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Perfect packings with complete graphs minus an edge

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Abstract

Let K_r^- denote the graph obtained from K_r by deleting one edge. We show that for every integer $r \geq 4$ there exists an integer $n_0 = n_0(r)$ such that every graph G whose order $n \geq n_0$ is divisible by r and whose minimum degree is at least $(1 - 1/\chi_{cr}(K_r^-))n$ contains a perfect K_r^--packing, i.e. a collection of disjoint copies of K_r^- which covers all vertices of G. Here $\chi_{cr}(K_r^-) = \frac{r(r-2)}{r-1}$ is the critical chromatic number of K_r^-. The bound on the minimum degree is best possible and confirms a conjecture of Kawarabayashi for large n.

1. Introduction

Given two graphs H and G, an *H-packing in G* is a collection of vertex-disjoint copies of H in G. An H-packing in G is called *perfect* if it covers all vertices of G. In this case, we also say that G contains an *H-factor*. The aim now is to find natural conditions on G which guarantee the existence of a perfect H-packing in G. For example, a famous theorem of Hajnal and Szemerédi [6] gives a best possible condition on the minimum degree of G which ensures that G has a perfect K_r-packing. More precisely, it states that every graph G whose order n is divisible by r and whose minimum degree is at least $(1 - 1/r)n$ contains a perfect K_r-packing. (The case $r = 3$ was proved earlier by Corrádi and Hajnal [4] and the case $r = 2$ follows immediately from Dirac's theorem on Hamilton cycles.)

Alon and Yuster [2] proved an extension of this result to perfect packings of arbitrary graphs H. They showed that for every $\gamma > 0$ and each graph H there exists an integer $n_0 = n_0(\gamma, H)$ such that every graph G whose order $n \geq n_0$ is divisible by $|H|$ and whose minimum degree is at least $(1-1/\chi(H)+\gamma)n$ contains a perfect H-packing. They observed that there are graphs H for

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which the error term γn cannot be omitted completely, but conjectured that it could be replaced by a constant which depends only on H. This conjecture was proved by Komlós, Sárközy and Szemerédi [10].

Thus one might think that just as in Turán theory – where instead of an H-packing one only asks for a single copy of H – the chromatic number of H is the crucial parameter when one considers H-packings. However, one indication that this is not the case is provided by the result of Komlós [9], which states that if one only requires an *almost* perfect H-packing (i.e. one which covers almost all of the vertices of G), then the relevant parameter is the critical chromatic number of H. Here the *critical chromatic number* $\chi_{cr}(H)$ of a graph H is defined as $(\chi(H)-1)|H|/(|H|-\sigma(H))$, where $\sigma(H)$ denotes the minimum size of the smallest colour class in a colouring of H with $\chi(H)$ colours and where $|H|$ denotes the order of H. Note that $\chi_{cr}(H)$ always satisfies $\chi(H)-1 < \chi_{cr}(H) \leq \chi(H)$ and is closer to $\chi(H)-1$ if $\sigma(H)$ is comparatively small. Building on this, in [11] it was shown that for some graphs H the critical chromatic number is even the relevant parameter for perfect packings, while for all other graphs the relevant parameter is the chromatic number. In order to state the precise result (Theorem 1) we need to introduce some notation. A colouring of a graph H is called *optimal* if it uses exactly $\chi(H)$ colours. Let $\ell := \chi(H)$. Given an optimal colouring c of H, let $x_1 \leq x_2 \leq \cdots \leq x_\ell$ be the sizes of the colour classes. Define $\mathcal{D}(c) = \{x_{i+1} - x_i \mid i = 1, \ldots, \ell - 1\}$. Let $\mathcal{D}(H)$ be the union of all the sets $\mathcal{D}(c)$ over all optimal colourings c of H. We define $\mathrm{hcf}_\chi(H)$ to be the highest common factor of the elements of $\mathcal{D}(H)$ (or $\mathrm{hcf}_\chi(H) := \infty$ if $\mathcal{D}(H) = \{0\}$). Define $\mathrm{hcf}_c(H)$ to be the highest common factor of the orders of all the components of H. For any graph H, if $\chi(H) \neq 2$, we say $\mathrm{hcf}(H) = 1$ if $\mathrm{hcf}_\chi(H) = 1$. If $\chi(H) = 2$, we say $\mathrm{hcf}(H) = 1$ if both $\mathrm{hcf}_c(H) = 1$ and $\mathrm{hcf}_\chi(H) \leq 2$.

Theorem 1 (*[11]*). *Given a graph H, let $\delta(H,n)$ denote the smallest integer k such that every graph G whose order n is divisible by $|H|$ and with $\delta(G) \geq k$ contains a perfect H-packing. Then*

$$\delta(H,n) = \begin{cases} \left(1 - \dfrac{1}{\chi_{cr}(H)}\right) n + O(1) & \text{if } \mathrm{hcf}(H) = 1, \\ \left(1 - \dfrac{1}{\chi(H)}\right) n + O(1) & \text{if } \mathrm{hcf}(H) \neq 1. \end{cases}$$

Here the $O(1)$ error term depends only on H and there are graphs H for which it cannot be omitted completely (see Proposition 4). Also, note that the upper bound on $\delta(H,n)$ in the case when $\mathrm{hcf}(H) \neq 1$ is the result in [10] mentioned earlier. The proof in [11] for the case when $\mathrm{hcf}(H) = 1$ gave a constant which was dependent on the constant in Szemerédi's regularity lemma, and is therefore huge.

Our main result shows that in the case when $H = K_r^-$, where $r \geq 4$, the error term in Theorem 1 can be omitted completely. (Recall that K_r^- denotes the graph obtained from K_r by deleting one edge.) Note that $\mathrm{hcf}(K_r^-) = 1$ for $r \geq 4$.

Theorem 2. *For every integer $r \geq 4$ there exists an integer $n_0 = n_0(r)$ such that every graph G whose order $n \geq n_0$ is divisible by r and whose minimum degree is at least*

$$\left(1 - \frac{1}{\chi_{cr}(K_r^-)}\right) n$$

contains a perfect K_r^--packing.

This theorem confirms a conjecture of Kawarabayashi [7] for large n. The case $r = 4$ of the conjecture (and thus of Theorem 2) was proved by Kawarabayashi [7]. By a result of Enomoto, Kaneko and Tuza [5], the conjecture also holds for the case $r = 3$ under the additional assumption that G is connected. (Note that K_3^- is just a path on 3 vertices and that in this case the required minimum degree equals $n/3$.) For completeness, in Proposition 3 we will give an explicit construction showing that the bound on the minimum degree in Theorem 2 is best possible.

