Fixed point index of iterations of local homeomorphisms of the plane: a Conley index approach

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Abstract

Let $U$ be an open subset of $\mathbb{R}^2$ and let $f: U \to \mathbb{R}^2$ be a local homeomorphism. Let $p \in U$ be a non-repeller fixed point of $f$ such that \{p\} is an isolated invariant set. We introduce a particular class of index pairs for \{p\} that we call generalized filtration pairs. The computation of the fixed point index of any iteration of $f$ at $p$ is quite easy once one knows a generalized filtration pair. The existence of generalized filtration pairs provides a short and elementary proof of a theorem of P. Le Calvez and J.C. Yoccoz (Ann. of Meth. 146 (1997) 241–293), and it also allows to compute the fixed point index of any iteration of arbitrary local homeomorphisms. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The problem of the existence of minimal homeomorphisms of $\mathbb{R}^m$ with a single point omitted, was one of the problems posed by Ulam that are included in the famous Scottish Book [5]. There are several partial answers (see [5]) but the problem remains open. As a consequence of the Brouwer translation theorem, it follows that there are no minimal homeomorphisms $f: \mathbb{R}^2 \to \mathbb{R}^2$. For the multipunctured plane the problem is much more complicated. Handel [3] proved the non-existence of minimal homeomorphisms of $\mathbb{R}^2 \setminus K$, where $K$ is a finite set with

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at least two points and more recently Le Calvez and Yoccoz [4] have completely solved this

Le Calvez and Yoccoz proved the non-existence of minimal homeomorphisms of \( \mathbb{R}^2 \setminus K \), for
any finite set \( K \), using the fixed point index theory. Given an orientation preserving local
homeomorphism \( f: U \subset \mathbb{R}^2 \to \mathbb{R}^2 \), they make a strong local study, near a fixed point \( p \) which
is a locally maximal invariant set and which is neither a sink nor a source, that allows them
to prove that there are integers \( q,r \geq 1 \) such that

\[
    i_{\mathbb{R}^2}(f^k, p) = \begin{cases} 
        1 - rq & \text{if } k \in r\mathbb{Z}, \\
        1 & \text{if } k \notin r\mathbb{Z}.
    \end{cases}
\]

While the Conley index provides a very short proof of the non-existence of minimal homeo-
morphisms of the multipunctured plane, it does not provide the deep local analysis needed to
prove this theorem. Nevertheless, we will show that taking a different class of index pairs the
keys for the computations appear naturally and a simpler proof can be given.

The main goal of this paper is to provide an alternative, shorter and simpler proof of the
above result. Our techniques are based on Conley index ideas and they can be applied for
arbitrary local homeomorphisms. We obtain in this way a complete general theorem that allows
one to compute the fixed point index of every iteration of any local homeomorphism at any
non-repeller fixed point which is a locally maximal invariant set. We introduce a special class
of index pairs, that we call generalized filtration pairs. A remarkable fact is that once we have
such an index pair, the computation of the fixed point index and the integers \( q \) and \( r \) follows
automatically. It is shown that these integers depend on the behavior of \( f \) in the exit set
of a given generalized filtration pair (see Example 1). More generally, the same arguments
also allow one to compute the fixed point index of iterations of homeomorphisms in isolating
neighborhoods of compact isolated invariant sets that admit generalized filtration pairs.

The principal result of this paper is the following:

**Main Theorem.** Let \( f: U \subset \mathbb{R}^2 \to \mathbb{R}^2 \) be a local homeomorphism. Let \( p \in U \) be a non-repeller
fixed point of \( f \) such that \( \{p\} \) is an isolated invariant set. Then there are an AR, \( D \), containing
a neighborhood \( V \subset \mathbb{R}^2 \) of \( p \), a finite subset \( \{q_1, \ldots, q_m\} \subset D \) and a map \( \tilde{f}: D \to D \) such
that \( \tilde{f}|_V = f|_V \) and for every \( k \in \mathbb{N} \), \( \text{Fix}(\tilde{f})^k \subset \{p, q_1, \ldots, q_m\} \).

Moreover,

(a) (Le Calvez–Yoccoz) If \( f \) preserves the orientation, then

\[
    i_{\mathbb{R}^2}(f^k, p) = \begin{cases} 
        1 - rq & \text{if } k \in r\mathbb{N}, \\
        1 & \text{if } k \notin r\mathbb{N},
    \end{cases}
\]

where \( k \in \mathbb{N} \), \( q \) is the number of periodic orbits of \( \tilde{f} \) (excluding \( p \)) and \( r \) is their period.

(b) Assume that \( f \) reverses the orientation.

(1) If \( \tilde{f} \) has no fixed points in \( \{q_1, \ldots, q_m\} \), then

\[
    i_{\mathbb{R}^2}(f^k, p) = \begin{cases} 
        1 & \text{if } k \text{ odd}, \\
        1 - 2q & \text{if } k \text{ even}.
    \end{cases}
\]
If $f'$ has exactly one fixed point in \( \{q_1, \ldots, q_m\} \), then
\[
i_{\mathbb{R}^2}(f^k, p) = \begin{cases} 0 & \text{if } k \text{ odd}, \\ 1 - (2q + 1) & \text{if } k \text{ even}. \end{cases}
\]

If $f'$ has two fixed points in \( \{q_1, \ldots, q_m\} \), then
\[
i_{\mathbb{R}^2}(f^k, p) = \begin{cases} -1 & \text{if } k \text{ odd}, \\ 1 - (2q + 2) & \text{if } k \text{ even}, \end{cases}
\]

where $q$ is the number of orbits of period 2 of $f'$ in \( \{q_1, \ldots, q_m\} \).

