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A Kuratowski-Type Theorem for the Maximum Genus of a Graph

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Let G be a finite connected graph. The genus of G, denoted by $\gamma(G)$, is the least integer n such that G can be imbedded in S_n. The maximum genus of G, denoted by $\gamma_M(G)$, is the largest integer k such that G can be 2-cell imbedded in S_k. This paper characterizes those graphs G for which $\gamma(G) = \gamma_M(G)$. As part of this characterization, it is shown that $\gamma_M(G) = 0$ if and only if G does not contain a subgraph isomorphic to a subdivision of one of two given graphs.

1. INTRODUCTION

In the papers [5] and [6], the authors have investigated the 2-cell (disk) imbeddings of a finite undirected connected graph G with no loops or multiple edges, on a compact orientable 2-manifold. Such a surface is homeomorphic to a sphere with k handles, which we denote by S_k.

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Corresponding to a given graph G having V vertices and E edges the genus k is finite, and if $\gamma(G)$ and $\gamma_M(G)$ denote, respectively, the minimum and maximum values of k, then 2-cell imbeddings of G occur on S_k if and only if $\gamma(G) \leqslant k \leqslant \gamma_M(G)$. The parameters $\gamma(G)$ and $\gamma_M(G)$ are called the *genus* and the *maximum genus of the graph* G, respectively.

In this paper a characterization is given for those connected graphs G for which $\gamma(G) = \gamma_M(G)$, so that G has a 2-cell imbedding for exactly one value of k. The result obtained in Theorem 4 below shows that k must be zero, and describes those graphs G for which $\gamma_M(G) = 0$. This is done in a manner reminiscent of the celebrated theorem of Kuratowski on the planarity of a graph, which states that a necessary and sufficient condition for $\gamma(G) = 0$ is that G have no subgraph isomorphic to a subdivision of either the complete graph K_5 or the complete bipartite graph $K_{3,3}$. As an application of Theorem 4, simplified proofs of two results of Duke [2] on Betti numbers are given.

2. Preliminaries

Euler's extended polyhedral formula $F - E + V = 2 - 2k$ applies to any 2-cell imbedding of a graph G on S_k, and may be written as $F = 1 + \beta(G) - 2k$, where $\beta(G) = E - V + 1$ is the circuit rank or Betti number of G, and F is the number of faces in the imbedding, where each face is homeomorphic to an open unit disk (a 2-cell). When the genus k is a minimum, the number $F = d(G)$ is called the *regional number* of G, and represents the maximum number of faces possible in any 2-cell imbedding. Since in general $F \geqslant 1$, we obtain at once an upper bound for the maximum genus from the inequality $\gamma_M(G) \leqslant [\beta/2]$, where $[x]$ denotes the greatest integer less than or equal to x.

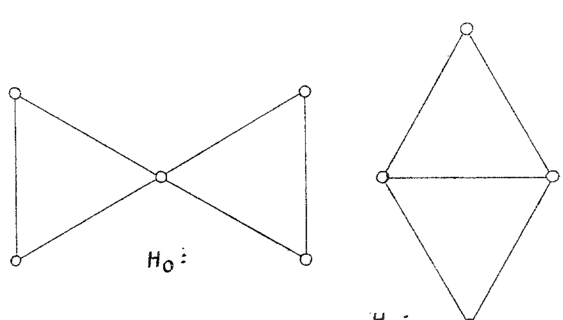

FIG. 1. The graphs H_0 and H_1.

There are two planar graphs H_0 and H_1, shown in Fig. 1, which play a decisive role in the characterization theorem (Theorem 4) soon to be developed. The graph H_0 consists of two triangles having a common vertex, and H_1 consists of two triangles having a common edge. In the proof of Theorem 4 we shall employ subdivisions of these graphs. A graph G_0 is said to be a *subdivision* of G if G_0 can be obtained from G in a finite number of steps, each consisting of the deletion of an edge uv and the addition of a new vertex w together with the edges uw and wv.

The following definition will also be required. A *cactus* is a connected (planar) graph in which each block is a cycle or an edge. Additional graph theory terminology may be found in Harary [4].

We now establish three theorems which will be employed in the proof of Theorem 4. A proof of the second of these is also contained in [6]. We use Edmond's technique of describing a 2-cell imbedding T, as presented in [3] (see also [7]). For each vertex v of G we choose a cyclic permutation P_v of the vertices adjacent to v, and obtain the faces of T by the rule that the directed edge (a, b) in the boundary of a face f is followed by the directed edge $(b, P_b(a))$. Here $P_b(a)$ denotes that vertex which follows vertex a cyclically in P_b.