Clearly, it would be desirable to characterize all those graphs for which the $O(1)$-error term in Theorem 1 can be omitted. However, we do not know what such a characterization might look like. By the Hajnal–Szemerédi theorem [6] the error term can be omitted for complete graphs. A result of Abbasi [1] implies that, for large n, it can be omitted for cycles. In [3] the first author describes a further class of graphs for which the ideas in this paper can be adapted to remove the error term completely for large n. On the other hand, Proposition 4 shows that the error term cannot be omitted if H is a complete ℓ-partite graph with $\ell \geq 3$ and at least $\ell - 1$ vertex classes of size at least 3. A larger class of graphs H for which this is the case is given in [3].

Algorithmic issues related to Theorem 1 are discussed in [12]. It was shown there that for any $\varepsilon > 0$ the perfect H-packing guaranteed by Theorem 1 can be found in polynomial time if the $O(1)$-error term is replaced by εn. Moreover, if the minimum degree condition on G is reduced a little below the threshold, then there are many graphs H for which the decision problem of whether G has a perfect H-packing becomes NP-complete.

2. Notation and preliminaries

Throughout this paper we omit floors and ceilings whenever this does not affect the argument. We write $e(G)$ for the number of edges of a graph G, $|G|$ for its order, $\delta(G)$ for its minimum degree, $\Delta(G)$ for its maximum degree, $\chi(G)$ for its chromatic number and $\chi_{cr}(G)$ for its critical chromatic number as defined in Section 1. We denote the degree of a vertex $x \in G$ by $d_G(x)$ and its neighbourhood by $N_G(x)$. Given a vertex set $A \subseteq V(G)$, we also write $N_A(x)$ for the set of all neighbours of x in A. We denote by $G[A]$ the subgraph of G induced by the vertex set A. Given disjoint sets $A, B \subseteq V(G)$, we denote by $e(A, B)$ the number of all edges between A and B and write $d(A, B) := e(A, B)/|A||B|$ for the density of the bipartite subgraph of G between A and B. We denote by $d(A) := e(A)/\binom{|A|}{2}$ the density of A.

For a graph H of chromatic number ℓ, define the *bottle graph $B^*(H)$ of H*, to be the complete ℓ-partite graph which has $\ell - 1$ classes of size $|H| - \sigma(H)$ and one class of size $(\ell - 1)\sigma(H)$. (Recall that $\sigma(H)$ is the smallest possible size of a colour class in an ℓ-colouring of H.) Thus $B^*(H)$ contains a perfect H-packing consisting of $\ell - 1$ copies of H. We will use B^* to denote $B^*(K_r^-)$ whenever this is unambiguous.

For completeness, we include the construction which shows that the bound on the minimum degree in Theorem 2 is best possible.

Proposition 3. *Let $r \geq 4$. Then for all $k \in \mathbb{N}$ there is a graph G on $n = kr$ vertices whose minimum degree is $\lceil (1 - 1/\chi_{cr}(K_r^-))n \rceil - 1$ but which does not contain a perfect K_r^--packing.*

Proof. We construct G as follows. G is a complete $(r-1)$-partite graph with vertex classes $U_0, \ldots, U_{r-2}$, where $|U_0| = k - 1$ and the sizes of all other classes are as equal as possible. It is easy to check that G has the required minimum degree. Moreover, every copy of K_r^- in G contains at least one vertex in U_0. Thus we can find at most $|U_0|$ pairwise disjoint copies of K_r^-

which therefore cover at most $(k-1)(r-1) < n - |U_0|$ vertices of $G - U_0$. Thus G does not contain a perfect K_r^--packing. □

Note that Proposition 3 extends to every graph H which is obtained from a K_{r-1} by adding a new vertex and joining it to at most $r-2$ vertices of the K_{r-1}. Since each such H is a subgraph of K_r^- and since $\chi_{cr}(H) = \chi_{cr}(K_r^-)$, it follows from this observation and from Theorem 2 that $\delta(H, n) = \lceil (1 - 1/\chi_{cr}(H))n \rceil$ if n is sufficiently large (where $\delta(H, n)$ is as defined in Theorem 1).

The following example shows that for a large class of graphs, the $O(1)$-error term in Theorem 1 cannot be omitted completely. The example is an extension of a similar construction in [10].

Proposition 4. *Suppose that H is a complete ℓ-partite graph with $\ell \geq 3$ such that every vertex class of H, except possibly its smallest class, has at least 3 vertices. Then there are infinitely many graphs G whose order n is divisible by $|H|$, whose minimum degree satisfies $\delta(G) = (1 - \frac{1}{\chi_{cr}(H)})n$ but which do not contain a perfect H-packing.*

Proof. Let σ denote the size of the smallest vertex class of H. Given $k \in \mathbb{N}$, consider the complete ℓ-partite graph on $n := k(\ell-1)|H|$ vertices whose vertex classes $A_1, \dots, A_\ell$ satisfy $|A_1| := (|H| - \sigma)k + 1$, $|A_\ell| := k(\ell-1)\sigma - 1$ and $|A_i| := (|H| - \sigma)k$ for all $1 < i < \ell$. Let G be the graph obtained by adding a perfect matching into A_1 or, if $|A_1|$ is odd, a matching covering all but 3 vertices and a path of length 2 on these remaining vertices. Observe that the minimum degree of G is $(1 - \frac{1}{\chi_{cr}(H)})n$.

Consider any copy H' of H in G. Suppose that H' meets A_ℓ in at most $\sigma - 1$ vertices. Then there is a colour class X of H' which meets A_ℓ but does not lie entirely in A_ℓ. So some vertex class of G must meet at least two colour classes of H'. Since H' is complete ℓ-partite, this vertex class must have some edges in it, and so must be A_1. However, A_1 cannot meet three colour classes of H', since it is triangle free. Thus every colour class of H' except X lies completely within one A_i. Furthermore, A_1 cannot contain two complete colour classes of H', since then $G[A_1]$ would have a vertex of degree 3, a contradiction. So A_1 meets X as well as another colour class Y of H'. Furthermore $X \setminus A_\ell \subseteq A_1$ and $Y \subseteq A_1$. Let $x \in X \cap A_1$. Then $Y \subseteq N_G(x)$ since $Y \subseteq N_{H'}(x)$. This implies that $|Y| \leq 2$ and so $\sigma = |Y| \leq 2$. Thus $|X| \geq 3$. Since at most $\sigma - 1 \leq 1$ vertices of X lie in A_ℓ this in turn implies that $|X \cap A_1| \geq 2$. As $X \cap A_1$ lies in the neighbourhood of any vertex from Y, we must have that $|X \cap A_1| = 2$. Thus $X \cap A_1$ can only lie in the neighbourhood of one vertex from Y. Hence $\sigma = |Y| = 1$. But then X avoids A_ℓ, a contradiction.