The paper is organized as follows: in Section 1 we prove the Main Theorem and we devote Section 2 to prove the existence of generalized filtration pairs. Section 2 also contains two theorems that prove, in the above setting, the existence of very special isolating blocks and filtration pairs that could be applied to the Conley Index Theory.

2. Preliminary results and the Main Theorem

Let $U \subset X$ be an open set. By a \textit{(local) semidynamical system} we mean a locally defined continuous map $f : U \to X$. A function $\sigma : \mathbb{Z} \to X$ is said to be a \textit{solution to $f$ through $x$} in $N \subset X$ if $f(\sigma(i)) = \sigma(i+1)$ for all $i \in \mathbb{Z}$, $\sigma(0) = x$ and $\sigma(i) \in N$ for all $i \in \mathbb{Z}$. The \textit{invariant part} of $N$, $\text{Inv}(N, f)$, is defined as the set of all $x \in N$ that admit a solution to $f$ through $x$ in $N$, i.e. the set of all $x \in N$ such that there is a full orbit $\gamma$ such that $x \in \gamma \subset N$.

A compact set $S \subset X$ is \textit{invariant} if $f(S) = S$. An invariant compact set $S$ is \textit{isolated with respect to $f$} if there exists a compact neighborhood $N$ of $S$ such that $\text{Inv}(N, f) = S$. The neighborhood $N$ is called an \textit{isolating neighborhood} of $S$.

Given $A \subset B \subset N$, $\text{cl}(A)$, $\text{cl}_B(A)$, $\text{int}(A)$, $\text{int}_B(A)$, $\partial(A)$ and $\partial_B(A)$ will denote the closure of $A$, the closure of $A$ in $B$, the interior of $A$, the interior of $A$ in $B$, the boundary of $A$ and the boundary of $A$ in $B$, respectively.

We consider the exit set of $N$ to be defined as
\[
N^- = \{ x \in N : f(x) \notin \text{int}(N) \}.
\]

The next definition is based on the notion of filtration introduced by Franks and Richeson [2] and it is the key for the direct computation of the fixed point index of any iteration of any homeomorphism of the plane. The reader can find the definition of filtration in Section 2.

\textbf{Definition 1.} Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a local homeomorphism. Let $p$ be a fixed point such that $\{p\}$ is an isolated invariant set. Suppose that $L \subset N$ is a compact pair contained in the interior of $U$. The pair $(N, L)$ is said to be a \textit{generalized filtration pair} for $f$ at $p$ provided $N$ and $L$ are each the closure of their interiors and

1. $N$ and $\partial(N \setminus L)$ are homeomorphic to a disc and $S^1$, respectively.
2. $\text{cl}(N \setminus L)$ is an isolating neighborhood of $\{p\}$.
(3) \( f(cl(N \setminus L)) \subset int(N) \).
(4) For any component \( L_i \) of \( L \), \( \partial_N(L_i) \) is an arc and there exists a topological disc \( B_i \) such that \( \partial_N(L_i) \subset B_i \subset L_i \), \( B_i \cap N^c \neq \emptyset \) and \( f(B_i) \cap cl(N \setminus L) = \emptyset. \)

The next theorem asserts the existence of generalized filtration pairs. We will give the proof in the next section.

**Theorem 1.** Let \( f : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a local homeomorphism. Let \( p \in U \) be a non-repeller fixed point of \( f \) such that \( \{p\} \) is an isolated invariant set. Then there exists a generalized filtration pair for \( f \) at \( p \).

Assume that \((N, L)\) is a generalized filtration for a local homeomorphism \( f \). \( L \) is the union of its components \( L = L_1 \cup L_2 \cup \cdots \cup L_m. \)

We have that

\[
f \mid cl(N \setminus L) : cl(N \setminus L) \rightarrow N
\]

and for every \( i \), there exists \( j \) such that \( f(\partial_N(L_i)) \subset int(L_j) \).

Consider the quotient space \( D = cl(N \setminus L) / \sim \) by identifying each \( \partial_N(L_i) \) to a point \( q_i \) (if \( i \neq j \) then \( q_i \neq q_j \)).

Take the map projection

\[
\pi : cl(N \setminus L) \rightarrow cl(N \setminus L) / \sim
\]

and a retraction

\[
r : N \rightarrow cl(N \setminus L)
\]

with \( r(x) = x \) if \( x \in cl(N \setminus L) \) and \( r \) retracts each \( L_i \) to \( \partial_N(L_i) \).

Now define \( f' = \pi \circ r \circ f \circ \pi^{-1} \),

\[
f' : cl(N \setminus L) / \sim \setminus \{q_1, \ldots, q_m\} \rightarrow cl(N \setminus L) / \sim.
\]

Then \( f' \) is continuous and, in a small enough neighborhood of \( p \), \( f' \equiv f \). Since \( f(\partial_N(L_i)) \subset int(L_j) \), \( f' \) admits a unique continuous extension

\[
\tilde{f} : cl(N \setminus L) / \sim \rightarrow cl(N \setminus L) / \sim
\]

such that \( \tilde{f}(U'(q_i)) = q_j \) for a neighborhood \( U'(q_i) \) of \( q_i \).