We often find it convenient to compare the face count for an imbedding T of G with the face count for an imbedding T' of a subgraph G'; we introduce the notations $F(T, G)$ and $F(T', G')$ for this purpose. The symbol $P_{v(T)}$ will designate the permutation P_v in the imbedding T.

THEOREM 1. *Any cycle in G can be taken as the boundary of a face, for an appropriately chosen imbedding T for G.*

Proof. To obtain the boundary:

$$(v_{i-1} v_i v_{i+1} \cdots),$$

select the permutation P_{v_i} at vertex v_i of the cycle as follows:

$$P_{v_i} : (v_{i-1}, v_{i+1}, \ldots).$$

Battle, Harary, Kodama, and Youngs [1] have shown that the genus of a graph is the sum of the genera of its blocks. The analogous result does not hold in general for the parameter γ_M, as shown in [5]; however the following theorem does afford a partial analog.

THEOREM 2. *Let H be a graph with n components $G_1, G_2, \ldots, G_n$, and G a connected graph obtained from H by the addition of $n - 1$ edges. Then*

$$\gamma_M(G) = \sum_{i=1}^{n} \gamma_M(G_i).$$

Proof. We establish the case $n = 2$. The general result then follows by induction. (We assume that no G_i is an isolated vertex; this trivial case is left to the reader.) For $n = 2$, add edge xy, joining x in G_1 to y in G_2. Let T_1 be any 2-cell imbedding of G_1, with $P_{x(T_1)}$: $(x_1, x_2, ...)$. Let T_2 be any 2-cell imbedding of G_2, with $P_{y(T_2)}$: $(y_1, y_2, ...)$. Then we may form a 2-cell imbedding T of G, described by:

$$P_{v(T)} = \begin{cases} P_{v(T_1)}, & \text{if } v \in V(G_1) \text{ and } v \neq x; \\ (x_1, y, x_2, ...), & \text{if } v = x; \\ (y_1, x, y_2, ...), & \text{if } v = y; \\ P_{v(T_2)}, & \text{if } v \in V(G_2) \text{ and } v \neq y. \end{cases}$$

There is a face f_1 in T_1 with boundary $(x_1 x x_2 \cdots)$, and a face f_2 in T_2 with boundary $(y_1 y y_2 \cdots)$. These two faces are combined in T for G into one face f with boundary $(x_1 x y y_2 \cdots y_1 y x x_2 \cdots)$. This already uses all the innovations in T as compared with T_1 and T_2, so all the other faces of T are the same as faces in T_1 and T_2. Hence we have

$$F(T, G) = F(T_1, G_1) + F(T_2, G_2) - 1. \tag{1}$$

Conversely, if we are given T for G, we reverse the above definitions and arguments to find T' for G_1 and T'' for G_2 with the property that

$$F(T', G_1) + F(T'', G_2) = F(T, G) + 1. \tag{2}$$

Now suppose that T^* is a 2-cell imbedding of G corresponding to $\gamma_M(G)$. Then $F(T^*, G) \leqslant F(T, G)$ for every 2-cell imbedding T of G. Using (2) we know there exists a 2-cell imbedding $(T^*)'$ for G_1 and a 2-cell imbedding $(T^*)''$ for G_2 with $F(T^*, G) = F((T^*)', G_1) + F((T^*)'', G_2) - 1$.

Let T_1^* be a 2-cell imbedding of G_1 corresponding to $\gamma_M(G_1)$, so that $F(T_1^*, G_1) \leqslant F(T_1, G_1)$ for every 2-cell imbedding T_1 of G_1. Similarly, let T_2^* be a 2-cell imbedding of G_2 corresponding to $\gamma_M(G_2)$, so that $F(T_2^*, G_2) \leqslant F(T_2, G_2)$ for every 2-cell imbedding T_2 of G_2. Using (1) we know there exists a 2-cell imbedding T of G with

$$F(T, G) = F(T_1^*, G_1) + F(T_2^*, G_2) - 1.$$

By the minimal properties of $F(T_1^*, G_1)$ and $F(T_2^*, G_2)$, it follows that

$$\begin{aligned} F(T, G) &= F(T_1^*, G_1) + F(T_2^*, G_2) - 1 \\ &\leqslant F((T^*)', G_1) + F((T^*)'', G_2) - 1 = F(T^*, G). \end{aligned}$$

By the choice of T^*, we know that $F(T^*, G) \leqslant F(T, G)$. Hence $F(T^*, G) = F(T, G)$, and we may make the following computation:

$$\begin{aligned} 2\gamma_M(G) &= 2 + E - V - F(T^*, G) \\ &= 2 + (E_1 + E_2 + 1) - (V_1 + V_2) - (F(T_1^*, G_1) + F(T_2^*, G_2) - 1) \\ &= (2 + E_1 - V_1 - F(T_1^*, G_1)) + (2 + E_2 - V_2 - F(T_2^*, G_2)) \\ &= 2\gamma_M(G_1) + 2\gamma_M(G_2). \end{aligned}$$

Division by 2 then completes the proof.

