## Mixed needlets

Daryl Geller ${ }^{\text {a,*, }}$, Domenico Marinucci ${ }^{\text {b }}$<br>${ }^{\text {a }}$ Department of Mathematics, Stony Brook University, Stony Brook, NY 11794-3651, United States<br>${ }^{\mathrm{b}}$ Department of Mathematics, University of Rome Tor Vergata, Italy

## A R T I CLE IN F O

## Article history:

Received 21 June 2010
Available online 12 October 2010
Submitted by M.M. Peloso

## Keywords:

Spherical harmonics
Line bundles
Spin needlets
Mixed needlets
Besov spaces
Cosmic microwave background radiation Polarization
Weak gravitational lensing


#### Abstract

The construction of needlet-type wavelets on sections of the spin line bundles over the sphere has been recently addressed in Geller and Marinucci (2008) [14], and Geller et al. $(2008,2009)[12,13]$. Here we focus on an alternative proposal for needlets on this spin line bundle, in which needlet coefficients arise from the usual, rather than the spin, spherical harmonics, as in the previous constructions. We label this system mixed needlets and investigate in full their properties, including localization, the exact tight frame characterization, reconstruction formula, decomposition of functional spaces, and asymptotic uncorrelation in the stochastic case. We outline astrophysical applications.


(C) 2010 Elsevier Inc. All rights reserved.

## 1. Introduction and motivations

A growing literature has been recently concerned with (random or deterministic) spin functions, i.e. sections of spin fiber bundles over the sphere (we defer a rigorous definition and more formal discussion to the next section). Actually the interest of the physical literature on such objects goes back for several decades, the seminal contributions going back to [42] and [20]. In these papers, spin spherical harmonics were introduced in the language of physicists and used for the analysis of gravitational radiation. Much more recently, spin functions have found a crucial role in the analysis of cosmological observations, in particular in connection with so-called Cosmic Microwave Background polarization data. Polarization is a peculiar imprint characterizing the electromagnetic radiation which was emitted at the age of recombination, some 13.7 billion years ago and in the immediate adjacency of the Big Bang; as such, it delivers information on a number of extremely important topics in the current landscape of physical and cosmological research, for instance on the existence of primordial gravitation waves, on the reionization era and on primordial non-Gaussianity. The literature on these issues is vast; we mention $[7,9,48,24]$ for an introduction, while a massive amount of observations are currently being collected by satellite experiments by NASA and ESA (WMAP and Planck, respectively). Spin fiber bundles will definitely be of the greatest interest for other areas of physical research in the next decade, for instance in the analysis of gravitational weak lensing on the images of galaxies [6]. We expect random sections of spherical fiber bundles to enjoy a growing relevance also outside the physical sciences, for instance in medical imaging.

[^0]Despite such a rich environment from the physical sciences, the interest in the mathematical literature on spin bundles on the sphere has grown only very recently. In particular, some efforts have been entertained to extend to sections of spin fiber bundles the construction of spherical wavelets of a needlet type. Scalar needlets were introduced for the sphere by [40,41]; the general case of compact Riemannian manifolds has been presented by [15-17]. The analysis of the asymptotic properties of scalar needlets in random circumstances has been started by [2,3], see also [30,1,38,31,4,26] for further developments and $[43,36,44,11,8,22,46,47,21]$, among others, for applications to cosmological data. Spin needlets were introduced by [14], in that paper, localization and uncorrelation properties are also addressed. The general case where the wavelet system need not be compactly supported in harmonic space is discussed by [18]; stochastic properties and related statistical procedures are investigated in [13], while applications to a CMB framework are provided by [12]. In [1], it is proved that spin needlets actually make up a tight frame system, with the same cubature points as the scalar case, and characterizations of Besov spaces on spin fiber bundles are discussed. Further results on the stochastic foundations of spin random fields are provided by [33] and [32].

The purpose of this paper is to consider an alternative construction for wavelets on spin fiber bundles. In particular, we focus on the case where the resulting needlet coefficients are (complex-valued) scalars, rather than spin quantities as in the spin needlet proposal. We label this system mixed needlets and investigate in full their properties, including localization, the exact tight frame characterization, reconstruction formula, decomposition of functional spaces, and asymptotic uncorrelation in the stochastic case; we outline also astrophysical applications. Concerning the latter, we stress in particular that mixed needlets allow for possibilities that were ruled out for the pure spin construction, such as the estimation of cross power spectra between scalar and spin components.

The plan of the paper is as follows: in Section 2 we review some background material on spin fiber bundles, and we discuss some equivalent definitions which have been provided in the literature. In Section 3 we explain the construction of spin needlets, while Section 4 is devoted to the investigation of mixed spin needlets. Section 5 discusses characterizations of Besov spaces, establishing the equivalence of spin and mixed spin needlets in this regard and investigating on the properties of functional spaces for underlying scalar functions. Section 6 is devoted to directions for future research, with particular reference to statistical applications.

## 2. Spin functions

### 2.1. Some definitions

We begin by summarizing some definitions and basic facts about spin functions. For more details, the reader may consult our article [14].