\( \tilde{f}(\{q_1, \ldots, q_m\}) \subset \{q_1, \ldots, q_m\} \). In fact, \( \tilde{f}(q_i) = q_j \) iff \( f(\partial_N(L_i)) \subset int(L_j) \).

Obviously,

\[
Fix_{cl(N \setminus L) / \sim}(\tilde{f}^k) \subset \{p, q_1, \ldots, q_m\}
\]

and since \( Inv(cl(N \setminus L), f) = \{p\} \), it is clear that

\[
Fix_{cl(N \setminus L) / \sim}((\tilde{f}^k)^k) \subset \{p, q_1, \ldots, q_m\}.
\]
Example 1. We will present some homeomorphisms where we will identify the fixed point index and the integers $r$ and $q$ of the theorem of Le Calvez–Yoccoz and our Main Theorem. We will offer examples of both the orientation preserving and orientation reversing cases.

Fig. 1.
Let $N = \{ x \in \mathbb{R}^2 : ||x|| \leq 2^{1/2} \}$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism generating the discrete dynamical system of Fig. 1a. Take $\varepsilon > 0$ big enough and let $L$ be the union of the $\varepsilon$-neighborhoods, in $N$, of $(1,1),(-1,1),(-1,-1)$ and $(1,-1)$ (see Fig. 1b).

Let $g,s : \mathbb{R}^2 \to \mathbb{R}^2$ be a $\pi/2$-rotation and a symmetry with respect to $\{ x - y = 0 \}$, respectively. $0 \in N$ is a non-repeller rest point, $N$ is an isolating neighborhood of $\{ 0 \}$ for $f$, $g \circ f$ and $s \circ f$. In all cases the pair $(N,L)$ is a generalized filtration pair (for an adequate $\varepsilon > 0$). Consider $q_i = \pi(L_i) \in \text{cl}(N \setminus L) / \sim$ for $i \in \{ 1, 2, 3, 4 \}$.

$$Fix_{\text{cl}(N \setminus L) / \sim}((f^i)^k) = \{ 0, q_1, q_2, q_3, q_4 \}.$$  

In this case, we have, apart from $0$, four period-one periodic orbits. Then $r = 1$ and $q = 4$. Therefore, for every $k \in \mathbb{N}$

$$i_{\mathbb{R}^2}(f^k,0) = -3.$$

$$Fix_{\text{cl}(N \setminus L) / \sim}((g \circ f)^k) = \begin{cases} \{ 0, q_1, q_2, q_3, q_4 \} & \text{if } k \in 4\mathbb{N}, \\ \{ 0 \} & \text{if } k \not\in 4\mathbb{N}. \end{cases}$$

Now $(g \circ f)'$ has, apart from $0$, a period-four periodic orbit. Then $r = 4$ and $q = 1$ and

$$i_{\mathbb{R}^2}((g \circ f)^k,0) = \begin{cases} -3 & \text{if } k \in 4\mathbb{N}, \\ 1 & \text{if } k \not\in 4\mathbb{N}. \end{cases}$$

On the other hand, $s \circ f$ is orientation reversing.

$$Fix_{\text{cl}(N \setminus L) / \sim}((s \circ f')^k) = \begin{cases} \{ 0, q_1, q_2, q_3, q_4 \} & \text{if } k \in 2\mathbb{N}, \\ \{ 0, q_1, q_3 \} & \text{if } k \not\in 2\mathbb{N}. \end{cases}$$

$(s \circ f)'$ has, apart from $0$, $q = 1$ periodic orbits of period two and two fixed points. Then, our Main Theorem will say that

$$i_{\mathbb{R}^2}((s \circ f)^k,0) = \begin{cases} -3 & \text{if } k \text{ even}, \\ -1 & \text{if } k \text{ odd}. \end{cases}$$

**Definition 2.** Let $\theta = \{ p_1, \ldots, p_s \} \subset \{ q_1, \ldots, q_m \}$ be a subset on which $f^i$ acts as a permutation. We say that $p_i, p_j \in \theta$ are adjacent if there is an arc $\gamma \subset \partial(N \setminus L) / \sim$ joining $p_i$ and $p_j$ such that $\gamma \cap \theta = \{ p_i, p_j \}$.

The next proposition proves that $f^i$ preserves adjacency in every subset $\theta$ on which it acts as a permutation. Given any two adjacent elements in $\theta$ we will find the arc in $\partial(N \setminus L) / \sim$ making their images adjacent.

**Proposition 1.** Let $\theta = \{ p_1, \ldots, p_s \} \subset \{ q_1, \ldots, q_m \}$ be a subset such that $f^i(\theta) = \theta$. If $p_i$ and $p_j$ are adjacent in $\theta$ then their images $p_{i+1}$ and $p_{j+1}$ are also adjacent.
**Proof.** Consider the arc, as in Definition 2, in $\partial(N\setminus L) \sim S^1$ joining $p_i$ and $p_j$. Denote this arc by $\overline{p_ip_j}$.

