mathcal{R}(f_{X^2}) (1,n) \beta)^{1}
\]

\[
= \mathcal{R}(f_{X^2})(1,n) \beta)^{1}
\]

The converse of Proposition 4.1 is false as the following example shows.

**Example 4.2.** Let \( f = h \vee g : (S^1 \vee S^2, x_0) \to (S^1 \vee S^2, x_0) \) be the map where \( h : S^1 \to S^1 \) is the flip map given by \( h(e^{i0}) = e^{-i0} \), \( g : S^2 \to S^2 \) is a map of degree 2, and the common point \( x_0 \) is \( e^{i0} \) in \( S^1 \), and a fixed point in \( S^2 \). If we take the path \( \omega \) to be the constant path at \( x_0 \), then from below \( \mathcal{R}(f_{X^2}) = \mathbb{Z}_2 \), and \( \mathcal{R}(f_{X^2}) = \mathbb{Z} \). Moreover the diagram

\[
\begin{array}{cccc}
\pi_1(X) & \xrightarrow{1-f_{X^2}} & \pi_1(X) & \xrightarrow{\mathcal{R}(f_{X^2})} \mathbb{Z}_2 \\
\downarrow 1-f_{X^2} & & \downarrow \mathcal{R}(f_{X^2}) & \downarrow 0 \\
\pi_1(X) & \xrightarrow{1-f_{X^2}} & \pi_1(X) & \xrightarrow{\mathcal{R}(f_{X^2})} \mathbb{Z}_2
\end{array}
\]

\[
\begin{array}{cccc}
\mathcal{R}(f_{X^2}) = \mathbb{Z}_2 & \xrightarrow{1-f_{X^2}} & \mathcal{R}(f_{X^2}) = \mathbb{Z}_2 & \xrightarrow{0} \mathcal{R}(f_{X^2}) = \mathbb{Z}
\end{array}
\]
is commutative. Since $f^n_\alpha$ is multiplication by $-1$, the arrow denoted by 0 is of course $\ell_{1,2}$.

Of the two classes in $R(f^n_\alpha) = \mathbb{Z}_2$, one is weakly common and the other not. However both ‘boost’ to the single weakly common class $[0]^2$ in $R(f^{2\alpha}_\alpha)$.

The contrapositive however gives us:

**Corollary 4.3.** If $[\alpha]^n \in R(f^{n_\alpha}_\alpha)$ reduces to $[\beta]^m \in R(f^{m_\alpha}_\alpha)$, and if $[\alpha]^n$ is a weakly non-common $n$-periodic point class of $f$ and $f \alpha$ then so also is $[\beta]^m$.

We are now ready to define the Nielsen type number for the $n$th iterate on the complement.

**Definition 4.4.** Let $f : (X, A) \to (X, A)$ be a map of a pair of compact ANRs. The relative Nielsen number for the $n$th iterate on the complement of $f : (X, A) \to (X, A)$ is the number

$$N\Phi_n(f ; X - A) = \min \{ h(S) : S \text{ a set of representatives of the set}$$
$$\text{of essential weakly non-common algebraic orbits of } f \text{ at levels } n \text{ or less} \}.$$ 

As the height is a positive integer (or zero), we see that there always exists at least one set $S$ of representatives for $f : (X, A) \to (X, A)$ (as in the definition) with $h(S) = N\Phi_n(f ; X - A)$. By Corollary 4.3 if $S$ is a minimal such set of representatives, then every element will be weakly non-common. If the empty set is such a set of representatives for $f : (X, A) \to (X, A)$ then $N\Phi_n(f ; X - A) = 0$.

The reader will have noticed that the definition of $N\Phi_n(f ; X - A)$ is complicated, and in practice it is not what we use for calculations. It is however convenient to use for proofs, as we see next in showing that $N\Phi_n(f ; X - A)$ has the required lower bound property.

**Proposition 4.5** (Lower bound property). If $f : (X, A) \to (X, A)$ is a map of a pair of compact ANRs, then $f$ has at least $N\Phi_n(f ; X - A)$ periodic points of least period $n$.

