A Note on the Positive Energy Solutions for Elliptic Equations Involving Critical Sobolev Exponents

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(Received July 2002; accepted August 2002)
Communicated by A. Ambrosetti

Abstract—In this paper, we study the Neumann problem for a class of semilinear elliptic equations
-\Delta u = |u|^{2^*} - 2u + \mu|u|^{q-2}u \quad \text{in } \Omega,
\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega,
where \Omega is the unit ball in \mathbb{R}^N centered at the origin, N \geq 3, 2^* = \frac{2N}{N-2}, 1 < q < 2, \mu > 0. By cutting the unit ball into angular sectors and using the variational method to deal with a mixed boundary value problem on such domains, we prove the existence of infinitely many nonradial solutions with positive energy for small \mu > 0.
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Keywords—Neumann problem, Semilinear elliptic equation, Critical point.

1. INTRODUCTION AND NOTATIONS

In this paper, we study multiplicity of solutions for the following semilinear elliptic equation:
-\Delta u = |u|^{2^*} - 2u + \mu|u|^{q-2}u, \quad \text{in } \Omega,
\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial \Omega.
(1.1)
where \Omega is the unit ball in \mathbb{R}^N centered at the origin, N \geq 3, \nu denotes the unit outward normal to boundary \partial \Omega, 2^* = \frac{2N}{N-2}, 1 < q < 2, \mu > 0.

Hence, the nontrivial solutions of (1.1) are equivalent to the nonzero critical points of the energy functional
I(u) = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{2^*} |u|^{2^*} - \frac{\mu}{q} |u|^q \right) \, dx.
(1.2)

Note that 2^* is the critical Sobolev exponent, and the embedding \text{H}^1(\Omega) \hookrightarrow \text{L}^{2^*}(\Omega) is not compact, which leads to that I does not satisfy the (PS) condition. This difficulty lies in obtaining existence of solutions, especially for existence of infinitely many solutions with positive energy.

*Supported by Special Funds for Major States Basic Research Projects of China (G1999075107) and Knowledge Innovation Program of CAS in China.

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doi:10.1016/S0893-9659(03)00151-4
Typeset by A4\LaTeX
In the last 20 years, Dirichlet and Neumann problems for semilinear equations have been studied extensively. By variant methods, for example, upper-lower solution method, variational method, degree theory, etc., people have obtained many important and interesting results (see [1-10], for example).

The Dirichlet problem with sublinear and superlinear terms
\[-\Delta u - |u|^{p-2}u + \mu|u|^{q-2}u, \quad \text{in } \Omega,\]
\[u = 0, \quad \text{on } \partial\Omega,\]
(1.3)
has been treated by Ambrosetti et al. [4]. Their main results are the following (where \(\Omega\) is a general bounded domain).

1. For all \(0 < q < 2 < p\), there exists \(A > 0\) such that, for all \(\mu \in (0, A)\), problem (1.3) has a minimal solution \(u_{\mu}\) satisfying \(J(u_{\mu}) < 0\).
2. Let \(1 < q < 2 < p \leq 2^*\). Then, for all \(\mu \in (0, A)\), problem (1.3) has a second solution \(u_{\mu} > u_{\mu}\).
3. Let \(1 < q < 2 < p \leq 2^*\). Then there exists \(\mu^* > 0\) such that for all \(\mu \in (0, \mu^*)\), problem (1.3) has infinitely many solutions such that \(J(u) < 0\).
4. Let \(1 < q < 2 < p < 2^*\). Then, for all \(\mu \in (0, \mu^*)\), problem (1.3) has also infinitely many solutions such that \(I(u) > 0\).

In the last section of [4], authors proposed some open problems, and the second one is whether problem (1.3) has infinitely many solutions with positive energy, when \(p = 2^*\), for \(\mu > 0\) small enough.

For (1.1), following the argument of [4], it is not difficult to obtain infinitely many solutions for (1.1) with negative energy (that is, \(I(u) < 0\)) since \(I(u)\) satisfies the (PS)_c condition for \(c < 0\). However, since \(I(u)\) satisfies the (PS)_c condition only for \(c \in (-\infty, (1/2N)S^{N/2} - \kappa_0\mu^{2^{*}/(2^{*}-q)})\), it is not easy to obtain solutions with positive energy and even difficult to obtain existence of infinitely many solutions. Since we are dealing with problems with critical Sobolev exponent and aiming to obtain solutions of (1.1) with positive energy, fountain theorem, and its dual theorem [5,11] are not applicable.