We let $\mathbf{N}$ be the north pole of the unit sphere $\mathbb{S}^{2}$, namely $(0,0,1)$, and we let $\mathbf{S}$ be the south pole, $(0,0,-1)$. Let

$$
U_{I}=\mathbb{S}^{2} \backslash\{\mathbf{N}, \mathbf{S}\}
$$

If $R \in S O$ (3), we define

$$
U_{R}=R U_{I}
$$

On $U_{I}$ we use standard spherical coordinates $(\vartheta, \varphi)(0<\vartheta<\pi,-\pi \leqslant \varphi<\pi)$, and analogously, on any $U_{R}$ we use coordinates $\left(\vartheta_{R}, \varphi_{R}\right)$ obtained by rotation of the coordinate system on $U_{I}$.

At each point $p$ of $U_{R}$ we let $\rho_{R}(p)$ be the unit tangent vector at $p$ which is tangent to the circle $\vartheta_{R}=$ constant, pointing in the direction of increasing $\varphi_{R}$. (This is well defined since $R \mathbf{N}, R \mathbf{S} \notin U_{R}$.) We think of $\rho_{R}(p)$ as being our "reference direction" at $p$, relative to the chart $U_{R}$.

If $p \in U_{R_{1}} \cap U_{R_{2}}$, we let
$\psi_{p R_{2} R_{1}}$ be the oriented angle from the reference direction $\rho_{R_{1}}(p)$ to $\rho_{R_{2}}(p)$.
(See [14] for a precise explanation of which is the oriented angle. For example, at $\mathbf{S}$, the oriented angle from $\vec{i}$ to $\vec{j}$ is $\pi / 2$; at $\mathbf{N}$, the oriented angle from $\vec{j}$ to $\vec{i}$ is $\pi / 2$.)

Since $R$ is conformal, the angle $\psi_{p R_{2} R_{1}}$ would be clearly be the same if we had instead chosen $\rho_{R}(p)$ to point in the direction of increasing $\vartheta_{R}$, for instance. Thus $\psi_{p R I}$ measures "the angle by which the tangent plane at $p$ is rotated if the coordinates are rotated by $R$ ".

Now say $\Omega \subseteq \mathbb{S}^{2}$ is open. Let

$$
\begin{aligned}
C_{s}^{\infty}(\Omega)=\left\{F=\left(F_{R}\right)_{R \in S O(3)}:\right. & \text { all } F_{R} \in C^{\infty}\left(U_{R} \cap \Omega\right), \text { and for all } R_{1}, R_{2} \in S O(3) \text { and all } p \in U_{R_{1}} \cap U_{R_{2}} \cap \Omega, \\
& \left.F_{R_{2}}(p)=e^{i s \psi} F_{R_{1}}(p), \text { where } \psi=\psi_{p R_{2} R_{1}}\right\} .
\end{aligned}
$$

(Heuristically, a physicist would think of $F_{R}$ as $F_{I}$ "looked at after the coordinates have been rotated by $R$ "; at $p$, it has been multiplied by $e^{i s \psi}$, which is how physicists think of spin quantities behaving after rotation.) Equivalently, $C_{S}^{\infty}(\Omega)$ consists of the smooth sections over $\Omega$ of the line bundle with transition functions $e^{i s \psi_{p R_{2} R_{1}}}$ from $U_{R_{1}}$ to $U_{R_{2}}$.

Similarly we can define $L_{s}^{2}(\Omega)$ (the $F_{R}$ need to be in $L^{2}$ ). There is a well-defined inner product on $L_{S}^{2}(\Omega)$, given by $\langle F, G\rangle=\left\langle F_{R}, G_{R}\right\rangle$; clearly this definition is independent of choice of $R$.

There is a unitary action of $S O(3)$ on $L_{s}^{2}\left(\mathbb{S}^{2}\right)$, given by $F \rightarrow F^{R}$, which is determined by the equation

$$
\begin{equation*}
\left(F^{R}\right)_{I}(p)=F_{R}(R p) \tag{1}
\end{equation*}
$$

We think of $F^{R}$ as a "rotate" of $F$. Thus we have two important relations: if we "rotate coordinates", we have

$$
F_{R_{2}}(p)=e^{i s \psi} F_{R_{1}}(p)
$$

while if we "rotate $F$ ", we have

$$
\left(F^{R}\right)_{I}(p)=F_{R}(R p)
$$

Physicists would say that spin $s$ quantities need to be multiplied by a factor $e^{i s \psi}$ when rotated.

### 2.2. Twisted bundles

Although we will not make use of it in this article, it may be useful to recast the previous discussion in a slightly different form, as suggested by [33]. View $S O$ (2) as a closed subgroup of $S O(3)$, with elements $k \in S O(2)$. This an Abelian subgroup, and assuming $k$ is parametrized by the Euler angle $\gamma \in[0,2 \pi]$ the irreducible representations of $S O(2)$ are well known to be one-dimensional and given by $W_{s}(k): \mathbb{C} \rightarrow \mathbb{C}, W_{s}(k) x=\exp (i s \gamma) x$, where $s \in \mathbb{N}$. Let $g \in S O$ (3), and consider the action

$$
k:\{S O(3) \times \mathbb{C}\} \rightarrow\left\{\mathbb{S}^{2}, \mathbb{C}\right\}, \quad k(g, x)=(g k, \exp (i s \gamma) x)
$$

We denote by $\mathcal{E}_{s}$ the quotient space of orbits of the above action; that is, two elements ( $g_{1}, x_{1}$ ) and ( $g_{2}, x_{2}$ ) belong to the same equivalence class if there exists $k \in S O(2)$ such that $\left(g_{2}, x_{2}\right)=\left(g_{1} k, W_{s}(k) x_{1}\right)$. For $s=0$, this is clearly isomorphic to $\left\{\mathbb{S}^{2}, \mathbb{C}\right\}$, i.e. the space of complex-valued functions on the sphere. For $s \neq 0$, we obtain indeed the same spin fiber bundle we defined before $\left\{\mathcal{E}_{s}, \pi, \mathbb{S}^{2}\right\}$, by taking the projection

$$
\pi: \mathcal{E}_{s} \rightarrow \mathbb{S}^{2}, \quad \pi(g, x)=g S O(2)
$$

where we denoted as usual $g S O(2)$ the equivalence class $\{g k, k \in S O(2)\}$; for $g \in S O(3)$, this is isomorphic to $S O(3) /$ $S O(2) \simeq \mathbb{S}^{2}$.