**Proof.** Note that since $f(A) \subseteq A$, then if $x \in \Phi(f^n)$ (and so $f^n(x) = x$) and also $x \in X - A$, then the whole orbit $\{x, f(x), \ldots, f^{n-1}(x)\}$ lies in $X - A$. Let $\Phi(f^n ; X - A) = \Phi(f^n) \cap (X - A)$. We begin by partitioning $\Phi(f^n ; X - A)$ as follows: two elements $x$ and $y$ of $\Phi(f^n ; X - A)$ belong to the same element of this partition if and only if there is a non-negative integer $q$ such that $x = f^q(y)$. We denote the elements of this partition by $Q^n, R^n, \ldots$. If $x \in Q^n$, then $Q^n$ can be written as

$$Q^n = \{x, f(x), \ldots, f^{m-1}(x)\},$$

and $m$ is the smallest positive integer such that $f^m(x) = x$. As we have seen $Q^n \subseteq X - A$.

Clearly $\#Q^n = m$; moreover, each $Q^n$ determines a geometric orbit $\{A(m)\}$ of $f$ of period $m$ in $\Phi(f^m)$. Since (via $\rho$), an algebraic orbit $\{\rho(A(m))\}$ in $R(f^{m_\alpha})$. Let $S$ be the set of all weakly non-common algebraic orbits determined in the above way. (Note that we may have $Q^n \subseteq X - A$ but the algebraic orbit determined by it may
fail to be weakly non-common.) From the general theory (see inequality (2)) $d(\langle A^{(k)} \rangle) \geq d(\langle A^{(k)} \rangle)$ so we see that

$\# \Phi(f^n X - A) \geq \sum m \geq \sum d(\langle A^{(m)} \rangle) \geq \sum \left( d(\langle [\alpha]^{(m)} \rangle) \right) = h(S)$

(where each sum is taken over the set of algebraic orbits $\{[\alpha]^{(m)} \in S\}$).

We need only show that $S$ is a set of weakly non-common $n$-representatives. Let $\{[\alpha]^{(p)}\}$ be an essential weakly non-common algebraic orbit where $p|n$, and let $x \in \Phi(f^p)$ be such that if $x \in A^{(p)}$ then

$\left[ \rho(A^{(p)}) \right]^{(p)} = \langle [\alpha]^{(p)} \rangle$.

Now $x \in Q^m$ for some $Q^m$; moreover, $m|p$ by the division algorithm and so,

$[m, p] \left[ \left[ \rho(A^{(m)}) \right]^{(m)} \right] = \langle [\alpha]^{(p)} \rangle$

by the commutativity of the second diagram in [8, 1.14]. That is $\langle [\alpha]^{(p)} \rangle$ is reducible to the element $\langle [\rho(A^{(m)})]^{(m)} \rangle$ of $S$. \qed

We finish this section by stating that the relative Nielsen type number for the $n$th iterate $N\Phi_n(f; X - A)$ satisfies the usual properties of Nielsen type numbers. The proofs are straightforward modifications of the general theory.

**Proposition 4.6** (Homotopy invariance). *If the maps $f, g : (X, A) \to (X, A)$ are homotopic, then $N\Phi_n(f; X - A) = N\Phi_n(g; X - A)$.*

**Proposition 4.7** (Commutativity). *If $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (X, A)$, then $N\Phi_n(g \circ f; X - A) = N\Phi_n(f \circ g; Y - B)$.*

**Proposition 4.8** (Homotopy type invariance). *If $f : (X, A) \to (X, A)$ and $g : (Y, B) \to (Y, B)$ are maps of the same (pairwise) homotopy type, then $N\Phi_n(f; X - A) = N\Phi_n(g; Y - B)$.*

5. Relations between the Nielsen complement numbers

In this section we relate the numbers $NP_n(f; X - A)$ and $N\Phi_n(f; X - A)$ to each other, and to $N(f^m; X - A)$ for $m|n$. The first result is obvious.