So any copy of H in G has at least σ vertices in A_ℓ. Thus any H-packing in G consists of less than $k(\ell-1)$ copies of H and therefore covers less than $k(\ell-1)(|H| - \sigma) < |G| - |A_\ell|$ vertices of $G - A_\ell$. So G does not contain a perfect H-packing. □

Note that the proof of Proposition 4 shows that if $|H| - \sigma$ is odd then we only need that every vertex class of H (except possibly its smallest class) has at least two vertices. Moreover, it is not hard to see that the conclusion of Proposition 4 holds for all graphs H which do not have a colouring with a vertex class of size $\sigma + 1$ (see [3] for details).

In the proof of Theorem 2 we will use the following observation about packings in almost complete $(q+1)$-partite graphs. It follows easily from the blow-up lemma (see e.g. [8]), but we also sketch how it can be deduced directly from Hall's theorem.

Proposition 5. *For all $q, r \in \mathbb{N}$ there exists a positive constant $\tau_0 = \tau_0(q, r)$ such that the following holds for every $\tau \leq \tau_0$ and all $k \in \mathbb{N}$. Let $H_{q,r}$ be the complete $(q + 1)$-partite graph with q vertex classes of size r and one vertex class of size 1. Let G^* be a $(q + 1)$-partite graph with vertex classes $V_1, \dots, V_{q+1}$ such that $|V_i| = kr$ for all $i \leq q$ and such that $|V_{q+1}| = k$. Suppose that for all distinct $i, j \leq q + 1$ every vertex $x \in V_i$ of G^* is adjacent to all but at most $\tau|V_j|$ vertices in V_j. Then G^* has a perfect $H_{q,r}$-packing.*

Proof. We proceed by induction on q. If $q = 1$ then we are looking for a perfect $K_{1,r}$-packing. So the result can easily be deduced from Hall's theorem with $\tau_0 = 1/2$. Now suppose that $q > 1$ and let $\tau_0(q, r) \ll \tau_0(q-1, r)$. As before, we can find a perfect $K_{1,r}$-packing in $G^*[V_q \cup V_{q+1}]$. Let G' be the graph obtained from G^* by replacing each copy K of such a $K_{1,r}$ with one vertex x_K and joining x_K to $y \in V_1 \cup \dots \cup V_{q-1}$ whenever y is adjacent to every vertex of K. Then G' contains a perfect $H_{q-1,r}$-packing by induction. Clearly, this corresponds to a perfect $H_{q,r}$-packing in G^*. □

3. Overview of the proof

Our main tool is the following result from [11]. It states that in the "non-extremal case", where the graph G given in Theorem 1 satisfies certain conditions, we can find a perfect packing even if the minimum degree is slightly smaller than required in Theorem 1. The conditions ensure that the graph G does not look too much like one of the extremal examples of graphs whose minimum degree is just a little smaller than required in Theorem 1 but which do not contain a perfect H-packing.

Theorem 6. *Let H be a graph of chromatic number $\ell \geq 2$ with $\mathrm{hcf}(H) = 1$. Let z_1 denote the size of the small class of the bottle graph $B^*(H)$, let z denote the size of one of the large classes, and let $\xi = z_1/z$. Let $\theta \ll \tau_0 \ll \xi, 1 - \xi, 1/|B^*(H)|$ be positive constants. There exists an integer n_0 such that the following holds. Suppose G is a graph whose order $n \geq n_0$ is divisible by $|B^*(H)|$ and whose minimum degree satisfies $\delta(G) \geq (1 - \frac{1}{\chi_{cr}(H)} - \theta)n$. Suppose that G also satisfies the following conditions:*

(i) *G does not contain a vertex set A of size $zn/|B^*(H)|$ such that $d(A) \leq \tau_0$.*
(ii) *If $\ell = 2$, then G does not contain a vertex set A with $d(A, V(G) \setminus A) \leq \tau_0$.*

Then G has a perfect H-packing.

By applying this theorem with $H := K_r^-$ (where $r \geq 4$), we only need to consider the extremal case, when there are large almost independent sets. (Note that if the order of the graph G given by Theorem 2 is not divisible by $|B^*(K_r^-)|$, we must first greedily remove some copies of K_r^- before applying Theorem 6. The existence of these copies follows from the Erdős–Stone theorem, and since we only need to remove a bounded number of copies, this will not affect any of the properties required in Theorem 6 significantly.)

Suppose that we have q such large almost independent sets. Then we will think of the remainder of the vertices of G as the $(q + 1)$th set. We will show in Section 4 that by taking out a few copies of K_r^- and rearranging these $q + 1$ sets slightly, we can achieve that these sets will induce an almost complete $(q + 1)$-partite graph. Furthermore, the proportion of the size of each of the first q of these modified sets to the size of the entire graph will be the same as for the large classes of the bottle graph $B^*(K_r^-)$ defined in Section 2.

Let B_1^* be the subgraph of $B^*(K_r^-)$ obtained by deleting q of the large vertex classes. Ideally, we would like to apply Theorem 6 to find a B_1^*-packing in the (remaining) subgraph of G induced by the $(q+1)$th vertex set. In a second step we would then like to extend this B_1^*-packing to a $B^*(K_r^-)$-packing in G, using the fact that the $(q+1)$-partite subgraph of G between the classes defined above is almost complete. This would clearly yield a K_r^--packing of G.

However, there are some difficulties. For example, Theorem 6 only applies to graphs H with $\text{hcf}(H) = 1$, and this may not be the case for B_1^* if it is bipartite. So instead of working with B_1^*, we consider a suitable subgraph B_1 of B_1^* which does satisfy $\text{hcf}(B_1) = 1$. Moreover, if B_1 is bipartite we may have to take out a few further carefully chosen copies of K_r^- from G to ensure that condition (ii) is also satisfied before we can apply Theorem 6 to the subgraph induced by the $(q+1)$th vertex set.