Let

$$L_iL_j = (\pi)^{-1}(p_ip_j) \subset \partial(N\setminus L).$$

$L_iL_j$ is an arc joining $a \in \partial_N(L_i)$ and $b \in \partial_N(L_j)$. Then, $f(L_iL_j) \subset int(N)$ is an arc.

Let $L_{i+1}$, $L_{j+1}$ be such that $f(\partial_N(L_i)) \subset int(L_{i+1})$ and $f(\partial_N(L_j)) \subset int(L_{j+1})$. Since $N^- \cap B_i \neq \emptyset$, we can construct an arc $\gamma_i$ in $B_i$ joining $a$ with $a' \in N^-$ such that $\gamma_i \cap N^- = \{a'\}$ and $\gamma_i \cap \partial_N(L_i) = \{a\}$.

$f(\gamma_i)$ is an arc, $f(\gamma_i) \subset L_{i+1}$, joining $f(a)$ with $\partial(N)$, $f(\gamma_i) \cap \partial(N) = \{f(a')\}$, $f(\gamma_i) \cap f(\partial_N(L_i)) = \{f(a)\}$.

Denote $K_{i+1} = f(\gamma_i)$. Consider a similar arc $\gamma_j$ for $L_j$, $K_{j+1} = f(\gamma_j)$.

It is clear that $f(L_iL_j) \cup K_{i+1} \cup K_{j+1}$ is an arc that decomposes $N$ into two components $U, V$ such that $U \cup f(L_iL_j) \cup K_{i+1} \cup K_{j+1}$ contains $f(\partial(N\setminus L))$ and $(V \cup f(L_iL_j) \cup K_{i+1} \cup K_{j+1}) \cap f(\partial(N\setminus L)) = f(L_iL_j)$. Let $R = V \cup f(L_iL_j) \cup K_{i+1} \cup K_{j+1}$ (the closed component such that $p \notin R$). $\partial(R)$ is a topological circle with $f(L_iL_j) \cup K_{i+1} \cup K_{j+1} \subset \partial(R)$ (see Fig. 2).

Consider

$$A = \partial(N) \cap R \subset \partial(N)$$

and define $\overline{p_{i+1}p_{j+1}} = (\pi \circ r)(A)$, an arc in $\partial(N\setminus L) \sim S^1$ joining $p_{i+1}$ and $p_{j+1}$.

We will show that $p_{i+1}$ and $p_{j+1}$ are adjacent (in $\theta$) because

$$\overline{p_{i+1}p_{j+1}} \cap \theta = \{p_{i+1}, p_{j+1}\}.$$
and
\[(\pi)^{-1}(p_k) = \partial_{N}(L_k) \subset (\pi)^{-1}(p_{i+1}p_{f+1}).\]

Let us write \((\pi)^{-1}(p_{i+1}p_{f+1}) = L_{i+1}L_{f+1}^{-1}\). \(f(\partial_{N}(L_{k-1})) \subset \text{int}(R)\). In fact, if \(x \in \partial_{N}(L_{k-1})\) is such that \(f(x) \in L_k \setminus \text{int}(R)\), there is an arc \(\gamma_x^y \subset B_{k-1}\) joining \(x \in \partial_{N}(L_{k-1})\) and \(y \in B_{k-1} \cap N^{-}\) such that \(\gamma_x^y \cap N^{-} = \{y\}\) and \(\gamma_x^y \cap \partial_{N}(L_{k-1}) = \{x\}\). It is clear that \(f(y) \in \partial(N)\) and \(f(\gamma_x^y) \subset L_k\).

Since \(f(x) \notin \text{int}(R)\) and \(f(y) \in \partial(N) \cap L_k \subset A \subset R\), there is \(z \in \gamma_x^y\) such that \(f(z) \in \partial_{N}(R) = f(L_{i}L_{j}) \cup K_{i+1} \cup K_{j+1}\). Then \(z \in (L_{i}L_{j} \cup \gamma_i \cup \gamma_f) \cap L_{k-1}\) and this is not possible.

Therefore, \(f(\partial_{N}(L_{k-1})) \subset \text{int}(R) \cap f(\partial(N \setminus L))\) and since \(f(L_{i}L_{j}) = f(\partial(N \setminus L)) \cap R\) we have a contradiction. \(\square\)

**Corollary 1.** *Under the above conditions,*

(a) *If \(f\) is orientation preserving then all the periodic orbits of \(\tilde{f}^r\), in \(\{q_1, \ldots, q_m\}\), have the same period.*

(b) *If \(f\) is orientation reversing then \(\tilde{f}^r\) has no more than two fixed points in \(\{q_1, \ldots, q_m\}\) and the period of its periodic points is \(\leq 2\).*

**Proof.** Let \(\theta\) be any subset of \(\{q_1, \ldots, q_m\}\) such that \(\tilde{f}^r(\theta) = \theta\). Take an orientation in \(\partial(N \setminus L) / \sim\). This orientation produces an order in \(\theta\) and, using Proposition 1, if \(f\) is orientation preserving (reversing) we have that \(\tilde{f}^r\) preserves adjacency and preserves (reverses) the order in \(\theta\).