**Proposition 5.1.** $N\Phi_1(f; X - A) = NP_1(f; X - A) = N(f; X - A)$.

We say that a map $f : (X, A) \to (X, A)$ is weakly non-common essentially reducible if every weakly non-common orbit $\{[\alpha]^{(p)}\}$ is reducible to orbits $\{[\beta]^{(p)}\}$ that are essential. Note that essentially reducible implies weakly non-common essentially reducible, but the converse is false (take any map on a space $X$ that is not essentially reducible, and then take $A = X$).
Let $\text{IENWO}_m(f)$ denote the set of irreducible essential weakly non-common orbits at level $m$. Clearly $h(\text{IENWO}_m(f)) = \text{NP}_m(f)$. Note that the set $\bigcup m \mid n \text{IENWO}_m(f)$ is always a subset of any set of weakly non-common $n$-representatives. With these observations the next results is immediate.

**Theorem 5.2.** Let $f : (X, A) \to (X, A)$ be a map.

$$N\Phi_n(f; X - A) \geq \sum_{m \mid n} \text{NP}_m(f; X - A).$$

If $f$ is weakly non-common essentially reducible then equality holds.

If under the condition of being weakly non-common essentially reducible, we apply the Möbius inversion formula to Proposition 5.2, then we obtain the next corollary.

**Corollary 5.3.** Let $n$ be a positive integer, and $f : (X, A) \to (X, A)$ a map which is weakly non-common essentially reducible. If $\mathcal{P}(n) = \{p_1, \ldots, p_k\}$ denotes the set of all distinct primes dividing $n$. Then

$$\text{NP}_n(f; X - A) = \sum_{\tau \subseteq \mathcal{P}(n)} (-1)^{\#\tau} N\Phi_{n: \tau}(f; X - A)$$

where $n : \tau = n(\prod_{p \in \tau} p)^{-1}$.

Our final results compare the periodic Nielsen numbers on the complement with each other, and with the ordinary Nielsen numbers on the complement of various iterates of $f$. We also give conditions under which the relative Nielsen number $N\Phi_n(f; X - A)$ for the complement for the $n$th iterate of a map of pairs $f : (X, A) \to (X, A)$ is equal to the relative Nielsen number $N(f^n; X - A)$ of $f^n$ on the complement. Since the ideas behind the proofs are not really new in this section, we omit them.

**Theorem 5.4.** If $f : (X, A) \to (X, A)$ is a self-map, then

$$N\Phi_n(f; X - A) \geq N(f^n; X - A).$$

Furthermore, if $f$ satisfies (a) $f$ is n-toral, (b) $\pi_1(X)$ is abelian, (c) $N(f^m) = R(f^m)$ for every $m \mid n$, then equality holds. In particular, if $X$ is a Jiang space, $L(f^m) \neq 0$ for every $m \mid n$, and if $\text{Fix} f^{m_0} = 0$, then equality holds.

Even when $N(f^n) \neq 0$ without some condition (such as $n$-torality) the inequality in Theorem 5.4 may be strict. This is demonstrated in the following example, which is a modification of [12, Example 4, p. 67] (see also [8, Example 2.3]).

**Example 5.5.** Let $X$ be the projective space $\mathbb{RP}^3$, considered as the quotient of $S^3 = \{(ue^{i\theta}, ve^{i\theta}) : u^2 + v^2 = 1\}$ modulo identification of antipodal points, and let $f : \mathbb{RP}^3 \to \mathbb{RP}^3$ be induced by the map $\tilde{f} : S^3 \to S^3$ given by $\tilde{f}(ue^{i\theta}, ve^{i\theta}) = (ue^{3i\theta}, ve^{3i\theta})$. Since $L(f) \neq 0$ there is a fixed point we denote by $x_0$. Letting $\omega$ be the constant path at
\(x_0\), we have that \(\pi = \pi_1(\mathbb{R}P^3) = \mathbb{Z}_2\), that \(f_{n}^{\text{nor}}\) is the identity, and that \(R(f_{n}^{\text{nor}}) \cong \mathbb{Z}_2\) for all \(n\). Moreover since \(f_{n}^{\text{nor}}\) is the identity, then each of the two orbits \(\langle [\alpha]^n \rangle\) consists of a single class \([\alpha]^n\), and \(t_{m,n}\) is multiplication by \(n|m\). In particular for \(n = 2\), \(t_{1,2}\) is not injective and so \(f\) is not \(n\) toral at least for this \(n\).