As an application of Theorem 2, we exhibit, for every natural number n, a connected graph G_n with $\gamma_M(G_n) = 2\gamma(G_n) = 2n$. Let n disjoint copies of the graph $K_{3,3}$ be joined together by $n - 1$ additional edges so as to form a connected graph G_n. By the result of Battle, Harary, Kodama, and Youngs, $\gamma(G_n) = n\gamma(K_{3,3}) = n$. Also, since $\gamma_M(K_{3,3}) = 2$ (see [6]), $\gamma_M(G_n) = n\gamma_M(K_{3,3}) = 2n$, by Theorem 2.

THEOREM 3. *If G has a 2-cell imbedding for which some vertex appears in the boundary of at least three distinct faces, then* $\gamma(G) \neq \gamma_M(G)$.

Proof. In [2, Theorem 3.2(iii)], Duke has shown that, if G has a 2-cell imbedding T in S_k such that some vertex appears in the boundary of at least three distinct faces, then G also has a 2-cell imbedding in S_{k+1}. Hence $\gamma(G) \neq \gamma_M(G)$.

3. An Analog of Kuratowski's Theorem

We are now prepared to state our principal result.

THEOREM 4. *Let G be a connected graph. Then the following four statements are equivalent*:

(i) $\gamma_M(G) = \gamma(G)$.

(ii) *G has no subgraph isomorphic to a subdivision of* H_0 *or* H_1.

(iii) *G is a cactus whose cycles are vertex disjoint.*

(iv) $\gamma_M(G) = 0$.

Proof. We establish the circular set of implications:

$$\text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(i)} \Rightarrow \text{(ii)}.$$

(ii) $\Rightarrow$ *(iii)*. By Kuratowski's theorem G is planar, since K_5 contains both H_0 and H_1 as subgraphs, and $K_{3,3}$ has a subgraph H_1' isomorphic to

a subdivision of H_1. Any block B of G which is not an edge must be a cycle, since by well-known properties of blocks (see, for example [4]) B would otherwise have an edge common to two cycles, implying that G has a subgraph isomorphic to a subdivision of H_1, contrary to (ii). Thus G is a cactus. Furthermore, the cycles of G are vertex disjoint, since otherwise G would contain a subgraph isomorphic to a subdivision of H_0, again violating (ii).

(iii) $\Rightarrow$ *(iv)*. Let G be a cactus with (disjoint) cycles C_i, $i = 1, 2,..., n$. Then $\beta(C_i) = 1$, and since $\gamma_M(C_i) \leqslant [\beta(C_i)/2]$, as noted earlier for connected graphs in general, then $\gamma_M(C_i) = 0, i = 1, 2,..., n$. The equation $\gamma_M(G) = \sum_{i=1}^{n} \gamma_M(C_i) = 0$ now follows from Theorem 2.

(iv) $\Rightarrow$ *(i)*. This is immediate, since $0 \leqslant \gamma(G) \leqslant \gamma_M(G)$.

(i) $\Rightarrow$ *(ii)*. We establish the contrapositive of this implication.

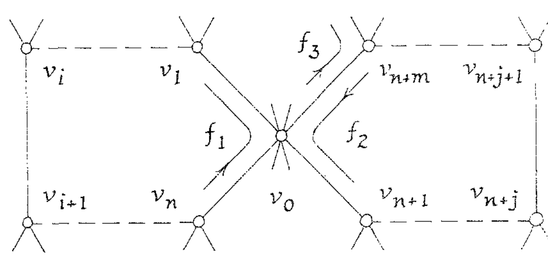

FIG. 2. The case H_0.

The case H_0. Assume that G contains a subgraph H_0' which is isomorphic to a subdivision of H_0. It is possible to choose a 2-cell imbedding T for G so that the vertex v_0 common to the two cycles of H_0' is in the boundary of at least three distinct faces (f_1, f_2, and f_3 in Fig. 2): we use Theorem 1 to obtain two faces containing v_0, each bounded by a cycle of H_0'. Since v_0 has degree at least four, it must appear in the boundary of at least one other face. Applying Theorem 3, we see that $\gamma(G) \neq \gamma_M(G)$.