When \(q = 2\), Comte and Knaap [8] considered problem (1.1), and obtained infinitely many solutions by cutting the unit ball into angular sectors.

In this paper, we take some ideas from [4,6,8] and obtain the following result.

**Theorem 1.1.** There exists \(\mu^* > 0\) such that, for every \(0 < \mu < \mu^*\), problem (1.1),(1.2) has infinitely many different nonradial solutions with positive energy.

Our strategy of proving Theorem 1.1 is as follows. First, we cut the unit ball angular sectors, then consider a mixed boundary problem set on the angular sectors, and finally prove the existence of infinitely many different nonradial solutions having positive energy. For convenience, as in [8], we move the center of the unit ball to the point \((0,0,\ldots,0,1)\), so that the origin lies on the boundary \(\partial B\). That is,
\[B = \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \mid x_1^2 + x_2^2 + \cdots + x_{N-1}^2 + (1 - x_N)^2 < 1\} .\]
(1.4)

Set
\[A_m = \{x \in B \mid \cos \left(\frac{\pi}{2m}\right)|x_{N-1}| < \sin \left(\frac{\pi}{2m}\right)(1 - x_N)\}, \quad m = 1, 2, \ldots .\]
(1.5)

The angle between the two planar boundaries is called the angle of the sector. So \(A_1\) is a half-ball, \(A_2\) an angular sector of angle \(\pi/2\), and \(A_3\) an angular sector of angle \(\pi/4, \ldots\).

We first consider the following mixed boundary problem:
\[-\Delta u - |u|^{p-2}u + \mu|u|^{q-2}u, \quad \text{in } A_m,\]
\[u = 0, \quad \text{on } \Gamma_0,\]
\[\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \Gamma_1,\]
(1.6)
where \(\Gamma_1 = \partial A_m \cap \partial B\).
After obtaining the existence of positive solutions of (1.6), we can construct nontrivial solutions of by suitable reflections.

Let us remark that unlike in [8], where \( q = 2 \), in our case, \( 1 < q < 2 \), the nonlinearity is sublinear at 0. We cannot obtain solutions of (1.6) by considering

\[
\text{W}^d \sim \left( s, \frac{1}{2} |u|^2 - \frac{1}{2^*} |u|^{2^*} - \frac{1}{q} |u|^q \right) dx,
\]

and

\[
J(u) = \int_{A_m} \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{2^*} |u|^{2^*} - \frac{1}{q} |u|^q \right) dx,
\]

\( \Gamma_m = \{ \gamma \in C([0,1], V(A_m)) \mid \gamma(0) = 0, J(\gamma(0)) < 0 \} \),

and

\[
c_m = \inf_{\gamma \in \Gamma_m} \max_{t \in [0,1]} J(\gamma(t)).
\]

For simplicity, we will always denote

\[
\|u\|_{A_m} = \left( \int_{A_m} (|\nabla u|^2 + u^2) \, dx \right)^{1/2}, \quad |\nabla u|_{2,A_m} = \left( \int_{A_m} |\nabla u|^2 \, dx \right)^{1/2},
\]

and \( C_0, C_1, C_2, \ldots \) denote (possibly different) positive constants.

2. PROOF OF THEOREM 1.1

Let \( S \) be the best Sobolev constant for the imbedding \( H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega) \), and \( D^{1,2}(R^N_+) = \{ u \in L^{2^*}(R^N_+) \mid |\nabla u| \in L^2(R^N_+) \} \), and it is well known that

\[
S(R^N_+) := \inf_{u \in D^{1,2}(R^N_+) \setminus \{0\}} \frac{|\nabla u|^2_{2,R^N_+}}{|u|^{2^*}_{2,R^N_+}} = 2^{-2/N} S.
\]

It is easy to verify that there exists \( k_0 > 0 \) independent of \( \mu > 0 \) such that

\[
\min_{t > 0} \left( \frac{1}{N} t^{2^*} - \mu \left( \frac{1}{q} - \frac{1}{2} \right) |B|^{1-q/2^*} t^q \right) = -k_0 \mu^{2^*/(2^*-q)}.
\]