Now \(X\) is a Jiang space and \(L(f^n) \neq 0\), so for any \(n\) every periodic point class of \(f\) is essential. If \(n = 2^r\) for some \(r > 0\), then \(\langle [\alpha]^n \rangle\) is irreducible, where \([\alpha]^n\) denotes the \(n\)-periodic point class of \(f^n\) which corresponds to the generator of \(\mathbb{Z}_2\). This shows that the single class \([\alpha]^n\) contains at least \(n\) periodic points by Lemma 2.4. If \(A = \{x_0\}\), and \(n = 2^r\), then for \(r \geq 1\),

\[
N\Phi_n(f; X - A) = N\Phi_n(f) - 1 = 2^{r+1} - 1, \quad \text{while } N(f^n; X - A) = 1.
\]

**Corollary 5.6.** Let \(n\) be a positive integer, \(f : (X, A) \to (X, A)\) a map, \(X\) a Jiang space, \(L(f^n) \neq 0\) for each \(m|n\), and \(\text{Fix } f^{\text{nor}} = 0\). Then

\[
N\Phi_n(f; X - A) = \sum_{\tau \leq \mathcal{P}(n)} (-1)^{\#\tau} N(f^{\tau}; X - A),
\]

where \(\mathcal{P}(n) = \{p_1, \ldots, p_k\}\) denotes the set of all distinct primes dividing \(n\), and \(n : \tau = n(\prod_{p \in \tau} p)^{-1}\).

For a given \(f : X \to X\) and a fixed natural number \(n\), recall from [11] that \(M(f, n)\) is defined by requiring that \(m \in M(f, n)\) iff \(m|n, N(f^m) \neq 0\) and if \(m|k|n\) with \(m \neq k\), then \(N(f^k) = 0\). The next theorem is an analogue of [11, 4.17].

**Theorem 5.7.** Let \(X\) be a Jiang space, and \(f : X \to X\) be a map which satisfies the following condition. For each \(m \in M(f, n)\), and for each \(k|m\) \(L(f^k) \neq 0\) and \(\text{Fix } f^{\text{nor}} = 0\). Then

\[
N\Phi_n(f; X - A) = \sum_{\mu \neq \mu \in M(f, n)} (-1)^{\#\mu - 1} N(f^{\xi(\mu)}; X - A),
\]

where \(\xi(\mu)\) is the greatest common divisor of all elements of \(\mu\).

The proof is essentially the same as the proof of [11, 4.17], but the proof there has some errors. A better proof can be found in [7, Theorem 5.8]. We will not repeat these ideas here.

**Example 5.8.** Let \(h : S^1 \to S^1\) be the flip map given by \(h(e^{i\theta}) = e^{-i\theta}\) given as the restriction of \(f\) in Example 4.2. Note that if \(x_0 = e^{i0}\) and \(\omega\) is the constant path, then the Reidemeister sets are the same as those of \(f\) in that example. In particular for odd \(n\), \(R(h^{\text{nor}}) = \mathbb{Z}_2\), and for even \(n\), \(R(h^{\text{nor}}) = \mathbb{Z}\). Unlike Example 4.2 for all even \(n\) all Reidemeister classes are inessential. In particular \(N(f^{2}; X - A) = 0\). From Theorem 5.7 \(N\Phi_2(h; X - A) = N(f; X - A) = 1\). In fact \(N\Phi_2(h; X - A) = 1\) for all \(n\), while \(N\Phi_{n}(h; X - A) = 1\), and for all \(n > 1\), \(N\Phi_{n}(h; X - A) = 0\).
6. Two examples

In this penultimate section we give the promised formula for $N\Phi_n(f; X - A)$ in Example 1.1, and show how the calculation of the $NP_n(f; X - A)$ are algorithmic. In addition we give an example where the subspace is not connected. We state the results of the first example as a theorem and give a specific case which shows how to use the algorithm.