4. Tidying up the classes

Let n and q be integers such that n is divisible by $r(r-2) = |B^*(K_r^-)|$ and such that $1 \le q \le r-2$. Note that in the case when $H := K_r^-$ the set A in condition (i) of Theorem 6 has size $\frac{r-1}{r(r-2)}n$. We say that disjoint vertex sets $A_1, \ldots, A_{q+1}$ are (q,n)-*canonical* if $|A_i| = \frac{r-1}{r(r-2)}n$ for all $i \le q$ and $|A_{q+1}| = \frac{n}{r} + (r-q-2)\frac{r-1}{r(r-2)}n = n - \sum_{i=1}^{q} |A_i|$. Note that in this case the graph $K(q,n)$ obtained from the complete graph on $\bigcup_{i=1}^{q+1} A_i$ by making each A_i with $i \le q$ into an independent set has a perfect $B^*(K_r^-)$-packing and thus also a perfect K_r^--packing.

Our aim in the following lemma is to remove a few disjoint copies of K_r^- from our given graph G in order to obtain a graph on n^* vertices which looks almost like $K(q,n^*)$. In the next section we will then use this property to show that this subgraph of G has a perfect K_r^--packing.

Lemma 7. *Let $r \ge 4$ and $0 < \tau \ll 1/r$. Then there exists an integer $n_0 = n_0(r,\tau)$ such that the following is true. Let G be a graph whose order $n \ge n_0$ is divisible by r and whose minimum degree satisfies $\delta(G) \ge (1 - \frac{1}{\chi_{cr}(K_r^-)})n$. Suppose that for some $1 \le q \le r-2$ there are q disjoint vertex sets $A_1, \ldots, A_q$ in G such that $|A_i| = \lceil \frac{r-1}{r(r-2)}n \rceil$ and $d(A_i) \le \tau$ for $1 \le i \le q$. Set $A_{q+1} := V(G) \setminus (A_1 \cup \cdots \cup A_q)$. Then there exist disjoint vertex sets $A_1^*, \ldots, A_{q+1}^*$ such that the following hold:*

(i) *If $G^* := G[\bigcup_{i=1}^{q+1} A_i^*]$ and $n^* := |G^*|$ then $r(r-2)$ divides n^*, and $G - G^*$ contains a perfect K_r^--packing. Furthermore, $n - n^* \le \tau^{1/3}n$.*
(ii) $|A_1^*| = |A_2^*| = \cdots = |A_q^*| = \frac{r-1}{r(r-2)}n^*$.
(iii) *For all $i, j \le q+1$ with $i \ne j$, each vertex in A_i^* has at least $(1-\tau^{1/5})|A_j^*|$ neighbours in A_j^*.*

Proof. Note that if n is divisible by $r(r-2)$ then the sets $A_1, \ldots, A_{q+1}$ are (q,n)-canonical. If n is not divisible by $r(r-2)$ then we will change the sizes of the A_i slightly as follows. Write $n = n' + kr$ where n' is divisible by $r(r-2)$ and $0 < k < r-2$. If $k \ge q$ then we do not change the sizes of the A_i. If $k < q$ then for each i with $k < i \le q$ we move one vertex from A_i to A_{q+1}. We still denote the sets thus obtained by $A_1, \ldots, A_{q+1}$. We may choose the vertices we move in such a way that the density of each A_i with $i \le q$ is still at most τ. Note that $\lceil \frac{r-1}{r(r-2)}n \rceil = \frac{r-1}{r(r-2)}n' + k + 1$. Thus both in the case when $k \ge q$ and in the case when $k < q$ the sets $A_1, \ldots, A_{q+1}$ can be obtained from (q,n')-canonical sets by adding kr new vertices as follows. For each $i \le \min\{k,q\}$ we add $k+1$ of the new vertices to the ith vertex set, for each i with $\min\{k,q\} < i \le q$ we add k new vertices to the ith vertex set and all the remaining new vertices are added to A_{q+1}. Let K be the graph obtained from the complete graph on $\bigcup_{i=1}^{q+1} A_i$ by making each A_i with $i \le q$ into an independent set. It is easy to see that $K(q, n')$ can be obtained from K by removing k disjoint copies of K_r^-. In particular, K has a perfect K_r^--packing. Note that if $k < q$ then this would not hold if we had not changed the sizes of the A_i. Later on we will use that in all cases we have

$$|A_i| \ge \frac{r-1}{r(r-2)} n' + k = \frac{r-1}{r(r-2)}(n - kr) + k \tag{1}$$

for all $i \le q$, where we set $n' := n$ and $k := 0$ if n is divisible by $r(r-2)$. Observe that $\chi_{cr}(K_r^-) = \frac{r(r-2)}{r-1}$ and so $\delta(G) \ge (1 - \frac{r-1}{r(r-2)})n$. Thus the minimum degree condition on G implies that the neighbours of any vertex might essentially avoid one of the A_i, for $i \le q$, but no more.

Now for each index i, call a vertex $x \in A_i$ *i-bad* if x has at least $\tau^{1/3}|A_i|$ neighbours in A_i. Note that, for $i \le q$, the number of i-bad vertices is at most $\tau^{2/3}|A_i|$ since $d(A_i) \le \tau$ for such i. Call a vertex $x \in A_i$ *i-useless* if, for some $j \ne i$, x has at most $(1 - \tau^{1/4})|A_j|$ neighbours in A_j. In this case the minimum degree condition shows that, provided $i \ne r-1$, x must have at least a $\tau^{1/3}$-fraction of the vertices in its own class as neighbours, i.e. x is i-bad. Thus every vertex that is i-useless is also i-bad for $i \ne r-1$. In particular, for each $i \le q$, there are at most $\tau^{2/3}|A_i|$ i-useless vertices.

For $i = q+1$ we estimate the number u_{q+1} of $(q+1)$-useless vertices by looking at the edges between A_{q+1} and $V(G) \setminus A_{q+1}$. We have

$$\begin{aligned} e(A_{q+1}, V(G) \setminus A_{q+1}) &\ge \sum_{i=1}^{q} \left\{ |A_i|\delta(G) - 2e(A_i) - \sum_{j \ne i, j \le q} |A_i||A_j| \right\} \\ &\ge q(|A_1| - 1)\delta(G) - q\tau|A_1|^2 - q(q-1)|A_1|^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} e(A_{q+1}, V(G) \setminus A_{q+1}) &\le u_{q+1}\{(q-1)|A_1| + (1 - \tau^{1/4})|A_1|\} + (|A_{q+1}| - u_{q+1})q|A_1| \\ &= q|A_1||A_{q+1}| - u_{q+1}\tau^{1/4}|A_1|. \end{aligned}$$

Combining these inequalities gives, after some calculations, that $u_{q+1} \le \tau^{2/3}|A_{q+1}|$. So in total the number of vertices which are i-useless for some i is at most $\tau^{2/3}n$.