### 2.3. Spin spherical harmonics

Next we need some facts about the spin spherical harmonics; again the reader may consult [14] for further details.
Let $f \in L^{2}\left(\mathbb{S}^{2}\right)$ be the space of square-integrable functions on the sphere; it is a well-known fact that the following spectral representation holds, in the $L^{2}$ sense:

$$
f(x)=\sum_{l m} a_{l m} Y_{l m}(x), \quad a_{l m}=\int_{\mathbb{S}^{2}} f(x) \bar{Y}_{l m}(x) d x
$$

or more formally

$$
L^{2}\left(\mathbb{S}^{2}\right)=\bigoplus_{l=0}^{\infty} \mathcal{H}_{l}
$$

where $\left\{\mathcal{H}_{l}\right\}$ are the linear spaces spanned by the standard spherical harmonics $\left\{Y_{l m}:-l \leqslant m \leqslant l\right\}$, which are certain eigenfunctions of the (positive) spherical Laplacian

$$
\Delta_{\mathbb{S}^{2}} Y_{l m}=l(l+1) Y_{l m}
$$

Explicit expressions for the $Y_{l m}$ may be found in [14].
We next define the spin-raising operator $\varnothing$ and the spin-lowering operator $\bar{\partial}$.
б, $\bar{\delta}$ are maps which take smooth spin $s$ functions to smooth spin $s+1$ (resp. $s-1$ ) functions, and which commute with the actions of the rotation group (1). On a smooth spin $s$ function $F$, we have $(\check{\partial} F)_{R}=\partial_{s R} F_{R},(\bar{\partial} F)_{R}=\bar{\partial}_{s R} F_{R}$, where

$$
\begin{align*}
& \partial_{s R} F_{R}(\vartheta, \varphi)=-\left(\sin \vartheta_{R}\right)^{s}\left[\frac{\partial}{\partial \vartheta_{R}}+\frac{i}{\sin \vartheta_{R}} \frac{\partial}{\partial \varphi_{R}}\right]\left(\sin \vartheta_{R}\right)^{-s} F_{R}\left(\vartheta_{R}, \varphi_{R}\right),  \tag{2}\\
& \bar{\partial}_{s R} F_{R}\left(\vartheta_{R}, \varphi_{R}\right)=-\left(\sin \vartheta_{R}\right)^{-s}\left[\frac{\partial}{\partial \vartheta_{R}}-\frac{i}{\sin \vartheta_{R}} \frac{\partial}{\partial \varphi_{R}}\right]\left(\sin \vartheta_{R}\right)^{s} F_{R}\left(\vartheta_{R}, \varphi_{R}\right) . \tag{3}
\end{align*}
$$

The spin $s$ spherical harmonics, defined for $l \geqslant|s|$, are then given by

$$
\begin{aligned}
& Y_{l m, s}=\left\{\frac{(l-s)!}{(l+s)!}\right\}^{1 / 2}(\text { ( })^{s} Y_{l m}, \quad \text { for } s>0, \\
& Y_{l m, s}=\left\{\frac{(l+s)!}{(l-s)!}\right\}^{1 / 2}(-\bar{\delta})^{-s} Y_{l m}, \quad \text { for } s<0,
\end{aligned}
$$

so that, if $l \geqslant|s|$,

$$
\begin{align*}
& \partial Y_{l m, s}=[(l-s)(l+s+1)]^{1 / 2} Y_{l m, s+1},  \tag{4}\\
& \bar{\jmath} Y_{l m, s}=-[(l+s)(l-s+1)]^{1 / 2} Y_{l m, s-1} \tag{5}
\end{align*}
$$

see also [51] and [39]. The $Y_{l m, s}$, for $l \geqslant|s|,-l \leqslant m \leqslant l$, form an orthonormal basis for $L_{s}^{2}$. In addition, $Y_{l m, s}$ is an eigenfunction of the (positive) spin spherical Laplacian

$$
\Delta_{s}= \begin{cases}-\bar{\partial} \partial & \text { if } s \geqslant 0  \tag{6}\\ -ð \bar{ð} & \text { if } s<0\end{cases}
$$

acting on smooth spin functions, with eigenvalue

$$
\begin{equation*}
e_{l s}=(l-|s|)(l+|s|+1) \tag{7}
\end{equation*}
$$

If $s=0$, then $\Delta_{s}$ is just the usual (positive) spherical Laplacian. The formal adjoint of $\varnothing$ (mapping smooth spin $s$ functions on $\mathbb{S}^{2}$ to smooth spin $s+1$ functions) is $-\overline{\bar{\delta}}$, so that $\Delta_{s}$ is formally self-adjoint on smooth spin $s$ functions.