**Theorem 6.1.** Let $(X, A) = (S^1 \times S^1, \Delta S^1)$ and $f : (X, A) \to (X, A)$ be as in Example 1.1, then

$$N\Phi_n(f; X - A) = N(f^n; X - A) = \begin{cases} 2^{2n} - 3 \cdot 2^n + 2 & \text{if } n \text{ is even}, \\ 2^{2n} - 2^n & \text{otherwise}, \end{cases}$$

and the $NP_n(f; X - A)$ can be obtained from these formulae by Möbius inversion. Furthermore $N\Phi_n(f; X - A) = M\Phi_n(f; X - A)$, and $NP_n(f; X - A) = MP_n(f; X - A)$ for this example.

**Proof.** The proof of the first part follows from Theorems 5.4 and 2.15 and the following lemma once we note that $L(f^n) \neq 0$ for all $n$, and with the choice of base points given in Example 3.11, that $\nu_A$ in Theorem 2.15 is the induced homomorphism $i_*$ on the Reidemeister groups. For the very last part we may simply observe this by counting, but see also the periodic Wecken properties of tori [14], and the last part of Section 7. □

**Lemma 6.2.** For $f$ as in the theorem we have that $\mathcal{R}(f^{n\alpha}_*) = \mathbb{Z}_{2^n-1}$, and

$$\mathcal{R}(f^{n\alpha}_*) = \begin{cases} \mathbb{Z}_{2^n-1} \times \mathbb{Z}_{2^n-1} & \text{if } n \text{ is even}, \\ \mathbb{Z}_{2^n-1} & \text{otherwise}. \end{cases}$$

In both cases $\text{Fix } f^{n\alpha}_* = 0$, and so $f$ is $n$-toral for all $n$. Furthermore $i_* : \mathcal{R}(f^{n\alpha}_*) \to \mathcal{R}(f^{n\alpha}_*)$ is an injective homomorphism for all $n$.

**Proof.** In fact we started the analysis in Example 3.11 where we discovered that $\mathcal{R}(f^{n\alpha}_*) = \mathbb{Z}_{2^n-1}$, and from the exact sequence of groups (2.1)

$$0 \to \text{Fix } f^{n\alpha}_* \to \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} \to \mathcal{R}(f^{n\alpha}_*) \to 0,$$

that for $n$ even that $\mathcal{R}(f^{n\alpha}_*) = \mathbb{Z}_{2^n-1} \times \mathbb{Z}_{2^n-1}$, and $\text{Fix } f^{n\alpha}_* = 0$.

For $n$ odd we see from the above sequence that $\mathcal{R}(f^{n\alpha}_*) = \mathbb{Z}_{2^{2n-1}}$ and $\text{Fix } f^{n\alpha}_* = 0$ as follows. For simplicity let us agree to equate the generator of $\pi_1(A)$ with $(1, 1)$, and the generators $\eta$ and $\xi$ of $\mathbb{Z} \times \mathbb{Z}$ given in Example 3.11 with $(1, 0)$, and $(0, 1)$, respectively. The matrix of $1 - f^{n\alpha}_*$ with respect to the standard basis is

$$F = \begin{pmatrix} 1 & -2^n \\ -2^n & 1 \end{pmatrix}. $$

Since $\det(F) \neq 0$, then $1 - f^{n\alpha}_*$ is injective and so $\text{Fix } f^{n\alpha}_* = \text{Ker}(1 - f^{n\alpha}_*) = 0$ as required. Thus from Proposition 3.10 $f^{n\alpha}_*$ is $n$-toral for all $n$. 
For \( n \) odd again, note that if \((x, y) \in \mathbb{Z} \times \mathbb{Z}\) then \((x, y) = x(1, -2^n) + \{2^n + y\}(0, 1)\), and so \((1, -2^n)\) and \((0, 1)\) are generators of \( \mathbb{Z} \times \mathbb{Z}\). Also \((1 - f^n)^n(x, y) = (x - 2^n y, y - 2^n x) = (x - 2^n y)(1, -2^n) + y(0, 1 - 2^{2n})\). Thus \((1, -2^n)\) and \((0, 1 - 2^{2n})\) generate \( \operatorname{Im}(1 - f^n)^n\). In particular \([1(-2^n)]^n = [0]^n\) in \( \mathcal{R}(f^n)\), so this group is cyclic with \([0, 1)]^n\) as a generator. To see that the order is \(2^{2n} - 1\), suppose that \((0, b)\) represents \([0, b)]^n = [0]^n\). Then \((0, b) = \lambda(1, -2^n) + \mu(0, 1 - 2^{2n})\) for some \(\lambda\) and \(\mu\). In fact \(\lambda = 0\) and \(b = \mu(1 - 2^{2n})\), so \(b\) is a multiple of \(1 - 2^{2n}\) as required.