LEMMA 2.1. For the \( k_0 \) defined in (2.2), then the functional \( J \) satisfies (PS)\(_c\) condition for \( c \) satisfying

\[
-\infty < c < \frac{1}{2N} S^{N/2} - k_0 \mu^{2^*/(2^*-q)}.
\]

PROOF. Assume \( (u_k) \subset V(A_m) \) is a sequence such that as \( k \to \infty \),

\[
J(u_k) \to c, \quad J'(u_k) \to 0;
\]

that is,

\[
\int_{A_m} \left( \frac{1}{2} |\nabla u_k|^2 - \frac{1}{2} |u_k|^{2^*} - \frac{1}{q} |u_k|^q \right) \, dx = c + O(1),
\]

\[
\int_{A_m} \left( \nabla u_k \cdot \nabla v - |u_k|^{2^*-2} u_k v - \mu |u_k|^{q-2} u_k v \right) \, dx = 0(1) \|v\|, \quad \forall v \in V(A_m).
\]
Here and hereafter, we use $O(1)$ to denote quantities which tend to zero as $k \to \infty$. Let $v = u_k$ in (2.5), and we have

$$
\int_{A_m} \left( |\nabla u_k|^2 - |u_k|^{2^*} - \mu |u_k|^q \right) \, dx = 0(1) ||u_k||. 
$$

(2.6)

From (2.4) and (2.5), we obtain

$$
\int_{A_m} \left( \frac{1}{n} |\nabla u_k|^2 - \mu \left( \frac{1}{q} - \frac{1}{2^*} \right) |u_k|^q \right) \, dx \leq C(1 + ||u_k||)
$$

Note that $1 < q < 2$ and norms $|\nabla u|_{2,A_m}$ and $||u||$ are equivalent in $V(A_m)$. It therefore follows from (2.4) and (2.6) that $||u_k|| \leq C$.

By choosing subsequence if necessary, we may assume that, as $k \to \infty$,

- $u_k \rightharpoonup u$, weakly in $V(A_m)$,
- $u_k \rightharpoonup u$, weakly in $L^2(A_m)$,
- $u_k \to u$, strongly in $L^\ell(A_m)$, for every $\ell \in (1,2^*)$,
- $u_k \to u$, a.e. on $A_m$.

It follows from (2.5) that $u$ satisfies

$$
\int_{A_m} \left( \nabla u \cdot \nabla v - |u|^{2^*-2} uv - \mu |u|^{q-2} uv \right) \, dx = 0, \quad \forall v \in V(A_m),
$$

and

$$
J(u) = \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{A_m} |u|^{2^*} \, dx - \mu \left( \frac{1}{q} - \frac{1}{2} \right) \int_{A_m} |u|^q \, dx.
$$

(2.7)

Set $v_k = u_k - u$. Brezis-Lieb lemma [12] leads to

$$
\int_{A_m} |u_k|^2 \, dx = \int_{A_m} |v_k|^2 \, dx + \int_{A_m} |u|^{2^*} \, dx + O(1).
$$

Obviously,

$$
\int_{A_m} |\nabla u_k|^2 \, dx = \int_{A_m} |\nabla v_k|^2 \, dx + \int_{A_m} |\nabla u|^2 \, dx + O(1).
$$

Hence, we obtain

$$
J(v) + \frac{1}{2} ||\nabla v_k||^2_{2,A_m} - \frac{1}{2^*} ||v_k||^2_{2^*,A_m} = c + O(1).
$$

(2.8)

Since $\langle J'(u_k), u_k \rangle \to 0$ as $k \to \infty$, we obtain as $k \to \infty$

$$
||\nabla v_k||^2_{2,A_m} - ||v_k||^2_{2^*,A_m} \to - ||\nabla u||^2_{2,A_m} + ||u||^2_{2^*,A_m} + \mu ||u||^q_{q,A_m} = -\langle J'(u), u \rangle = 0.
$$

We may therefore assume that, as $k \to \infty$,

$$
||\nabla v_k||^2_{2,A_m} \to b, \quad ||v_k||^2_{2^*,A_m} \to b,
$$

where $b$ is a nonnegative constant.