For $l \geqslant|s|$, we let $\mathcal{H}_{l s}$ denote the linear span of the $Y_{l m, s}$ for $-l \leqslant m \leqslant l$. $\mathcal{H}_{l s}$ is the eigenspace of $\Delta_{s}$ for the eigenvalue $e_{l s}$, and the direct sum of the $\mathcal{H}_{l s}$ (for $l \geqslant|s|,-l \leqslant m \leqslant l$ ) is all of $L_{s}^{2}\left(\mathbb{S}^{2}\right)$.

We make some elementary observations about the eigenvalues $e_{l s}$.
If $l \geqslant|s|$,

$$
\begin{equation*}
e_{l s}=l(l+1)-|s|(|s|+1) \leqslant l(l+1)=e_{l 0} \tag{8}
\end{equation*}
$$

and for any $l, l^{\prime}$ (always nonnegative),

$$
\begin{equation*}
\sqrt{e_{l 0}}+\sqrt{e_{l^{\prime}, 0}}<\sqrt{e_{l+l^{\prime}+1,0}} \tag{9}
\end{equation*}
$$

and for any $l \geqslant 0$ and any $s$,

$$
\begin{equation*}
e_{l 0} \leqslant e_{l+|s|, s} \tag{10}
\end{equation*}
$$

Here (8) and (10) are trivial. To prove (9), write $e_{l+l^{\prime}+1,0}=e_{l 0}+e_{l^{\prime}, 0}+2(l+1)\left(l^{\prime}+1\right)$, then square both sides of (9), to see that the inequality is equivalent to

$$
\sqrt{l l^{\prime}(l+1)\left(l^{\prime}+1\right)}<(l+1)\left(l^{\prime}+1\right)
$$

which is evident.
Note that, for a spin $s$ function $F_{s}$, we may speak unambiguously of the number $\left|F_{s}(x)\right|$, for any $x \in \mathbb{S}^{2}$. We now prove the following inequality, by imitating the familiar method of proof in the case $s=0$ :

Lemma 1. Say $l \geqslant|s|$, and that $Y \in \mathcal{H}_{l s}$. Then

$$
\begin{equation*}
\|Y\|_{\infty} \leqslant \sqrt{\frac{2 l+1}{4 \pi}}\|Y\|_{2} \tag{11}
\end{equation*}
$$

In particular, for any $m$,

$$
\begin{equation*}
\left\|Y_{l m, s}\right\|_{\infty} \leqslant \sqrt{\frac{2 l+1}{4 \pi}} \tag{12}
\end{equation*}
$$

Proof. Let $s^{+}=\max (s, 0)$. In Section 5 of [14] we showed that $Z_{l, s}$, defined by

$$
\begin{equation*}
Z_{l, s}=(-1)^{s^{+}}\left[\frac{2 l+1}{4 \pi}\right]^{1 / 2} Y_{l,-s, s} \tag{13}
\end{equation*}
$$

which we called the $s$-zonal harmonic for $\mathcal{H}_{l s}$, has quite similar properties to the usual zonal harmonic (for $\mathcal{H}_{10}$ ). Those properties, and (1) imply that the following argument is valid, just as in the case $s=0$ : Say $p \in \mathbb{S}^{2}$, and choose $R \in S O$ (3) with $R \mathbf{N}=p$. Then

$$
|Y(p)|=\left|Y^{R}(N)\right|=\left|\left\langle Y^{R}, Z_{l, s}\right\rangle\right| \leqslant\left\|Y^{R}\right\|_{2}\left\|Z_{l, s}\right\|_{2}=\|Y\|_{2}\left\langle Z_{l, s}, Z_{l, s}\right\rangle^{1 / 2}=\|Y\|_{2}\left|Z_{l, s}(\mathbf{N})\right|^{1 / 2}
$$

But

$$
\left|Z_{l, s}(\mathbf{N})\right|=\left|\sum_{m=-l}^{l} Y_{l m, s}(\mathbf{N}) \overline{Y_{l m, s}(\mathbf{N})}\right|=\frac{2 l+1}{4 \pi},
$$

in fact, for any $x \in \mathbb{S}^{2}$, and any $R^{\prime} \in S O$ (3), one has

$$
\begin{equation*}
\sum_{m=-l}^{l} Y_{l m, s R^{\prime}}(x) \overline{Y_{l m, s R^{\prime}}(x)}=\frac{2 l+1}{4 \pi} \tag{14}
\end{equation*}
$$

This completes the proof.
For $L \geqslant|s|$, let

$$
\begin{equation*}
V_{L, s}=\bigoplus_{l=|s|}^{L} \mathcal{H}_{l s} \tag{15}
\end{equation*}
$$

We note the following Bernstein-type lemma, adapted from [19].
Lemma 2. A smooth spin $s$ function $F$ is in $V_{L, s}$ if and only if there exist $A>0$ and $B<e_{L+1, s}$ such that for every nonnegative integer $N$,

$$
\begin{equation*}
\left\|\left(\Delta_{S}\right)^{N} F\right\|_{2} \leqslant A B^{N} \tag{16}
\end{equation*}
$$