It remains to show that \(i_* : \mathcal{R}(f^n) \to \mathcal{R}(f^n)\) is injective. For \( n \) even this is easy since the induced map \(i_*\) is the inclusion of \(\mathbb{Z}_{2^{n-1}}\) into the diagonal of \(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^{n-1}}\). For \( n \) odd we note that the generator of \( \mathcal{R}(f^n) = \mathbb{Z}_{2^{n-1}}\) is taken by \(i_*\) into \([1, 1)]^n = [(1, -2^n)]^n + [(0, 2^n + 1)]^n = [(0, 2^n + 1)]^n\), an element of order \(2^n - 1\) in \(\mathbb{Z}_{2^{n-1}}\).

We illustrate next that the \(NP_n(f)\) can be calculated by algorithm from Theorems 6.1 and 5.6.

**Example 6.3.** We calculate \(NP_{30}(f ; X - A)\) for Example 1.1. In fact noting that \(M (f, 30) = [2, 3, 5]\) in Theorem 5.6, we have, from that theorem and from Theorem 6.1, that

\[
NP_{30}(f ; X - A) = N(f^{30} ; X - A) - N(f^{15} ; X - A) - N(f^{10} ; X - A) - N(f^5 ; X - A)
+ N(f^3 ; X - A) + N(f^2 ; X - A) - N(f ; X - A)
= 1,152,921,500,310,864,090.
\]

We give a second example using the so called “Banana” example from [9]. It illustrates what happens when the subspace is not connected.

**Example 6.4 (The “Banana Example”).** Let \(X\) be the solid torus \(X = S^1 \times D^2\). First we define a self-map \(g\) of the core \(S^1 \times \{0\}\) of \(X\). We regard the core as the subset \(\{e^{i\theta} : 0 \leq \theta < 2\pi\}\) of \(C\), and let \(g\) be the standard map of degree 3 given by \(g(e^{i\theta}) = e^{3i\theta}\).

There are eight points of \(g\) which are fixed points of \(g^2\) and hence have least period 2 or less for \(g\), namely \(e^{j\pi/4}\) for \(j = 0, 1, \ldots, 7\). Choose \(x_0 = e^{i\theta}\) as the basepoint, and the constant path as \(\alpha\), then by taking the obvious path from \(x_0\) to \(e^{j\pi/4}\) we see that \(\rho([e^{j\pi/4}]) = [j]^2\) in \(\mathcal{R}(g^2)\) for \(j = 0, 1, \ldots, 7\). The irreducible periodic point classes of period 2 of \(g\) are those represented by a point \([j]^2\) for which \(j\) is not divisible by 1 + \(\deg(g) = 4\), and all periodic point classes of period 2 are essential. Next we select integers \(d_j\) for \(j = 0, 1, \ldots, 7\), and define a subspace \(A\) and a self-map \(f_A\) of \(A\) which will depend on the \(d_j\)’s (in [9] specific choices are sometimes made for the \(d_j\)’s, but the choices do not affect the periodic points on the complement, so we leave them arbitrary).

We let \(A = S^1_1 \cup S^1_2 \cup S^1_3 \cup \cdots \cup S^1_7\) be the subspace of \(X = S^1 \times D^2\) which consists of the eight boundary circles of the disks \(e^{j\pi/4} \times D^2\), with \(j = 0, 1, \ldots, 7\). The map \(f_A\) maps each circle of \(A\) to another circle of \(A\) in a fashion compatible with the map \(g\), i.e., the range of \(S^1_j\) is \(S^1_k\), where \(k = 3j \mod 8\), and so that the restriction of \(f_A\) to \(S^1_j\)
is a standard map of degree \( d_j \). From the maps \( g \) and \( f_A \) we obtain a self-map of the subspace \( (S^1 \times \{0\}) \cup A \) of \( X \). We can now extend this to a map \( f : (X, A) \to (X, A) \) by defining \( f \) on each of the eight pieces \( [e^{i\theta} \times \mathbb{D}^2; j\pi/4 \leq \theta \leq (j + 1)\pi/4] \) of \( X \). Each such piece is shaped like a piece of a banana, and the map \( f \) maps each piece into the union of three pieces so that each piece is stretched according to the map \( g \) and “twisted” according to the map \( f_A \) until a continuous map \( f : (X, A) \to (X, A) \) (determined up to homotopy) is obtained. Our aim is to calculate \( N\Phi_n(f; X - A) \) and \( N\Phi_n(f; X - A) \) for all \( n \).