Given $j \ne i$, call a vertex $x \in A_i$ *j-exceptional* if x has at most $\tau^{1/3}|A_j|$ neighbours in A_j. Thus every such vertex is also i-useless, and therefore i-bad if $i < r-1$. Furthermore, if $i = r-1$, then an exceptional vertex in A_i is also i-bad. So all exceptional vertices are bad.

Now if for some $i \ne j$ there exists an i-bad vertex $x \in A_i$ and an i-exceptional vertex $y \in A_j$, then let us swap x and y. (Note that a vertex is not i-exceptional for more than one i.) Having done this, since there are not too many exceptional vertices, we will still have that each non-bad vertex in A_i has at most $2\tau^{1/3}|A_i|$ neighbours in A_i, each non-useless vertex in A_i still has at least $(1 - 2\tau^{1/4})|A_j|$ neighbours in each A_j with $j \ne i$ and each non-i-exceptional vertex still has at least $\tau^{1/3}|A_i|/2$ neighbours in A_i. We will also have that for any i for which i-exceptional vertices exist, there are no i-bad vertices.

We now wish to remove all the exceptional vertices by taking out a few disjoint copies of K_r^- which will cover them. For simplicity, we will split the argument into two cases. In both cases we will repeatedly remove $r-2$ disjoint copies of K_r^- at a time. We say that such a collection of $r-2$ copies *respects the proportions of the A_i* if altogether these copies meet each A_i with $i \le q$ in exactly $r-1$ vertices.

Case 1. $q \leq r - 3$

In this case the minimum degree condition ensures that no vertex is $(q+1)$-exceptional. To deal with the j-exceptional vertices for $j \leq q$ we will need the fact that we can find a reasonably large number of disjoint copies of K_{r-1-q} in $G[A_{q+1}]$. To prove this fact, observe that

$$\delta(G[A_{q+1}]) \geq \delta(G) - \sum_{i=1}^{q} |A_i| \geq |A_{q+1}| - \frac{r-1}{r(r-2)} n \tag{2}$$

and

$$\frac{r-1}{r(r-2)} \frac{n}{|A_{q+1}|} \overset{(1)}{\leq} \frac{r-1}{r(r-2)} \frac{1}{\frac{1}{r} + (r-q-2)\frac{r-1}{r(r-2)}} \leq \frac{1}{r-q-2} - c(r) \tag{3}$$

where $c(r) > 0$ is a constant depending only on r. Combining these results gives

$$\delta(G[A_{q+1}]) \geq \left(1 - \frac{1}{r-q-2} + c(r)\right) |A_{q+1}|. \tag{4}$$

Thus we can apply Turán's theorem repeatedly to find at least $\frac{c(r)}{r-q-1}|A_{q+1}|$ disjoint copies of K_{r-q-1} in $G[A_{q+1}]$.

Now for each $i \leq q+1$ in turn, consider the exceptional vertices $x \in A_i$. Suppose that x is j-exceptional. First move x into A_j. Note that the minimum degree condition on G means that x is joined to almost all vertices in A_ℓ for every $\ell \neq j$. We greedily choose a copy of K_r^- covering x and one other vertex in A_j, $r-q-1$ vertices in A_{q+1} and one vertex in all other classes, where all vertices other than x were chosen to be non-useless. (Indeed, to find such a copy of K_r^- we first choose a copy of K_{r-q-1} in A_{q+1} which lies in the neighbourhood of x and which consists of non-useless vertices. Then we choose all the remaining vertices.) Remove this copy of K_r^-. Also greedily remove $r-3$ further disjoint copies of K_r^- such that together all these copies of K_r^- respect the proportions of the A_i. Proceed similarly for all the exceptional vertices. For each exceptional vertex we are removing $r-2$ copies of K_r^-, so in total we are removing at most $r(r-2)\tau^{2/3}n$ vertices.

Case 2. $q = r - 2$

In this case, the exceptional vertices in A_{r-1} need special attention since we cannot simply move them into another class without making A_{r-1} too small. So we proceed as follows. For each $i \leq r-2$, let s_i be the number of i-exceptional vertices in A_{r-1}. Whenever $s_i > 0$ we will find a matching of size s_i in $G[A_i]$. To see that such a matching exists, consider a maximal matching in A_i and let m denote the size of this matching. Note that

$$e(A_i) \leq 2m\Delta(A_i) \leq 2m2\tau^{1/3}|A_i|$$

since the presence of i-exceptional vertices guarantees that no vertex in A_i is i-bad. Also

$$\begin{aligned} e(A_i) &\geq \frac{1}{2}\{\delta(G)|A_i| - (n - |A_i| - s_i)|A_i| - s_i 2\tau^{1/3}|A_i|\} \\ &\geq \frac{|A_i|}{2}\left\{|A_i| - \frac{r-1}{r(r-2)}n + s_i(1 - 2\tau^{1/3})\right\} \\ &\overset{(1)}{\geq} \frac{|A_i|}{2}\left\{s_i(1-2\tau^{1/3}) - \frac{k}{r-2}\right\}. \end{aligned}$$

Since $k \leq r-3$ and $\tau \ll 1/r$, comparing these two bounds on $e(A_i)$ gives $m \gg s_i$ whenever $s_i > 0$. So we may pick a matching M_i with s_i edges in A_i, all of whose vertices are non-useless (since no vertices in A_i are bad). Now for each i in turn, we will remove the i-exceptional vertices in A_{r-1} using this matching. For each such vertex $x \in A_{r-1}$, pick an edge $yz \in M_i$. Swap x with y; we now no longer consider x to be exceptional. Then greedily find a copy of K_r^- which meets A_{r-1} precisely in y, which meets A_i precisely in z and which contains two vertices in some A_j with $j \neq i, r-1$ (such a j exists since $r \geq 4$), and one vertex in each other A_j. All these vertices will be chosen to be non-useless, and all (except y and z) will avoid each M_j. Remove this copy of K_r^-. Then also greedily take out $r-3$ further disjoint copies of K_r^-, avoiding the M_j and all useless vertices, in such a way that altogether they respect the proportions of the A_i. Note that we can find these copies greedily since the $(q+1)$-partite graph induced by the A_i is almost complete. We continue doing this until no exceptional vertices are left in A_{r-1}. The fact that M_i has s_i edges ensures that we will always have an edge left in the appropriate matching for each exceptional vertex in A_{r-1}.