Proof. If $F \in V_{L, s}$, we surely have (16), with $A=\|F\|_{2}, B=e_{L, s}$, by the orthogonality of the eigenspaces $\mathcal{H}_{l s}$.
For the converse, say we have (16). Suppose $l \geqslant L+1$, and $Y \in \mathcal{H}_{l, s}$; it suffices to show that $\langle F, Y\rangle=0$. But, since $\Delta_{s}$ is formally self-adjoint, for any $N$ we have

$$
|\langle F, Y\rangle|=e_{l s}^{-N}\left|\left\langle F, \Delta_{s}^{N} Y\right\rangle\right|=e_{l s}^{-N}\left|\left\langle\Delta_{s}^{N} F, Y\right\rangle\right| \leqslant A\left(\frac{B}{e_{l s}}\right)^{N}\|Y\|_{2}
$$

Since $B<e_{L+1, s} \leqslant e_{l s}$, this yields $\langle F, Y\rangle=0$ upon letting $N$ go to infinity. This completes the proof.
From this we find the following important product property. (The case $r=-s$ was first proved in [1] by developing the ideas of the subsection which follows. Here instead we adapt arguments from [19].)

## Lemma 3.

$$
V_{K, r} V_{L, s} \subseteq V_{K+L+|r+s|, r+s}
$$

Proof. First note that the product of a smooth spin $r$ function with a smooth spin $s$ function is a smooth spin $r+s$ function. Note next that if $F, G$ are respectively smooth spin $p$ and spin $q$ functions, then

$$
\begin{equation*}
\partial(F G)=(ð F) G+F(\partial G), \tag{17}
\end{equation*}
$$

similarly for $\bar{\delta}$ in place of $\partial$. (17) follows at once from (2), once we note that for any $p, R$, as differential operators

$$
\partial_{p R}=\partial_{0 R}+p \cot \vartheta_{R},
$$

and similarly for $q$ or $p+q$ in place of $p$.
To prove the lemma, it suffices to show that if $|r| \leqslant k \leqslant K$ and $|s| \leqslant l \leqslant L$, and if $F=Y_{k \mu, r}$ and $G=Y_{l m, s}$ for some $\mu, m$, then $F G \in V_{k+l+|r+s|, r+s}$.

Iterating (17) and the companion equation for $\overline{\bar{\delta}}$, we find that we can write

$$
\begin{equation*}
\left(-\Delta_{S}\right)^{N}(F G)=\sum_{j=0}^{2 N} T_{j} \tag{18}
\end{equation*}
$$

where
each $T_{j}$ is a sum of $\binom{2 N}{j}$ terms of the form $(D F)\left(D^{\prime} G\right)$, where $D$ (resp. $D^{\prime}$ ) is a $j$-fold (resp. $(2 N-j)$-fold) product of $\bar{\delta}$ 's and $\bar{\varnothing}$ 's, in some order.

Note that the constants appearing on the right sides of (4) and (5) are equal to $\pm \sqrt{e_{l, s}}$ or $\pm \sqrt{e_{l,-s}}$. Thus, by (8), if $D$ is as above,

$$
D F=0 \quad \text { or } \quad b Y_{k \mu, r^{\prime}} \quad \text { for some } r^{\prime}, \text { where }|b| \leqslant e_{k 0}^{j / 2}
$$

Thus, notation as above, by Lemma 1 we have

$$
\left\|(D F)\left(D^{\prime} G\right)\right\|_{2} \leqslant\|D F\|_{2}\left\|D^{\prime} G\right\|_{\infty} \leqslant \sqrt{\frac{2 l+1}{4 \pi}} e_{k 0}^{j / 2} e_{l 0}^{(2 N-j) / 2}
$$

so that, by the binomial theorem,

$$
\left\|\Delta_{s}^{N}(F G)\right\|_{2} \leqslant \sqrt{\frac{2 l+1}{4 \pi}} \sum_{j=0}^{2 N}\binom{2 N}{j} e_{k 0}^{j / 2} e_{l 0}^{(2 N-j) / 2}=\sqrt{\frac{2 l+1}{4 \pi}} B_{1}^{2 N}
$$

where

$$
B_{1}=\sqrt{e_{k 0}}+\sqrt{e_{l 0}}<\sqrt{e_{k+l+1,0}}
$$

by (9). Set $B=B_{1}^{2}$. By (10), $B<e_{k+l+|r+s|+1, r+s}$. By Lemma 2, we see that $F G \in V_{k+l+|r+s|, r+s}$, as desired.

### 2.4. Connection with Wigner D matrices

Next, we shall explain the connection of spin spherical harmonics with Wigner D matrices. This connection provides an alternative point of view, but it is not necessary for the rest of the article. For further details on this connection, the reader may consult [13] and [33].

It is well known that the elements $D_{m 0}^{l}, m=-l, \ldots, l$, of Wigner's $D$ matrices are proportional to the standard spherical harmonics $Y_{l m}$. It turns out that this equivalence holds in much greater generality, in fact one has that (compare [33] for a discussion of phase conventions)

$$
\begin{equation*}
Y_{l m, s}(\vartheta, \varphi)=(-1)^{m^{+}} \sqrt{\frac{2 l+1}{4 \pi}} \overline{D_{m,-s}^{l}}(\varphi, \vartheta, 0) \tag{19}
\end{equation*}
$$

Here in place of $Y_{l m, s}, \varphi, \vartheta$, we should have written $Y_{l m, s R}, \varphi_{R}, \vartheta_{R}$ throughout, but we drop the reference to the choice of chart for ease of notation whenever this can be done without the risk of confusion.