**Proposition 6.5.** Let \( f \) be as in Example 6.4, then

\[
N\Phi_n(f; X - A) = N(f^n; X - A) = \begin{cases} 3^n - 3 & \text{if } n \text{ is odd}, \\ 3^n - 9 & \text{otherwise}. \end{cases}
\]

Moreover Möbius inversion holds, and the \( N\Phi_n(f; X - A) \) can be calculated as in Corollary 5.6. In particular, \( N\Phi_1(f; X - A) = N\Phi_2(f; X - A) = 0 \).

**Proof.** Note first that the map \( g \) is a retract of \( f \), and so \( f \) and \( g \) have the same Nielsen theory. It should be clear that \( f \) satisfies the conditions of Theorem 5.4 that allow us to equate \( N\Phi_n(f; X - A) \) with \( N(f^n; X - A) \). Thus we can use Theorem 2.15 again, but this time we need to use the full definition of the \( \nu_k \). We choose basepoints and paths, as required for Definition 3.2 as follows: choose \( x_0 \) to be \( (e^{i0}, e^{i0}) \), and \( \omega \) as the constant path. For \( j = 0, 1, \ldots, 7 \) we choose \( a_j \) to be \( (e^{j\pi i/4}, e^{i0}) \), and \( \omega_j \) as constant paths. Finally for \( j = 0, 1, \ldots, 7 \) we choose \( u_j \) to be the shortest path counterclockwise from \( x_0 \) to \( a_j \). We do the calculations for \( n = 2r \), the odd case is similar. Consider first the case \( r = 1 \). Observe that the inclusion of \( A_j \) into \( X \) factors through a contractible space. Thus we see from the definitions that for \( j = 0, 1, \ldots, 7 \), and for any \( [\alpha]j^2 \in \mathcal{R}(f_{j*}^{2r\alpha}) \) that \( \nu_j^2([\alpha]j^2) = [j]^2 \) (note that we use superscripts on the \( \nu_j \) as in Proposition 4.1 to indicate the level). Since the images of the various \( \nu_j^2 \) are distinct, and comprise the whole of \( \mathcal{R}(f_{j*}^{2r\alpha}) \) we have, from Theorem 2.15, that \( N(f^2; X - A) = 0 \).

For \( n = 2r \) with \( r > 1 \) we see as above that the images of the various \( \mathcal{R}(f_{j*}^{2r\alpha}) \) are singletons \( [\alpha_j]j^{2r} \) say. To see that they too are all distinct consider the following commutative diagram (from the proof of Proposition 4.1).

\[
\begin{array}{ccc}
\mathcal{R}(f_{j*}^{2r\alpha}) & \xrightarrow{1_{2r}} & \mathcal{R}(f_{j*}^{2r\alpha}) \\
\nu_j^2 & \downarrow & \nu_j^2 \\
\mathbb{Z}_{2r} & \cong & \mathcal{R}(f_{j*}^{2r\alpha}) \xrightarrow{1_{2r}} \mathcal{R}(f_{j*}^{2r\alpha}) \cong \mathbb{Z}_{32r} - 1
\end{array}
\]
Now $[\alpha j]^2r = v_j^2r(t_{2,2r}([\alpha j]^2))$ for any $[\alpha j]^2 \in R(f_{j^2r}^{(2r)})$. From the commutativity of the diagram, and the fact that $v_j^2r([\alpha j]^2) = [j]^2$, we see that $[\alpha j]^2r = t_{2,2r}([j]^2)$. From Theorem 2.1 $f$ is $n$ toral for all $n$, so from Proposition 3.10 $t_{2,2r}$ is injective and the images of the $R(f_{j^2r}^{(2r)})$ are distinct as required. The result follows. \hfill $\Box$