Now we use the following inequality due to Lemma 2.1 in [13] (see also [14]).

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with $C^1$ boundary. Then for every $\varepsilon > 0$, we have

$$
|\nabla u|^2_{2,\Omega} + C_\varepsilon |u|^2_{2,\Omega} \geq \left( 2^{2^* - 2/N} \varepsilon - \varepsilon \right) |u|_{2^*,\Omega}^2, \quad \forall u \in H^1(\Omega).
$$

(2.9)

where $C_\varepsilon$ is a positive constant depending only on the diameter of the domain $\Omega$ and $\varepsilon$. 
Because the boundary of $A_m$ is not up to $C^1$, we extend functions in $V(A_m)$ to the unit ball $B$ by defining

$$\tilde{u}(x) = \begin{cases} u(x), & \text{if } x \in A_m, \\ 0, & \text{if } x \in B \setminus A_m. \end{cases}$$

Thus, $\tilde{u} \in H^1(B)$ and $B$ is of class $C^1$. In addition, we also have

$$|\nabla \tilde{u}|_{2,B} = |\nabla u|_{2,A_m}, \quad |\tilde{u}|_{2^*,B} = |u|_{2^*,A_m}.$$ 

Therefore, from (2.9), we obtain

$$|\nabla v_k|_{2,A_m}^2 + C_\varepsilon |v_k|_{2,A_m}^2 \geq \left( 2^{-2/N} S - \varepsilon \right) |v_k|_{2^*,A_m}^2.$$ 

Then, we have as $k \to \infty$, $b \geq (2^{-2/N} S - \varepsilon) b^{2/2^*}$. If $b = 0$, the proof is complete. Assume $b \neq 0$. Then $b^{2/N} \geq 2^{-2/N} S - \varepsilon$, and therefore, we obtain $b \geq (1/2) S^{N/2}$ due to the arbitrariness of $c$. Combined with (2.2), (2.7), and (2.8), we have

$$c = J(u) + \left( 1 - \frac{1}{2^*} \right) b$$

$$= \left( 1 - \frac{1}{2^*} \right) \left( b + |u|_{2^*,A_m}^{2^*} \right) - \mu \left( \frac{1}{q} - \frac{1}{2} \right) |u|_{q,A_m}^q$$

$$\geq \frac{1}{2N} S^{N/2} + \frac{1}{N} |u|_{2^*,A_m}^{2^*} - \mu \left( \frac{1}{q} - \frac{1}{2} \right) |B|^{1-q/2^*} |u|_{2^*,A_m}^{2^*}$$

$$\geq \frac{1}{2N} S^{N/2} - k_0 \mu^{2^*/(2^*-q)}.$$ 

We obtain a contradiction since $c < (1/2N) S^{N/2} - k_0 \mu^{2^*/(2^*-q)}$.

**Lemma 2.2.** There exists $\mu_0 > 0$ independent of $m$ such that, for any $0 < \mu < \mu_0$, there is a nonnegative function $v \in V(A_m) \setminus \{0\}$ satisfying

$$\sup_{t \geq 0} J(tv) < \frac{1}{2N} S^{N/2} - k_0 \mu^{2^*/(2^*-q)}.$$

**Proof.** Define, for $\varepsilon > 0$, $u_\varepsilon(x) = \psi(x) / (\varepsilon + |x|^2)^{(N-2)/2}$, where $\psi(x)$ is a smooth cut-off function which satisfies $\psi \equiv 1$ in a neighborhood of the origin; $\psi \equiv 0$ outside $B_m$, where $B_m$ is defined as

$$B_m = \left\{ x \in \mathbb{R}^N \mid |x| < \sin \left( \frac{\pi}{2m} \right) \right\}.$$ 