Now for all other exceptional vertices, proceed using the argument for the case when $q \leq r-3$. In this way we will remove all the exceptional vertices.

So in both cases we will obtain sets $A'_1, \ldots, A'_{q+1}$ not containing any exceptional vertices. We now want to remove any remaining useless vertices. Before dealing with the exceptional vertices, each useless but non-exceptional vertex in A_i had at least $\tau^{1/3}|A_j|/2$ neighbours in A_j for each $j \neq i$. Also, we had at most $\tau^{2/3}n$ useless vertices, and therefore also at most this many exceptional vertices. So we have taken out at most $r(r-2)\tau^{2/3}n$ vertices. Thus each remaining vertex $x \in A'_i$ still has at least $\tau^{1/3}|A'_j|/3$ neighbours in A'_j for each $j \neq i$, which is much larger than the number of j-useless vertices.

Ideally, for a useless vertex $x \in A'_i$ we would like to pick neighbours in each other class greedily so that together these vertices form a copy of K_r^- with, say, two vertices in A'_1, $r-q-1$ vertices in A'_{q+1} and one vertex in each other A'_j. The problem is that the neighbours of x may avoid a substantial proportion of A'_{q+1}, and so in particular may not include any of the copies of K_{r-q-1} which we know are contained in A_{q+1} (and therefore in A'_{q+1}).

So instead, we proceed as follows. We first deal with all the vertices which have too few neighbours in A'_{q+1}. Let U be the set of vertices in $A'_1 \cup \cdots \cup A'_q$ which originally had at most $(1-\tau^{1/4})|A_{q+1}|$ neighbours in A_{q+1}. In particular, all these vertices are useless. Note that a vertex $x \in U \cap A'_i$ (where $i \leq q$) still has at least $\tau^{1/3}|A'_i|/3$ neighbours in A'_i. For each such vertex x in turn we proceed as follows. We first move x into A'_{q+1}. Then we will greedily find a copy of K_r^- which avoids x and meets each A'_j with $j \leq q$ in precisely one vertex. Note that similarly as in (4) one can show that

$$\delta(G[A'_{q+1}]) \geq \left(1 - \frac{1}{r-q-2} + \frac{c(r)}{2}\right)|A'_{q+1}|. \tag{5}$$

So we may apply the Erdős–Stone theorem to find the necessary copy of K_{r-q}^- in A'_{q+1} avoiding x as well as all the $(q+1)$-useless vertices. We can extend it to the desired copy of K_r^-, also avoiding all the useless vertices. Remove this copy of K_r^-. In effect, we have removed two vertices from A'_i (one vertex in the copy of K_r^- and x), $r-q-1$ vertices from A'_{q+1} and one vertex from each other A'_j. We can also find $r-3$ further disjoint copies of K_r^- in such a way that altogether these copies respect the proportions of the A'_i. Remove these copies. Repeating this for each vertex $x \in U$, in total we move or remove at most $\tau^{1/2}n$ vertices. We denote by A''_i the sets thus obtained from the A'_i.

The effect of moving the vertices of U and taking out these copies of K_r^- is that all vertices (except those in A''_{q+1}) are joined to almost all of A''_{q+1}. The vertices in U may now be $(q+1)$-useless, but are certainly non-exceptional.

Now consider any useless vertex $x \in A''_i$ where $i \neq q+1$. Let A''_j be the vertex set in which x has the lowest number of neighbours, not including $j = i, q+1$. (Note that such a j exists since if $q = 1$, a useless vertex $x \in A_1$ would have been in U, so we would already have dealt with it.) Pick non-useless neighbours y and z of x in A''_j. (Such neighbours exist since x was not j-exceptional.) Recall that each of x, y and z is joined to almost all of A''_{q+1}. Since A''_{q+1} is almost as large as A_{q+1} it follows that many of the copies of K_{r-q-1} chosen after (4) lie in the common neighbourhood of x, y and z, and so form a copy of K^-_{r-q+2} together with x, y and z. Pick such a copy. Now note that the choice of j implies that x is joined to at least $|A''_\ell|/3$ vertices in A''_ℓ for each $\ell \neq i, j, q+1$. So we can greedily extend this copy of K^-_{r-q+2} to a copy of K_r^- in G by picking one non-useless vertex in every other A''_ℓ. We then greedily find $r-3$ further disjoint copies of K_r^- avoiding all the useless vertices so that together with the copy just found, these copies of K_r^- respect the proportions of the A''_i. Remove all these copies of K_r^-.

For a $(q+1)$-useless vertex x, we perform a similar process, except that x is already in A''_{q+1}, so we find non-useless neighbours y and z of x in A''_j and find a copy of K_{r-q-1} in A''_{q+1} which contains x and lies in the common neighbourhood of y and z. We can do this since (5) implies that

$$\delta(G[A''_{q+1}]) \geq \left(1 - \frac{1}{r-q-2} + \frac{c(r)}{3}\right) |A''_{q+1}|.$$

(Note that in particular this bound applies to the degree of x in A''_{q+1}.) So we can successively pick common non-useless neighbours of x, y and z in A''_{q+1} to construct the necessary K_{r-q-1} containing x. Together with y and z this forms a copy of K^-_{r-q+1} which we extend suitably to a copy of K_r^-. As before we then find further disjoint copies of K_r^- such that together all these copies respect the proportions of the A''_i. We can repeat this process until no useless vertices are left. The fact that there are not too many useless vertices will ensure that all our calculations remain valid.