7. Concluding remarks

In this paper we have presented a theory of periodic points on the complement that extends the fixed point theory case presented by the second author in [15]. As is pointed out in [17] for the fixed point case, there are 16 possible Nielsen type numbers which can be described as follows. For a given map $f : (X, A) \to (X, A)$ and a given fixed point class $A$ of $f$, we can ask four questions. Firstly the question of $A$ being essential or not, secondly the question of $A$ containing an essential fixed point class of $f$ and $f_A$ or not, thirdly the question of $A$ containing an inessential fixed point class of $f$ and $f_A$ or not, and lastly the question of $A$ assuming its index in $A$ or not. Since each question has two possible answers there are $2^4$ possible numbers that can be defined by asking each of the four questions in turn. Each combination of yes or no answers gives rise to a Nielsen type number for $f$. The case $n = 1$ in this paper (which is of course the same as [15]) concerns the answers ‘yes, no, no, and no’, respectively. Of course, when we generalize to periodic points we can ask a fifth question, namely is $A$ irreducible or not. An affirmative answer roughly corresponds to the $NP_n(f)$ numbers, while a negative one does not quite so easily correspond to the $N\Phi_n(f)$ numbers. It is however certainly related to them. As pointed out in a 1993 talk in Cortona [6] this makes 32 potential Nielsen type numbers for periodic points. We are not done yet however, since these 32 possibilities do not take into account the possible generalizations to periodic points, of the non-surplus theory on the complement, also due to the second author [16]. This last theory uses a local Nielsen number defined on the space $X - A$, it can detect points that do not show up under any of the 16 ordinary Nielsen numbers outlined above.

For the fixed point case four of the sixteen Nielsen type numbers are always zero (see [17]). Thus both in the fixed point case, and here for periodic points, not all of the indicated numbers are deserving of study. Perhaps the most interesting of the remaining periodic numbers is the generalization of non-surplus fixed point theory. This however also appears to be the most difficult. Let us consider the example from [16], that motivates non-surplus theory.

**Example 7.1.** Let $X = S^1$, $A = \{ \pm 1 \}$ and let $f : S^1 \to S^1$ be the map defined by $f(e^{i\theta}) = e^{2i\theta}$ if $0 \leq \theta \leq \pi$, and $f(e^{i\theta}) = e^{-2i\theta}$ if $\pi \leq \theta \leq 2\pi$. Note that $4\pi/3$ is a fixed point, that has a local non zero index on $X - A$. This point comprises a “non-surplus” fixed point class (there is no path in $X - A$ which satisfied Lemma 2.9). This basically means that the class cannot be moved to $A$ by a relative homotopy of $f$. Thus this fixed point class is an essential fixed point class for the complement in this theory. The non-surplus Nielsen number is the number of such classes, and is a lower bound for the number of fixed points.
on $X - A$, it may be greater than $N(f; X - A)$. In this example the non-surplus fixed point number is one, while $N(f; X - A) = 0$.

In considerations of the iterates of $f$ we see for example that $4\pi/5$ and $8\pi/5$ are periodic points of least period 2. It is furthermore reasonably clear that these points neither disappear nor move to $A$ under relative homotopies of $f$ (note again the difference between the consideration of relative homotopies of $f$ and those of $f^2$). The greatest (but not the only) difficulty seems to be that two periodic points which may be in the same non-surplus class of $f^n$ on the complement may be taken to different components of the complement upon iteration by $f$. At the time of writing the question of the existence of a satisfactory surplus periodic numbers is an open question.

The reader will also have noted that there are no minimum theorems in this paper. Part of reason for this is that the so called Wecken theory (the theory that gives condition under which the appropriate Nielsen number is a strict lower bound for then corresponding minimum numbers) is still in its infancy for the periodic numbers. At the time of writing the only spaces that we are sure are (ordinary) periodic Wecken are tori (see the work of You [14], and also comments about the situation in [5]). Of course Example 1.1 is a torus, and as it can be seen in that example the Nielsen number and the actual number of fixed points on the complement coincide.

References