From the choice of radius of $B_m$, we see that $u_\varepsilon|_{\Gamma_0} = 0$, and $u_\varepsilon \in V(A_m)$. From [6,8], and after simple calculations, we have for $N = 3$,

$$|\nabla u_\varepsilon|_{2,A_m}^2 = \frac{K_1}{2} \varepsilon^{-1/2} \left( 1 - \frac{\pi}{K_1} | \log \varepsilon |^{1/2} + D_m \left( \varepsilon^{1/2} \right) \right),$$

and for $N \geq 4$, we have

$$|\nabla u_\varepsilon|_{2,A_m}^2 = \frac{K_1}{2} \varepsilon^{-(N-2)/2} \left( 1 - L \varepsilon^{1/2} + F_m(\varepsilon) \right),$$

and

$$|u_\varepsilon|_{2^*,A_m}^2 = \frac{K_2}{2-(N-2)/N} \varepsilon^{-(N-2)/2} \left( 1 - \frac{N-3}{N+1} L \varepsilon^{1/2} + G_m(\varepsilon) \right).$$
\[ L = (N - 2)^2 K_1 \int_{R^{N-1}} \frac{|x|^4}{(1 + |x|^2)^N} \, dx, \quad K_1 = (N - 2)^2 \int_{R^{N}} \frac{|x|^2}{(1 + |x|^2)^N} \, dx, \]

\[ K_2 = \left( \int_{R^{N}} \frac{1}{(1 + |x|^2)^{N}} \, dx \right)^{-(N-2)/N}, \quad \frac{K_1}{K_2} = S \text{ is the best constant,} \]

and \(|D_m(\epsilon^{1/2})| \leq Ce^{1/2}, \ |E_m(\epsilon^{1/2})| \leq Ce^{1/2}, \ |F_m(\epsilon)| \leq Ce, \ |G_m(\epsilon)| \leq Ce\), for some \(C > 0\) independent of \(m\).

Assume the maximum of \(J(tu_\epsilon)\) is achieved at a point \(t_\mu\). Then,

\[
\sup_{t \geq 0} J(tu_\epsilon) = J(t_\mu u_\epsilon)
\]

\[
\begin{align*}
&= \frac{t^2}{2} \left( \nabla u_\epsilon \right)_{2,A_m}^2 - \frac{t^2}{2} \left( \frac{\mu}{2} \right)^* u_\epsilon \left( 2 + \frac{\mu}{2} \right)_{2,A_m}^2 - \frac{\mu t^2}{q} |u_\epsilon|_{q,A_m}^q \\
&\leq \frac{t^2}{2} \left( \nabla u_\epsilon \right)_{2,A_m}^2 - \frac{t^2}{2} \left( \frac{\mu}{2} \right)^* u_\epsilon \left( 2 + \frac{\mu}{2} \right)_{2,A_m}^2 \\
&\leq \sup_{t \geq 0} \left( \frac{t^2}{2} \left( \nabla u_\epsilon \right)_{2,A_m}^2 - \frac{t^2}{2} \left( \frac{\mu}{2} \right)^* u_\epsilon \left( 2 + \frac{\mu}{2} \right)_{2,A_m}^2 \right) \\
&= \frac{1}{N} \left( \left( \frac{\nabla u_\epsilon}{|u_\epsilon|_{2,A_m}^2} \right)_{2,A_m}^2 \right)^{N/2} \\
&= \frac{1}{2N} S^{N/2} \left( \frac{1 - LE^{1/2} + F_m(\epsilon)}{1 - (N - 3)/(N + 1)LE^{1/2} + G_m(\epsilon)} \right)^{N/2} \\
&< \frac{1}{2N} S^{N/2} \left( 1 - 2N \right)^{N/2} \\
&\leq \frac{1}{2N} S^{N/2} \left( 2SL \right)^{N/2} \epsilon^{N/4},
\end{align*}
\]

for any \(\epsilon \in (0, \epsilon_1]\), where \(\epsilon_1\) is a small positive number independent of \(m\).

Hence, from (2.15), we choose \(0 < \epsilon_0 < \epsilon_1\), and have

\[
\sup_{t \geq 0} J(tu_\epsilon) < \frac{1}{2N} S^{N/2} - \frac{1}{2N} \left( 2SL \right)^{N/2} \epsilon_0^{N/4}.
\]

It remains only to verify that

\[
\frac{1}{2N} \left( 2SL \right)^{N/2} \epsilon_0^{N/4} > k_0 \mu^{2/(2* - q)}.
\]

Obviously, there exists a small positive constant \(\mu_1\) independent of \(m\) such that for any \(0 < \mu < \mu_1\), (2.16) holds.