(a) Assume that \(f\) is orientation preserving and let \(\theta_1 = \{p_{11}, p_{12}, \ldots, p_{1r}\}\) and \(\theta_2 = \{p_{21}, p_{22}, \ldots, p_{2s}\}\) be two periodic orbits of \(\tilde{f}^r\) in \(\{q_1, \ldots, q_m\}\). Let \(\theta = \theta_1 \cup \theta_2\). Without loss of generality we can suppose that \(p_{11} < p_{21}\) are adjacent. Then, from Proposition 1, \(p_{11} = (\tilde{f}^r)^r(\{p_{11}\}) < (\tilde{f}^r)^r(\{p_{21}\})\) are adjacent. Therefore \((\tilde{f}^r)^r(\{p_{21}\}) = p_{21}\) and \(r \geq s\). In the same way we have that \(s \geq r\).

(b) Suppose that \(f\) is orientation reversing. Let \(p_1, p_2 \in \{q_1, \ldots, q_m\}\) be two fixed points of \(\tilde{f}^r\). If there is another fixed point \(p_3\), we have an order in \(\theta = \{p_1, p_2, p_3\}\), \(p_1 < p_3 < p_2\). Then their images satisfy the following inequality \(p_1 < p_3 < p_2 < \tilde{f}^r(p_3) < p_1\) and \(p_3\) is not fixed.

Now take a period \(r \geq 2\) periodic orbit \(\theta\) of \(\tilde{f}^r\) in \(\{q_1, \ldots, q_m\}\). Let \(p_1 < p_2\) be adjacent elements in \(\theta\). Then \(\tilde{f}^r(p_2) < \tilde{f}^r(p_1)\) are also adjacent. If \(r > 2\) we consider \(p_3 \in \theta\) such that \(p_2 < p_3\) are adjacent. Consequently, \(\tilde{f}^r(p_3) < \tilde{f}^r(p_2)\) are consecutive. Using an induction argument we will get adjacent elements \(p_i < p_{i+1} \leq \tilde{f}^r(p_{i+1}) < \tilde{f}^r(p_i)\). Then \(\{p_{i+1}, \tilde{f}^r(p_{i+1})\}\) is a periodic orbit of period \(\leq 2\) and we have a contradiction. \(\square\)

**Proof of the Main Theorem.** (a) \(\{q_1, \ldots, q_m\}\) decomposes into eventually periodic and periodic points. Using Corollary 1 there are \(q \geq 0\) (\(q \geq 1\) if \(\{p\}\) is not an attractor) periodic orbits, all of them of period \(r \geq 1\).
Then, using that $D = cl(N \setminus L)/\sim$ is an AR,

$$1 = i_{cl(N \setminus L)/\sim}((f^\ell)^k, cl(N \setminus L)/\sim) - i_{cl(N \setminus L)/\sim}((f^\ell)^k, p) = \sum_{q_i \in Fix((f^\ell)^k)} i_{cl(N \setminus L)/\sim}((f^\ell)^k, q_i).$$

Note that $i_{cl(N \setminus L)/\sim}((f^\ell)^k, p) = i_{R^2}((f^k, p)$ and, since $(f^\ell)^k$ is constant in a small neighborhood of $q_i \in Fix((f^\ell)^k)$, $i_{cl(N \setminus L)/\sim}((f^\ell)^k, q_i) = 1$ for every $q_i \in Fix((f^\ell)^k)$.

Therefore,

$$i_{R^2}(f^k, p) = 1 - \sum_{q_i \in Fix((f^\ell)^k)} i_{cl(N \setminus L)/\sim}((f^\ell)^k, q_i) = \begin{cases} 1 - rq & \text{if } k \in r\mathbb{N}, \\ 1 & \text{if } k \notin r\mathbb{N}. \end{cases}$$

(b) Analogous to (a). □

Remark 1. Note that the above theorem only deals with the iterations $f^k$ for $k \in \mathbb{N}$ while the theorem of Le Calvez–Yoccoz applies for arbitrary iterations $f^k, k \in \mathbb{Z}$ of an orientation preserving local homeomorphism. Our techniques are also valid for $k < 0$, for fixed points that are neither attractors nor repellers because it can be proved that $f$ and $f^{-1}$ admit generalized filtration pairs at $p$ having the same properties. The proof of this fact is not difficult but tedious and we will not give it here.

Remark 2. Let $f : U \subset \mathbb{R}^2 \to f(U) \subset \mathbb{R}^2$ be a local homeomorphism. Let $p \in U$ be a non-repeller fixed point of $f$ such that $\{p\}$ is an isolated invariant set. Then, $i_{R^2}(f^k, p)$ is a bounded function of $k$. Next corollary is a similar result in our context.

Corollary 2. Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a local homeomorphism. Let $p \in U$ be a non-repeller fixed point of $f$ such that $\{p\}$ is an isolated invariant set. Then, $i_{R^2}(f^k, p)$ is a bounded function of $k$.

3. Existence of generalized filtration pairs

Definition 3. A compact set $N$ is called an **isolating block** if,

$$f(N) \cap N \cap f^{-1}(N) \subset int(N).$$
Definition 4 (See Franks and Richeson [2]). Let $S$ be an isolated invariant set and suppose $L \subset N$ is a compact pair contained in the interior of the domain of $f$. The pair $(N,L)$ is called a filtration pair for $S$ provided $N$ and $L$ are each the closure of their interiors and

1. $\text{cl}(N \setminus L)$ is an isolating neighborhood of $S$,
2. $L$ is a neighborhood of $N^-$ in $N$ and
3. $f(L) \cap \text{cl}(N \setminus L) = \emptyset$.