Many of the properties of spin spherical harmonics follow easily from their proportionality to elements of Wigner's $D$ matrices. Indeed, for instance, their orthonormality

$$
\int_{\mathbb{S}^{2}} Y_{l m, s}(p) \bar{Y}_{l^{\prime} m^{\prime}, s}(p) d p=\int_{0}^{2 \pi} \int_{0}^{\pi} Y_{l m, s}(\vartheta, \varphi) \bar{Y}_{l^{\prime} m^{\prime}}(\vartheta, \varphi) \sin \vartheta d \vartheta d \varphi=\delta_{l}^{l^{\prime}} \delta_{m}^{m^{\prime}}
$$

is immediate. Also, viewing spin-spherical harmonics as functions on the group $S O(3)$ (i.e. identifying $p=(\vartheta, \varphi)$ as the corresponding rotation by means of Euler angles), and using (19) and the group addition properties we obtain easily

$$
\sum_{m=-l}^{l} Y_{l m, s}(p) \overline{Y_{l m, s}\left(p^{\prime}\right)}=\frac{2 l+1}{4 \pi} \sum_{m} \overline{D_{m,-s}^{l}}(\varphi, \vartheta, 0) D_{m,-s}^{l}\left(\varphi^{\prime}, \vartheta^{\prime}, 0\right)=\frac{2 l+1}{4 \pi} \overline{D_{s,-s}^{l}}\left(\psi\left(p, p^{\prime}\right)\right),
$$

where $\psi\left(p, p^{\prime}\right)$ denotes the composition of the two rotations (explicit formulae can be found in [49]). In the special case $p=p^{\prime}$, we recover (14).

### 2.5. E and $M$ modes

For a smooth spin function $F$ on $\mathbb{S}^{2}$, we have the expansion

$$
\begin{equation*}
F=\sum_{l} \sum_{m} a_{l m, s} Y_{l m, s} \tag{20}
\end{equation*}
$$

with rapid decay of the $a_{l m, s}$ in $l$. From (20), a further, extremely important characterization of spin functions was first introduced by [42], see also [10] and [14] for a more mathematically oriented treatment. In particular, it can be shown that there exists a scalar complex-valued function

$$
\begin{equation*}
g(\vartheta, \varphi)=\mathfrak{R}\{g\}+i \Im\{g\} \tag{21}
\end{equation*}
$$

such that,

$$
\begin{equation*}
F_{s}=F^{E}+i F^{M}=\sum_{l m} a_{l m ; E} Y_{l m, s}+i \sum_{l m} a_{l m ; M} Y_{l m, s} \tag{22}
\end{equation*}
$$

where

$$
F^{E}=(ð)^{S} \mathfrak{R}\{g\}, \quad F^{M}=(ð)^{S} \Im\{g\} .
$$

Note that $a_{l m, s}=a_{l m ; E}+i a_{l m ; M}$, where $a_{l m ; E}=\bar{a}_{l,-m ; E}, a_{l m ; M}=\bar{a}_{l,-m ; M}$. It is also readily seen that

$$
\begin{aligned}
a_{l m, s}+\bar{a}_{l,-m, s} & =a_{l m ; E}+i a_{l m ; M}+a_{l m ; E}-i a_{l m ; M}=2 a_{l m ; E} \\
a_{l m, s}-\bar{a}_{l,-m, s} & =a_{l m ; E}+i a_{l m ; M}-a_{l m ; E}+i a_{l m ; M}=2 i a_{l m ; M}
\end{aligned}
$$

In the cosmological literature, $\left\{a_{l m ; E}\right\}$ and $\left\{a_{l m ; M}\right\}$ are labelled the $E$ and $M$ modes (or the electric and magnetic components) of CMB polarization.

## 3. Spin needlets

We now recall the construction of spin needlets, see $[14,12,18,13,1]$ for further details and discussions. Fix a "dilation parameter" $B>0 ; B$ is often chosen to be 2 , but it is sometimes useful to let it take other values. Let $\phi$ be a $C^{\infty}$ function on $\mathbb{R}$, symmetric and decreasing on $\mathbb{R}^{+}$, supported in $|\xi| \leqslant 1$, such that $0 \leqslant \phi(\xi) \leqslant 1$ and $\phi(\xi)=1$ if $|\xi| \leqslant \frac{1}{B}$. Let

$$
\begin{equation*}
b^{2}(\xi)=\phi\left(\frac{\xi}{B}\right)-\phi(\xi) \geqslant 0 \tag{23}
\end{equation*}
$$

Note that $\operatorname{supp} b \subseteq[1 / B, B]$, and that

$$
\begin{equation*}
\sum_{j} b^{2}\left(\frac{\xi}{B^{j}}\right)=\lim _{j \rightarrow \infty} \phi\left(\frac{\xi}{B^{j}}\right)=1 \quad \text { for all } \xi>0 \tag{24}
\end{equation*}
$$