The case H_1. Assume now that G contains a subgraph H_1' isomorphic to a subdivision of H_1. Again using Theorem 1, we select a 2-cell imbedding T for G as shown in Fig. 3. Here Theorem 1 guarantees that we can bound one face, $f_3 = (v_0 v_1 \cdots v_{i-1} v_i v_{i+1} \cdots v_n)$, with a cycle of H_1'. We consider two subcases:

Subcase 1. In the imbedding T for G, suppose that edge $v_0 v_{n+1}$ is in the boundary of two distinct faces, f_1 and f_2 (see Fig. 3). Then v_0 is in the

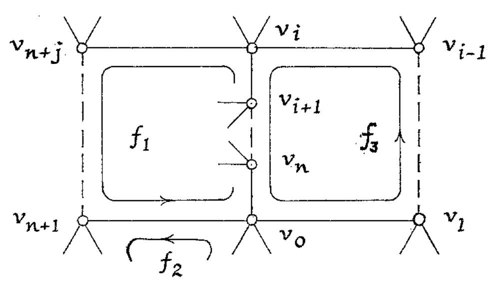

FIG. 3. The case H_1.

boundary of these two faces and is also in the boundary of the face f_3, so that $\gamma(G) \neq \gamma_M(G)$, by Theorem 3.

Subcase 2. Now suppose that directed edges (v_0, v_{n+1}) and (v_{n+1}, v_0) occur in the boundary of the same face of T:

$$(v_{n+1}v_0v_n \cdots v_0v_{n+1} \cdots v_{i+1}v_iv_{n+j} \cdots).$$

Then we have, in particular, $P_{v_0(T)} : (v_{n+1}, v_n, v_1, ...)$ and $P_{v_i(T)}$: $(v_{i-1}, v_{i+1}, v_{n+j}, ...)$; see Fig. 3, where $f_1 = f_2$ for this subcase. We modify the imbedding T for G to T' for G as follows:

$$P_{v(T')} = \begin{cases} (v_n, v_{n+1}, v_1, ...), & \text{if } v = v_0; \\ (v_{i-1}, v_{n+j}, v_{i+1}, ...), & \text{if } v = v_i; \\ P_{v(T)}, \text{ otherwise.} \end{cases}$$

Then v_0 is in the boundary of at least the following three faces of T':

$$(v_{n+1}v_0v_1 \cdots v_{i-1}v_iv_{n+j} \cdots),$$

$$(v_0v_{n+1} \cdots v_{i+1}v_i \cdots v_iv_{i+1} \cdots v_n),$$

and

$$(v_0v_n \cdots).$$

The latter two faces are distinct, since the removal of edge v_0v_n from T' leaves a 2-cell imbedding of $G - v_nv_0$ with v_0 and v_n in the boundary of a common face. Adding v_0v_n within that face gives the two distinct faces described above, for a total of at least three distinct faces containing v_0. Now applying Theorem 3 again, we see that $\gamma(G) \neq \gamma_M(G)$. This completes the proof.

4. Applications

We use Theorem 4 to provide short proofs for two theorems of Duke [2, Theorems 4.1 and 4.3], relating the regional number $d(G)$ and the Betti number $\beta(G)$ of a connected graph G. We combine these two theorems into the following:

THEOREM 5. *Let G be a connected graph. Then*:

(i) $d(G) = 1$ *if and only if* $\beta(G) = 0$;

(ii) $d(G) = 2$ *if and only if* $\beta(G) = 1$.

Proof. (i) if $\beta(G) = 0$, then $\gamma_M(G) \leqslant [\beta(G)/2]$ implies $\gamma_M(G) = 0$, so that $\gamma(G) = 0$. From $d(G) = 1 + \beta(G) - 2\gamma(G)$, $d(G) = 1$. Conversely, if $d(G) = 1$, then $\beta(G) = 2\gamma(G)$, and $\gamma_M(G) \leqslant \beta(G)/2 = \gamma(G)$. Thus $\gamma_M(G) \leqslant \gamma(G)$, so that $\gamma_M(G) = \gamma(G) = 0$, by Theorem 4, and $\beta(G) = 0$.

(ii) if $\beta(G) = 1$, then $\gamma_M(G) \leqslant [\beta(G)/2]$ implies $\gamma_M(G) = 0$, so again $\gamma(G) = 0$. From $d(G) = 1 + \beta(G) - 2\gamma(G)$, $d(G) = 2$. Conversely, if $d(G) = 2$, then $\beta(G) = 2\gamma(G) + 1$, and $\gamma_M(G) \leqslant \beta(G)/2 = \gamma(G) + 1/2$. Thus $\gamma_M(G) \leqslant \gamma(G)$, so again $\gamma_M(G) = \gamma(G) = 0$, by Theorem 4, and $\beta(G) = 1$.

It is evident that, when (i) is satisfied, G is a tree; when (ii) holds, G is a planar graph having exactly one cycle.

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