If \(N = 3\), from (2.11),(2.12), similar to the proof of (2.15) we have

\[
\frac{|\nabla u_\epsilon|_{2,A_m}^2}{|u_\epsilon|_{2,A_m}^2} = \frac{(K_1/2)\epsilon^{1/2} (1 - (\pi/K_1)) \log \epsilon^{1/2} + D_m(\epsilon^{1/2})}{(K_2/2^{1/3}) \epsilon^{-1/2} (1 + E_m(\epsilon^{1/2}))} \\
\leq 2^{-2/3} \left( \frac{\pi}{K_1} \right) \log \epsilon^{1/2} + C \epsilon^{1/2} \\
\leq 2^{-2/3} \left( \frac{\pi}{K_1} \right) \log \epsilon^{1/2} + C \epsilon^{1/2} + C \epsilon^2 \\
\leq 2^{-2/3} \left( \frac{\pi}{2K_1} \right) \log \epsilon^{1/2}, \quad \text{for any } 0 < \epsilon < \epsilon_2,
\]

where \(\epsilon_2\) is a positive constant independent of \(m\).
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Therefore, we can choose $0 < \epsilon_0 < \epsilon_2$, and $\mu_2 > 0$ independent of $m$ such that for any $0 < \mu < \mu_2$,

$$
\sup_{t \geq 0} J(tu_{\epsilon_0}) \leq \frac{1}{3} \left( \frac{\|\nabla u_{\epsilon_0}\|_{L^2}^2}{\|u_{\epsilon_0}\|_{L^2}^2} \right)^{3/2} < \frac{1}{6} S^{3/2} \left( 1 - \frac{\pi}{2K_1} \log \epsilon_0 \right)^{3/2} \leq \frac{1}{6} S^{3/2} - \frac{1}{6} \left( \frac{\pi S}{2K_1} \log \epsilon_0 \right)^{3/4} \leq \frac{1}{6} S^{3/2} - k_0 \mu^{6/(6-q)}.
$$

(2.17)

Therefore, set $\lambda_0 = \min(\mu_1, \mu_2)$ such that for any $0 < \mu < \lambda_0$, there exists a nonnegative function $v \in V(A_m) \setminus \{0\}$ satisfying (2.10), and the proof of Lemma 2.2 is complete.

**Proof of Theorem 1.1.** Let $\lambda_m$ and $\psi_m$ be, respectively, the first eigenvalue and the corresponding eigenfunction of the problem

$$
-\Delta \psi = \lambda \psi, \quad \text{in } A_m,
$$

$$
\psi = 0, \quad \text{on } \Gamma_0,
$$

$$
\frac{\partial \psi}{\partial \nu} = 0, \quad \text{on } \Gamma_1.
$$

The eigenfunction $\psi_m$ can be expressed in terms of the spherical Bessel function and $\lambda_m > 0$, $\lambda_m \to \infty$ as $m \to \infty$ (see [8,15]).

For any $u \in V(A_m)$,

$$
J(u) = \int_{A_m} \left( \frac{1}{2} \|
abla u\|^2 - \frac{1}{2} |u|^{2^*} - \frac{\mu}{q} |u|^q \right) \, dx \geq \frac{\lambda_m}{2(\lambda_m + 1)} \|u\|^2 - C_1 \|u\|^{2^*} - C_2 \mu \|u\|^q.
$$

(2.18)

Since $\lambda_m > 0$, $\lambda_m \to \infty$, as $m \to \infty$, there exists a positive constant $\delta_0$ independent of $m$ such that $\lambda_m/2(\mu_m + 1) \geq 2\delta_0$. Therefore, we have

$$
J(u) \geq 2\delta_0 \|u\|^2 - C_1 \|u\|^{2^*} - C_2 \mu \|u\|^q.
$$

(2.19)

From (2.9), we know that positive constants $C_1, C_2$ are also independent of $m$.