Of course the sum on the left side of (24) is zero if $\xi=0$.
Let $T$ be a positive self-adjoint operator on a Hilbert space $\mathcal{H}$, and let $P$ be the projection onto the null space of $T$. It is a special case of Theorem 2.1(b) of [15], that we may use the spectral theorem to replace $\xi$ by $T$ in (24), obtaining that

$$
\begin{equation*}
\sum_{j} b\left(\frac{\sqrt{T}}{B^{j}}\right)^{2}=I-P \tag{25}
\end{equation*}
$$

where the sum converges strongly.
We take $\mathcal{H}=L_{s}^{2}, T=\Delta_{s}, P \mathcal{H}=\mathcal{H}_{|s|, s}$. Thus, if $F=\sum_{l} \sum_{m} a_{l m, s} Y_{l m, s} \in \mathcal{H}$, then

$$
\begin{equation*}
b\left(\frac{\sqrt{T}}{B^{j}}\right) F=\sum_{l} \sum_{m} b\left(\frac{\sqrt{e_{l s}}}{B^{j}}\right) a_{l m, s} Y_{l m, s} \tag{26}
\end{equation*}
$$

From this, it is easy to check (25) directly. Note also that $P \mathcal{H}$ is finite-dimensional (in fact, $2|s|+1$-dimensional). Note moreover that $b\left(\frac{\sqrt{T}}{B j}\right) \equiv 0$ for $j$ sufficiently negative, specifically if $B^{2 j}<e_{|s|+1, s}$, the smallest positive eigenvalue of $\Delta_{s}$.

For $x \in U_{R}$, let

$$
\begin{equation*}
\Lambda_{j}(x, y, R)=\sum_{l} \sum_{m} b\left(\frac{\sqrt{e_{l s}}}{B^{j}}\right) Y_{l m, s R}(x) \bar{Y}_{l m, s}(y) \tag{27}
\end{equation*}
$$

Then evidently, if $F(y)=\sum_{l} \sum_{m} a_{l m, s} Y_{l m, s}(y) \in \mathcal{H}$, we have

$$
\begin{equation*}
\left[b\left(\frac{\sqrt{T}}{B^{j}}\right) F\right]_{R}(x)=\sum_{l} \sum_{m} b\left(\frac{\sqrt{e_{l s}}}{B^{j}}\right) a_{l m, s} Y_{l m, s R}(x)=\int \Lambda_{j}(x, y, R) F(y) d y \tag{28}
\end{equation*}
$$

Here the integral is over $\mathbb{S}^{2}$. It is important to note that, in the notation of Section 2.3,

$$
\begin{equation*}
b\left(\frac{\sqrt{T}}{B^{j}}\right) F \in V_{L_{j s}, s}, \tag{29}
\end{equation*}
$$

where $L_{j s}$ (:= $L_{j}$ if $s$ is understood) is the largest integer with $e_{L_{j} s} \leqslant B^{2(j+1)}$. In particular

$$
\begin{equation*}
L_{j} \sim B^{j} \tag{30}
\end{equation*}
$$

as $j \rightarrow \infty$.
Now, take $F \in(I-P) \mathcal{H}$. Applying both sides of (25) to $F$, and take the inner product with $F$. We find

$$
\begin{equation*}
\|F\|_{L_{s}^{2}}^{2}=\sum_{j}\left\|b\left(\frac{\sqrt{T}}{B^{j}}\right) F\right\|_{L_{s}^{2}}^{2}=\sum_{j} \int\left|b\left(\frac{\sqrt{T}}{B^{j}}\right) F\right|^{2}(x) d x \tag{31}
\end{equation*}
$$

while, as long as $x \in U_{R}$,

$$
\begin{equation*}
\left|b\left(\frac{\sqrt{T}}{B^{j}}\right) F\right|^{2}(x)=\left|\int \Lambda_{j}(x, y, R) F(y) d y\right|^{2} \tag{32}
\end{equation*}
$$

By (29) and Lemma 3, $\left|b\left(\frac{\sqrt{T}}{B^{j}}\right) F\right|^{2} \in V_{2 L_{j}, 0}$. That is, it is the restriction to the sphere of an ordinary polynomial of degree at most $2 L_{j} \sim B^{j}$. Accordingly, as noted by [1] it is possible to follow the method used in [40] in the case $s=0$ : By familiar results for polynomials on the sphere, then, there is a constant $c>0$, such that for each $j$, there is a $c /\left(L_{j}+1\right)-$ net ${ }^{1}\left\{\xi_{j k}\right\}$ of points on the sphere, and cubature weights $\left\{\lambda_{j k}\right\} \sim\left(L_{j}+1\right)^{-2}$, such that for every polynomial $q$ of degree at most $2 L_{j}$,

$$
\begin{equation*}
\int q(x) d x=\sum_{k} \lambda_{j k} q\left(\xi_{j k}\right) \tag{33}
\end{equation*}
$$

Thus, for $F \in(I-P) \mathcal{H}$, and provided $\xi_{j k} \in R_{j k}$, we in fact have

$$
\begin{equation*}
\|F\|_{L_{s}^{2}}^{2}=\sum_{j} \sum_{k} \lambda_{j k}\left|\int \Lambda_{j}\left(\xi_{j k}, y, R_{j k}\right) F(y) d y\right|^{2} \tag{34}
\end{equation*}
$$

In other words, for $F \in(I-P) L_{s}^{2}$,

$$
\begin{equation*}
\|F\|_{L_{s}^{2}}^{2}=\sum_{j} \sum_{k}\left|\left\langle F, \psi_{j k, s}\right\rangle\right|^{2} \tag{35}
\end{equation*}
$$

where the spin needlets $\psi_{j k, s}$ are defined by

$$
\begin{equation*}
\psi_{j k, s}(y)=\sqrt{\lambda_{j k}} \bar{\Lambda}_{j}\left(\xi_{j k}, y, R_{j k}\right)=\sqrt{\lambda_{j k}} \sum_{l} b\left(\frac{\sqrt{e_{l s}}}{B^{j}}\right) \sum_{m} Y_{l m, s}(y) \bar{Y}_{l m, s R_{j k}}\left(\xi_{j k}\right) \tag{36}
\end{equation*}
$$