Theorem 2. Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a local homeomorphism. Let $p \in U$ be a fixed point of $f$ such that $\{p\}$ is an isolated invariant set. Then, there exists an arbitrary small isolating block $N$, such that $N$ is homeomorphic to a disc and $\{p\} = \text{Inv}(N,f) \subset \text{int}(N)$.

Proof. Let $U_0$ be any open subset such that $\text{cl}(U_0) \subset U$ is a disc and $\{p\} = \text{Inv}(\text{cl}(U_0),f) \subset U_0$.

Let $M$ be a compact (smooth) manifold, $M \subset \text{cl}(U_0)$, isolating block for $\{p\} \subset \text{int}(M)$ (see Theorem 3.7 in [2] for a proof of the existence of $M$).

$M$ is a disc with a finite amount of holes $\{D_1,\ldots,D_n\}$.

From now onwards $D(X)$ will denote the subset obtained by filling the holes of $X$, then $D(M) = M \cup (\bigcup_{i=1}^n D_i)$.

Let us study the behaviour of $f$ in $\{D_1,\ldots,D_n\}$. There are three possible cases:

(A) There is $D_0 \in \{D_1,\ldots,D_n\}$ such that

$$f(D_0) \subset M \quad \text{or} \quad f^{-1}(D_0) \subset M.$$

(B) There is $D_0 \in \{D_1,\ldots,D_n\}$ such that

$$f(D_0) \not\subset D(M) \quad \text{or} \quad f^{-1}(D_0) \not\subset D(M).$$

(C) Neither of the above cases, i.e. for every $D_j$,

$$f(D_j) \subset D(M), \quad f(D_j) \cap \text{int} \left( \bigcup_{i=1}^n D_i \right) \neq \emptyset$$

and

$$f^{-1}(D_j) \subset D(M), \quad f^{-1}(D_j) \cap \text{int} \left( \bigcup_{i=1}^n D_i \right) \neq \emptyset.$$

In any of the above cases, we will transform $M$ into a manifold $M_1$ with the same properties of $M$ having at least one hole less.

Case (A). Take $D_0$ such that $f(D_0) \subset M$ (if $f^{-1}(D_0) \subset M$, the argument is similar).

$M_1 = M \cup D_0$ is an isolating block. Indeed, since $M$ is an isolating block, $f^{-1}(D_0) \cap M = \emptyset$.

On the other hand, $f^{-1}(D_0) \not\subset D_0$ because $\text{cl}(U_0)$ is an isolating neighborhood of $\{p\}$, then $f^{-1}(D_0) \cap M_1 = \emptyset$. Let $x \in \partial(M_1) \subset \partial(M)$, then (1) $f(x) \not\in M$ or (2) $f^{-1}(x) \not\in M$.

In case (1) $f(x) \not\in M_1$ because if $f(x) \in M_1 \setminus M$, then $f(x) \in \text{int}(D_0)$ and we get a contradiction with the fact that $f^{-1}(D_0) \cap M_1 = \emptyset$.

In case (2) it follows that $f^{-1}(x) \not\in M_1$ because if $f^{-1}(x) \in M_1 \setminus M$, then $y = f^{-1}(x) \in \text{int}(D_0)$ and $x = f(y) \in \text{int}(M) \cap \partial(M)$.
Case (B). Take $D_0$ with $f(D_0) \not\subset D(M)$ (if $f^{-1}(D_0) \not\subset D(M)$ the proof is analogous).

There are $D_0'$ (probably $D_0' \neq D_0$) and an arc $\gamma : I \to f(M) \setminus D(M)$, such that $\gamma(0) = p_1$, $\gamma(1) = q_1$, $\gamma((0, 1)) \subset int(f(M))$ and

\[ p_1 \in \partial(f(D_0')) \setminus D(M), \]
\[ q_1 \in \partial(f(D(M))) \setminus D(M). \]

Take a small enough neighborhood $\gamma_\varepsilon$ of $\gamma(I)$, such that $\gamma_\varepsilon$ is a disc and $\gamma(I) \subset int(f(M))(\gamma_\varepsilon) \subset f(M) \setminus D(M)$.

Then $f^{-1}(p_1) \in \partial(D_0')$, $f^{-1}(q_1) \in \partial(D(M))$, $f^{-1}(\gamma((0, 1))) \subset int(M)$ and $f^{-1}(\gamma(I))$ is an arc in $M$ joining the boundary of $D_0'$ with the boundary of the disc $D(M)$.

Let $V' \subset M \setminus \gamma(I)$. For instance $V' = B_{\varepsilon'}(f^{-1}(\gamma(I))) \cap M$, with $\varepsilon' > 0$ small enough to get $V' \subset M$, $f^{-1}(\gamma(I)) \subset V' \subset f^{-1}(\gamma_\varepsilon)$ and

\[ M_1 = M \setminus V'. \]

to be a connected manifold. $M_1$ has $n - 1$ holes because $D_0'$ has disappeared. In order to show that $M_1$ is an isolating block, if $x \in \partial(M_1)$, then $x \in \partial(M)$ or $x \in V'$. In the first case, $f(x) \not\in M$ or $f^{-1}(x) \not\in M$, then $f(x) \not\in M_1$ or $f^{-1}(x) \not\in M_1$. If $x \in V'$, we have that $f(x) \in \gamma_\varepsilon \subset f(M) \setminus M$, then $f(x) \not\in M_1$.