Finally, if $k > 0$, we remove k further disjoint copies of K_r^- to ensure that the sets $A^*_1, \dots, A^*_{q+1}$ thus obtained from the A''_i are (q, n^*)-canonical where $n^* := |A^*_1 \cup \dots \cup A^*_{q+1}|$. This can be done because of our modification of the A_i at the beginning of the proof. Since the A^*_i contain neither exceptional nor useless vertices and since we have not removed too many vertices, it is easy to check that the A^*_i satisfy all the conditions of the lemma. □

5. Proof of Theorem 2

Recall that $B^* = B^*(K_r^-)$ denotes the bottle graph of K_r^-. Fix constants $0 < \tau_1 \ll \tau_2 \ll \dots \ll \tau_{r-1} \ll 1/r$. Let G be the graph given in Theorem 2. Let $q \leq r-2$ be maximal such that the conditions of Lemma 7 are satisfied with $\tau := \tau_q$. As already observed in Section 3, by Theorem 6 we may assume that $q \geq 1$. To prove Theorem 2, we apply first Lemma 7 with this choice of q to obtain a subgraph G^* of G and a $(q, |G^*|)$-canonical partition $A^*_1, \dots, A^*_{q+1}$ of $V(G^*)$. Our definition of q will ensure that if $q \neq r-3$ then the graph induced by A^*_{q+1} does not look like one of the extremal graphs and so we can apply Theorem 6 to it in order to find a perfect B_1-packing, where B_1 is the spanning subgraph of B^*_1 defined below. (Recall that B^*_1 is the $(r-q-1)$-partite subgraph of B^* obtained by deleting q of the large vertex classes.) In the case when $q = r - 3$ the graph $G^*[A^*_{q+1}]$ might violate condition (ii) of Theorem 6. So in this case we will apply Theorem 6 to the "almost-components" of $G^*[A^*_{q+1}]$ instead.

Recall that $A^*_1, \ldots, A^*_q$ all have the same size, which is a multiple of $r - 1$ (the size of a large class of the bottle graph B^*). The size of A^*_{q+1} is a multiple of $|B^*_1|$. Our aim is to find a perfect B_1-packing in $G^*[A^*_{q+1}]$, where B_1 is the graph consisting of q vertex-disjoint copies of K_{r-q-1} together with $r-q-2$ vertex-disjoint copies of K^-_{r-q}. We think of these copies as being arranged into an $(r - q - 1)$-partite graph with one vertex set of size $r - 2$ and $r - q - 2$ vertex sets of size $r - 1$. Thus $B_1 \subseteq B^*_1$ and the vertex classes of B_1 have the same sizes as those of B^*_1. This B_1-packing in $G^*[A^*_{q+1}]$ will then be extended to a perfect K^-_r-packing in G^*.

Lemma 8. *We can take out from G^* at most $\tau^{1/3}n^*$ disjoint copies of K^-_r to obtain subsets $A^\diamond_1, \ldots, A^\diamond_{q+1}$ of $A^*_1, \ldots, A^*_{q+1}$ and a subgraph $G^\diamond$ of G^* such that the sets $A^\diamond_1, \ldots, A^\diamond_{q+1}$ are $(q, |G^\diamond|)$-canonical and such that $G^\diamond[A^\diamond_{q+1}]$ contains a perfect B_1-packing.*

Proof. Note that in the case when $q = r - 2$ the graph B_1 just consists of $r - 2$ isolated vertices, and the existence of a perfect B_1-packing is trivial since $r - 2$ divides $|A^*_{r-1}|$. In the case when $q \le r - 3$ the proof of Lemma 8 will invoke the non-extremal result, Theorem 6, with τ_{q+1} playing the role of τ_0 there. It is for this reason that we will need the term $-\theta n$ in the minimum degree condition in Theorem 6. Finally, note that $\mathrm{hcf}(B_1) = 1$ (even in the case when B_1 is bipartite, i.e. when $q = r - 3$). Let $s := r - q - 1 \ge 2$. Thus B_1 is an s-partite graph. Observe that $\chi_{cr}(B_1) = \chi_{cr}(B^*_1) = \frac{s(r-1)-1}{r-1}$. Using (i) and (ii) of Lemma 7, similarly as in (2) and the first inequality in (3) one can show that

$$\delta(G[A^*_{q+1}]) \ge \left(1 - \frac{1}{\chi_{cr}(B_1)} - \tau_q^{1/4}\right)|A^*_{q+1}| = \left(\frac{(s-1)(r-1)-1}{s(r-1)-1} - \tau_q^{1/4}\right)|A^*_{q+1}|. \tag{6}$$

So the minimum degree condition of Theorem 6 is satisfied with $\theta := \tau_q^{1/4} \ll \tau_{q+1}$. Our choice of q implies that $G^*[A^*_{q+1}]$ satisfies condition (i) of Theorem 6 (with $\tau_0 := \tau_{q+1}$). Thus in the case when $s > 2$ we can apply Theorem 6 to find a perfect B_1-packing in $G^*[A^*_{q+1}]$.

So we only need to consider the case when $s = 2$. In this case B_1 is the bipartite graph consisting of $r - 3$ disjoint edges and one path of length 2, and we are done if condition (ii) of Theorem 6 holds. So suppose not and we do have some set $C_1 \subseteq A^*_{q+1}$ with $d(C_1, A^*_{q+1} \setminus C_1) \le \tau_{q+1}$. Define $C_2 := A^*_{q+1} \setminus C_1$. Then there is a vertex $x \in C_1$ which has at most $\tau_{q+1}|C_2| \le \tau_{q+1}|A^*_{q+1}|$ neighbours in C_2. Together with (6) this shows that $|C_1| > \delta(G^*[A^*_{q+1}]) - \tau_{q+1}|A^*_{q+1}| \ge |A^*_{q+1}|/3$. Similarly, $|C_2| > |A^*_{q+1}|/3$.

We now aim to show that by moving a few vertices, we can achieve that each vertex in C_1 has few neighbours in C_2 and vice versa. (This in turn will imply that the graphs induced by both C_1 and C_2 have large minimum degree.) Call a vertex $x \in C_i$ in *useless* if it has at most $|C_i|/3$ neighbours in C_i. By (6) every such x has at least $|C_j|/3$ neighbours in the other class C_j. Furthermore, the low density between C_1 and C_2 shows that there are at most $\tau_{q+1}^{3/4}|A^*_{q+1}|$ useless vertices. We move each useless vertex into the other class and still denote the classes thus obtained by C_1 and C_2. Then $d(C_1, C_2) \le \tau_{q+1}^{2/3}$. Now call a vertex x in either class *bad* if it has at least a $\tau_{q+1}^{1/6}$-fraction of the vertices in the other class as neighbours. Clearly there are at most $\tau_{q+1}^{1/2}|A^*_{q+1}|$ bad vertices. For each bad vertex $x \in C_i$ in turn we greedily choose a copy of B_1 in C_i containing x such that these copies are disjoint for distinct bad vertices. (Use that $\delta(G^*[C_i]) \geq |C_i|/4$ for $i = 1, 2$ and the fact that B_1 consists only of edges and a path of length 2 to see that such copies can be found.) By removing these copies of B_1, we end up with two sets C_1' and C_2' which do not contain bad vertices. So each vertex in C_1' has at most $2\tau_{q+1}^{1/6}|C_2'|$ neighbours in C_2' and vice versa. Since $|C_i'| \geq |A_{q+1}^*|/4$ for $i = 1, 2$ (and thus also $|C_i'| \leq 3|A_{q+1}^*|/4$ for $i = 1, 2$) this in turn implies that

$$\delta(G^*[C_i']) \overset{(6)}{\geq} \left(1 - \frac{1}{\chi_{cr}(B_1)} - \tau_{q+1}^{1/7}\right)\frac{4|C_i'|}{3} > \left(1 - \frac{1}{\chi_{cr}(B_1)}\right)|C_i'|. \tag{7}$$