Set $\mu_3 = 2^{q-2} C_2^{-1} \delta_0^{1+4(N-n)(2-q)/4} C_1^{-(N-n)(2-q)/4}$, $\rho_0 = (\delta_0 C_1) (N-2)/4$, $\mu^* = \min(\mu_0, \mu_3)$, where $\mu_0$ is in Lemma 2.2, and

$$
\|u\| = \rho, \quad \delta_0 \|u\|^{2^*} - C_1 \|u\|^{2^*} = \delta_0 \rho^2 - C_1 \rho^{2^*} > 0, \quad \text{and} \quad \delta_0 \|u\|^q - C_2 \mu \|u\|^q = \delta_0 \rho^2 - C_2 \mu^q > 0.
$$

Then for any $0 < \mu < \mu^*$, $\mu \in (\rho_0/2, \rho_0)$, we have

$$
b := \inf_{\|u\| = \rho} J(u) > 0 = J(0).
$$

In addition, as $\rho \to \infty$,

$$
J(tu) = \int_{A_m} \left( \frac{t^2}{2} \|
abla u\|^2 - \frac{t^{2^*}}{2} |u|^{2^*} - \frac{\mu t^q}{q} |u|^q \right) \, dx \to -\infty,
$$

where the function $v$ is from Lemma 2.2.

Hence, there exists $t_0 > 0$ such that $\|tu_0\| > \rho$ and $J(t_0 u) < 0$. By the mountain pass theorem and Lemma 2.2, there exists a sequence $(u_k) \subset V(A_m)$ satisfying as $k \to \infty$,

$$
J(u_k) \to c_m > 0, \quad J'(u_k) \to 0.
$$
In addition, $c_m \leq \sup_{t \in [0,1]} J(tu) \leq \sup_{t \geq 0} J(tu) < (1/2N)S^{N/2} - k_0 \mu^{2^*/(2^* - 2)},$ by Lemma 2.1, $(u_k)$ has a converging subsequence, still denoted by $(u_k)$, such that

$$u_k \rightarrow u, \quad \text{strongly in } V(A_m).$$

Consequently, $u$ is a critical point of $J$ and satisfies problem (1.6), and therefore, $c_m$ is a critical value of the functional $J$.

To obtain a solution defined on the unit ball with positive energy, we use the following observation. Let $\Omega$ be a domain in $\mathbb{R}^N$, which is symmetric with respect to a hyperplane $\{x_N = 0\}$, $\Omega^+ = \Omega \cap \{x_N > 0\}, \Omega^- = \Omega \cap \{x_N < 0\}$, and $\Gamma_0 = \{x \in \Omega \mid x_N = 0\}$. Suppose $u$ is a solution of the following problem:

$$-\Delta u = f(u), \quad \text{in } \Omega^+,$$

$$u = 0, \quad \text{on } \Gamma_0,$$

where $f$ is a real, odd, and continuous function.

Define

$$\tilde{u}(x', x_N) = \begin{cases} u(x', x_N), & \text{in } \Omega^+, \\ -u(x', -x_N), & \text{in } \Omega^-, \\ 0, & \text{on } \Gamma_0. \end{cases}$$

Then $\tilde{u}$ satisfies

$$-\Delta u = f(u), \quad \text{in } \Omega.$$

Now we apply this principle to the solution $u$ of problem (1.6). Let $A_m'$ be the reflection of $A_m$ in one of the planar boundaries. On $A_m \cup A_m'$, we can define the function $\tilde{u}$ such that $\tilde{u} = u$ on $A_m$ and $\tilde{u}$ is antisymmetric with the plane of reflection. Now let $A_m''$ be the reflection of $A_m \cup A_m'$ in one of the planar boundaries and $\tilde{u}$ the function defined on $A_m \cup A_m' \cup A_m''$ such that $\tilde{u} = \tilde{u}$ on $A_m \cup A_m'$ and $\tilde{u}$ is antisymmetric with the planar of reflection. Repeating this procedure, after finite steps, we finally obtain a function defined on the whole unit ball $B$, denoted by $u_m$. Clearly, $u_m$ satisfies the Neumann condition on the boundary $\partial B$, and thus, it is a solution of problem (1.1). In addition, $I(u_m) = k_m J(u) > 0$, $k_m$ is certain positive integer. That is, $u_m$ is also a solution having positive energy. Since for every $m = 1, 2, \ldots,$ problem (1.6) admits a solution with positive energy, and these solutions are different by maximum principle, hence there exist infinitely many different nonradial solutions of problem (1.1) with positive energy, and the proof of Theorem 1.1 is complete.

REFERENCES