Case (C). Let $n_0 \in \mathbb{Z}$ such that $|n_0| \in \mathbb{N}$ is the minimum natural number with

\[ f^{n_0}(D_1 \cup \cdots \cup D_n) \not\subset D(M). \]

It is clear that $n_0$ exists because $Inv(cl(U_0), f) = \{p\}$. $|n_0| \geq 2$.

Without loss of generality we can assume that $n_0$ is positive. Let $D_0 \in \{D_1, \ldots, D_n\}$ such that $f^{n_0}(D_0) \not\subset D(M)$. Let $p_{n_0} = f^{n_0}(p_0) \in f^{n_0}(int(D_0))$, with $p_{n_0} \not\in D(M)$.

There are $D_{i_1}, \ldots, D_{i_n}$ such that

\[ f(D_{i_1}) \cap int(D_{i_1}) \neq \emptyset, \ldots, f(D_{i_n}) \cap int(D_{i_n}) \neq \emptyset. \]

Define $A_0(M) = D_1 \cup \cdots \cup D_n$ and $A_1(M) = A_0(M) \cup V_1$ where $V_1$ is a compact manifold, homeomorphic to $f(A_0(M))$. We can choose $V_1 \subset int(f(A_0(M)))$, transversal to $A_0(M)$ and near enough to $f(A_0(M))$ to get $p_1 = f(p_0) \in int(V_1)$ and from (1) $A_1(M)$ having $\leq n$ connected components. Then $A_1(M)$ is a finite amount of discs, each of them with a finite amount of holes ($A_1(M)$ is a compact manifold).

We have that $D(A_1(M))$ is a finite union of discs. There are two possible situations for $p$:

(a) $p \in cl(D(A_1(M)) \setminus A_1(M))$,
(b) $p \not\in cl(D(A_1(M)) \setminus A_1(M))$.

If (a), then $p \in G$, where $G$ is a hole of some connected component $[A_1(M)]_p$ of $A_1(M)$. In this case, define

\[ M_1 = G \setminus \bigcup_{i=1}^{n} int(D_i). \]

It is clear that $p \in int(G)$ because $p \not\in A_1(M)$. In fact, $p \in int(M_1)$.
$M_1$ is a compact manifold with at most $n-1$ holes ($[A_1(M)]_p$ contains at least one).

Let us show that $M_1$ is an isolating block. Take $x \in \partial(M_1)$, then $x \in \partial(A_1(M))$ or $x \in \partial(A_0(M))$.

If $x \in \partial(A_0(M))$, there is $i \in \{1, \ldots, n\}$ such that $x \in \partial(D_i)$. Then $f(x) \notin M$ or $f^{-1}(x) \notin M$.

Since $M_1 \subset M$, we have that $f(x) \notin M_1$ or $f^{-1}(x) \notin M_1$.

If $x \in \partial(V_1)$, it follows that $f^{-1}(x) \in \text{int}(A_0(M))$. Then $f^{-1}(x) \notin M$.

If (b), then $p \notin D(A_1(M))$. Define

$$M_1 = M \setminus \text{int}(D(A_1(M))).$$

It follows that $M_1 \subset M$, $A_0(M) = D_1 \cup \cdots \cup D_n = D(A_0(M)) \subset D(A_1(M))$ with $p \in \text{int}(M_1)$.

Moreover $\{p_0, f(p_0), f^2(p_0)\} \subset D(A_1(M))$ (the holes of $M_1$). $M_1$ has at most, $n$ holes.

Let us prove that $M_1$ is an isolating block. Take $x \in \partial(M_1)$. Then $x \in \partial(M)$ or $x \in \partial(A_1(M)) \setminus \partial(M)$.

In the first situation, we have that $f(x) \notin M$ or $f^{-1}(x) \notin M$. Since $M_1 \subset M$, we obtain the desired conclusion.

If $x \in \partial(V_1)$, it follows that $f^{-1}(x) \in \text{int}(A_0(M))$ and consequently $f^{-1}(x) \notin M_1$.

If $n_0 > 2$ the components of $D(A_1(M))$ satisfy, in $M_1$, the same conditions of case (C). Repeating the same argument we construct

$$A_1(M_1), \text{ } D(A_1(M_1)) \text{ and } M_2$$

with $p \in \text{int}(M_2)$, where $M_2$ is a compact manifold and isolating block. $D(A_1(M_1))$ is the family of holes of $M_2$ with $\leq n$ connected components. Moreover, $\{p_0, f(p_0), f^2(p_0)\} \subset D(A_1(M_1))$ and $n_0 - 1$ is the minimum natural number such that $f^{n_0-1}(D(A_1(M_1))) \notin M_1 \cup D(A_1(M_1)) = M \cup D(A_0(M))$.

By induction, we obtain $A_1(M_{n_0-2}), D(A_1(M_{n_0-2}))$ and $M_{n_0-1}$ compact manifold, isolating block, with $p \in \text{int}(M_{n_0-1})$, $D(A_1(M_{n_0-1}))$ the family of holes of $M_{n_0-1}$ having $\leq n$ components and $\{p_0, f(p_0), \ldots, f^{n_0-1}(p_0)\} \subset D(A_1(M_{n_0-2}))$.