We now aim to take out a few further copies of K_r^- from G^* to ensure that both $|C_1'|$ and $|C_2'|$ are divisible by $|B_1|$. As observed at the beginning of this section, $|A_{q+1}^*|$ is divisible by $|B_1|$. Thus $|C_1'| + |C_2'|$ is also divisible by $|B_1|$. Assume first that $|C_1'| = m|B_1| - 1$ for some $m \in \mathbb{N}$. We aim to remove $2(r-2)$ disjoint copies of K_r^- from G^* in such a way that we remove $2(r-1)$ vertices from every A_i^* with $i \leq r-3$, $(r-1)+(r-2)-1$ vertices from C_1' and $(r-1)+(r-2)+1$ vertices from C_2'. Then the sizes of the remaining subsets of C_1' and C_2' will be divisible by $|B_1|$. Moreover, since the A_i^* were $(q, |G^*|)$-canonical, and since altogether we remove $2((r-1)+(r-2))$ vertices from A_{q+1}^*, the remaining subsets will still induce a canonical partition of the remaining subgraph of G^*.

The way we remove the above copies of K_r^- is as follows: Greedily find $r-2$ disjoint copies of K_r^- with two vertices in C_1', two vertices in A_i^* and one vertex in each A_j^* with $1 \leq j \leq r-3$ and $j \neq i$. For each of these copies of K_r^- the index i will be different except that $i = 1$ will be chosen twice. Also find $r-4$ disjoint copies of K_r^- with two vertices in C_2', two vertices in A_i^* and one vertex in each A_j^* with $1 \leq j \leq r-3$ and $j \neq i$. The choices of i will be between 2 and $r-3$, and no i will be chosen twice. Finally, find two copies of K_r^- with three vertices in C_2' and one in each A_i^* for $1 \leq i \leq r-3$.

In the general case (i.e. when $|C_i'| \equiv t \mod |B_1|$), we simply repeat this procedure t times to even out the residues modulo $|B_1|$ between $|C_1'|$ and $|C_2'|$. We denote the remaining subsets by $A_i^\diamond$ and $C_i^\diamond$ and the remaining subgraph by $G^\diamond$. We only need to perform the above procedure at most $|B_1| - 1$ times, so we are taking out a bounded number of copies of K_r^-, which will not affect any of the vertex degrees significantly. Thus each $G^\diamond[C_i^\diamond]$ satisfies the minimum degree condition in Theorem 6. Indeed, the first inequality in (7) shows that

$$\delta(G^\diamond[C_i^\diamond]) \geq \left(1 - \frac{1}{\chi_{cr}(B_1)} - \tau_{q+1}^{1/8}\right)\frac{4|C_i^\diamond|}{3} \geq \left(\frac{2}{5} - \tau_{q+1}^{1/8}\right)\frac{4|C_i^\diamond|}{3} \geq \frac{51}{100}|C_i^\diamond|. \tag{8}$$

This bound on the minimum degree also shows that each $C_i^\diamond$ cannot contain an almost independent set of size $|C_i^\diamond|/2$, so condition (i) of Theorem 6 is satisfied with room to spare. To see that condition (ii) also holds, observe that if $C_i^\diamond$ is partitioned into S_1 and S_2, where $0 < |S_1| \leq |C_i^\diamond|/2 \leq |S_2|$, then the neighbours of any vertex in S_1 cover a significant proportion (at least $1/50$) of S_2, and so $d(S_1, S_2) \geq 1/50$. So condition (ii) is satisfied too. Thus we can apply Theorem 6 to each of the subgraphs of $G^\diamond$ induced by $C_1^\diamond$ and $C_2^\diamond$ to find perfect B_1-packings in $G^\diamond[C_1^\diamond]$ and $G^\diamond[C_2^\diamond]$. Adding back into $A_{q+1}^\diamond$ the vertices in the copies of B_1 which were removed when dealing with the bad vertices (and letting $G^\diamond$ denote the subgraph of G induced by the modified $A_i^\diamond$), we still have a perfect B_1-packing in $G^\diamond[A_{q+1}^\diamond]$, and $G - G^\diamond$ consists of those copies of K_r^- which we removed. Thus $G^\diamond$ and the $A_i^\diamond$ are as required in the lemma. $\square$

Our aim now is to extend the perfect B_1-packing in $G^\diamond[A^\diamond_{q+1}]$ to a perfect K_r^--packing in $G^\diamond$. To do this, we define a $(q+1)$-partite auxiliary graph J, whose vertices are the vertices in $A^\diamond_i$ for all $1 \le i \le q$ together with all the copies of B_1 in the perfect B_1-packing of $G^\diamond[A^\diamond_{q+1}]$. There will be an edge between vertices from the $A^\diamond_i$'s whenever there was one in G, and a vertex $x \in A^\diamond_i$ for $1 \le i \le q$ will be joined to a copy of B_1 whenever x was joined to all the vertices of this copy in G.

Let $H_{q,r-1}$ denote the complete $(q+1)$-partite graph with q classes of size $r-1$ and one class of size 1. We wish to find a perfect $H_{q,r-1}$-packing in J. It is easy to see that this then yields a perfect K_r^--packing in $G^\diamond$ and thus, together with all the copies of K_r^- chosen earlier, a perfect K_r^--packing in G.

The existence of such a perfect $H_{q,r-1}$-packing follows immediately from Proposition 5. To see that we can apply this proposition, note that Lemma 7(iii) implies that in G^* each vertex is adjacent to almost all vertices in the other vertex classes and this remains true in $G^\diamond$ since we only deleted a small proportion of the vertices after applying Lemma 7. It follows immediately that every vertex in J is adjacent to almost all vertices in the other vertex classes of J. Note also that the vertex classes of J have the correct sizes since the sets $A^\diamond_1, \ldots, A^\diamond_{q+1}$ are $(q, |G^\diamond|)$-canonical. This completes the proof of Theorem 2.

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