Since $p_{n_0} = f(f^{n_0-1}(p_0)) \notin M \cup D(A_0(M))$ and $M_{n_0-1} \cup D(A_1(M_{n_0-2})) = M \cup D(A_0(M))$, it follows that

$$p_{n_0} \notin M_{n_0-1} \cup D(A_1(M_{n_0-2})).$$

Therefore we are in the setting of case (B) and the proof is finished. $\square$

**Theorem 3.** Let $f : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a local homeomorphism and let $p \in U$ be a non-repeller fixed point of $f$ such that $\{p\}$ is an isolated invariant set. Then, there exists a filtration pair $(N, L)$ for $\{p\}$, such that $N$ and the components of $L$ are topological discs.

**Proof.** Let $N$ be the isolating block constructed in Theorem 2. Using Theorem 3.7 in [2], there is $L = L_1 \cup \cdots \cup L_n$, such that $(N, L)$ is a filtration pair and $L$ is a two-dimensional manifold. Then, $L$ is a finite disjoint union of $n$ discs, each of them having a finite amount of holes.

Given the disc $G$, a hole of $L_i$, first, we will check that $p \notin G$.

Indeed, otherwise we can take the smallest hole, $G_0$, of $p$. Since $f(L) \cap cl(N \setminus L) = \emptyset$, it follows that $f(\partial(G_0)) \cap cl(N \setminus L) = \emptyset$. Since $p$ is a non-repeller fixed point, one gets immediately a contradiction.
Now, take \((N, D(L))\), where \(D(L)\) is the family of discs obtained by filling the holes of \(L\). In order to check that \((N, D(L))\) is a filtration pair, only a proof of the fact that \(f(D(L)) \cap c\l(N \setminus D(L)) = \emptyset\) is needed.

Let \(\{D(L_1), \ldots, D(L_n)\}\) be the discs obtained by filling the holes of \(L_i\). Then \(D(L) = \bigcup_{i=1}^{n} D(L_i)\). Since \(c\l(N \setminus D(L)) \subset c\l(N \setminus L)\), we have that \(f(L) \cap c\l(N \setminus D(L)) = \emptyset\).

Let us write \(c\l(D(L_i) \setminus L_i) = D_1^i \cup \cdots \cup D_p^i\) the finite union of discs.

Then,

\[
 f(D(L)) \cap c\l(N \setminus D(L)) = \left( \bigcup_{j=1}^{p(1)} D_1^j \right) \cup \cdots \cup \left( \bigcup_{j=1}^{p(n)} D_n^j \right) \cap c\l(N \setminus D(L)).
\]

Since \(\partial(D_j^i) \subset \partial(N(L))\), from the properties of filtration pair it follows that \(f(\partial(D_j^i)) \subset \text{int}(L) \subset \text{int}(D(L))\).

Therefore, \(f(D_j^i) \subset \text{int}(D(L))\). Consequently, \(f(D_j^i) \cap c\l(N \setminus D(L)) = \emptyset\) and \(f(D(L)) \cap c\l(N \setminus D(L)) = \emptyset\). \(\Box\)

**Proof of Theorem 1.** Let \(f : U \subset \mathbb{R}^2 \to \mathbb{R}^2\) be a local homeomorphism, \(\text{Inv}(U, f) = \{p\}\). Take \((N, L)\), a filtration pair for \(\{p\}\), as in Theorem 3. \(N\) is a disc, isolating block, and \(L = L_1 \cup \cdots \cup L_n\) is a disjoint union of discs. Without loss of generality, we can assume that \(L_j \cap N^- \neq \emptyset\) for every \(j \in \{1, \ldots, n\}\).

Divide \(\{L_1, \ldots, L_n\}\) into two classes:

1. The \(L_i\) such that \(N \setminus L_i\) is not connected. We will say that these \(L_i\) are *transversal* with respect to \(N\).
2. The \(L_i\) such that \(N \setminus L_i\) is connected.

Denote by \(\{L_1, \ldots, L_p\}\) the family of the transversal components of \(L\) that can be connected with \(p\) using a path in \(N\) without intersection with any other transversal component of \(L\) and let \(\{L_{p+1}, \ldots, L_m\}\) be the family of non-transversal components of \(L\) that can be connected with \(p\) using a path in \(N\) without intersection with any other transversal component of \(L\).

For each \(i \in \{1, \ldots, p\}\) define

\[
\overline{L}_i = N \setminus \text{c.c.}(N \setminus L_i, p),
\]

where \(\text{c.c.}(N \setminus L_i, p)\) is the component of \(N \setminus L_i\) containing \(p\).

Consider the pair \((N, \overline{L})\) with

\[
\overline{L} = \left( \bigcup_{i=1}^{p} L_i \right) \cup \left( \bigcup_{i=p+1}^{m} L_i \right).
\]

Each component of \(\overline{L}\) is a non-transversal disc and \((N, \overline{L})\) is a generalized filtration pair. \(\Box\)

**Remark 3.** Note that the proofs of Theorems 1–3 are also valid if we replace \(\{p\}\) by a non-repeller invariant continuum \(K\) such that there is a disc \(D\) with \(K = \text{Inv}(D, f) \subset \text{int}(D)\).
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