Since $e_{|s|, s}=0$, each $\psi_{j k, s} \in(I-P) L_{s}^{2}$. Consequently the $\left\{\psi_{j k, s}\right\}$ are a tight frame for $(I-P) L_{s}^{2}$.
For $F$ as above, we also define its spin needlet coefficients by

$$
\begin{equation*}
\beta_{j k, s}=\left\langle F, \psi_{j k, s}\right\rangle=\sqrt{\lambda_{j k}} \sum_{l} b\left(\frac{\sqrt{e_{l s}}}{B^{j}}\right) \sum_{m} a_{l m, s} Y_{l m, s R_{j k}}\left(\xi_{j k}\right) \tag{37}
\end{equation*}
$$

By general frame theory, if $F \in(I-P) L_{s}^{2}$, we have the reconstruction formula

$$
\begin{equation*}
F=\sum_{j} \sum_{k} \beta_{j k, s} \psi_{j k, s} \tag{38}
\end{equation*}
$$

Remarks. 1. The choice of $R_{j k}$ does not affect any of the terms on the right side of (35) or (38). For this reason, and for simplicity we will sometimes omit the $R_{j k}$ subscript in the formulas (36) and (37) for $\psi_{j k, s}$ and $\beta_{j k, s}$, when this can be done without causing confusion.
2. We are ignoring the finite-dimensional space $P L_{s}^{2}$. This is acceptable, because in astrophysical applications, the interest is in high frequencies.
3. One can use more general $b$, than those we used in (23), to construct spin wavelets on the sphere, as in [14,18]. This leads to nearly tight frames with other interesting properties. For instance, one can arrange for the support of the frame elements at scale $B^{-j}$ to be contained in a geodesic ball of radius $C B^{-j}$ (for some fixed $C$ ).
4. We will use the following notation and observations in Section 5. Let us set

$$
\begin{equation*}
Q_{j}=b\left(\frac{\sqrt{T}}{B^{j}}\right)^{2} \tag{39}
\end{equation*}
$$

[^1]Following the arguments of (32)-(35), but now without summing over $j$, we have that for $F \in L_{s}^{2}$,

$$
\begin{equation*}
\left\langle Q_{j} F, F\right\rangle=\sum_{k}\left|\left\langle F, \psi_{j k, s}\right\rangle\right|^{2} . \tag{40}
\end{equation*}
$$

After polarizing this identity, we see that for $F \in L_{s}^{2}$,

$$
\begin{equation*}
Q_{j} F=\sum_{k}\left\langle F, \psi_{j k, s}\right\rangle \psi_{j k, s} . \tag{41}
\end{equation*}
$$

In (36), $b\left(\frac{\sqrt{e_{l s}}}{B j}\right)=0$ unless $\frac{\sqrt{e_{l s}}}{B j} \in(1 / B, B)$. Thus, for all $j$,

$$
\begin{equation*}
\psi_{j k, s} \in V_{L_{j}} \ominus V_{L_{j-2}} . \tag{42}
\end{equation*}
$$

Accordingly,

$$
\begin{equation*}
Q_{j}: L_{s}^{2} \rightarrow V_{L_{j}} \ominus V_{L_{j-2}} \quad \text { and } \quad Q_{j} \equiv 0 \quad \text { on }\left[V_{L_{j}} \ominus V_{L_{j-2}}\right]^{\perp} . \tag{43}
\end{equation*}
$$

For any integer $N$, let

$$
\begin{equation*}
P_{N}=\sum_{j=-\infty}^{N} Q_{j} \tag{44}
\end{equation*}
$$

As we know, the sum in (44) is actually finite, and, by (25), $P_{N} \rightarrow I-P$ strongly as $N \rightarrow \infty$. By this, (42), and (43), we have

$$
\begin{equation*}
\psi_{j k, s}=\left(Q_{j-1}+Q_{j}+Q_{j+1}\right) \psi_{j k, s}, \tag{45}
\end{equation*}
$$

for all $j, k$.

## 4. Mixed needlets

We now present a construction of a different tight frame for spin functions, which we shall call mixed needlets.
For now we work with ordinary $L^{2}$ functions. Let $r$ be a fixed integer. Let $\mathcal{H}_{l}$ be the space of spherical harmonics on $\mathbb{S}^{2}$ of degree $l$, and let $\mathcal{H}^{r}=\bigoplus_{l=|r|}^{\infty} \mathcal{H}_{l} \subseteq L^{2}$. We consider (25) with $\mathcal{H}=\mathcal{H}^{r}$ and with $T$ replaced by $\tilde{T}=\Delta-|r|(|r|+1)$, so that $P \mathcal{H}^{r}=\mathcal{H}_{|r|}$. Since $e_{l r}=e_{l 0}-|r|(|r|+1)$, we have that $\tilde{T} Y_{l m}=e_{l r} Y_{l m}$ for $l \geqslant|r|$.

In this situation, (25) leads to evident modifications of (26)-(34). To modify the equations, we simply replace $T$ by $\tilde{T}$, and take $s=0$, the sole exceptions being that we write $e_{l r}, e_{L_{j r} r}$ instead of $e_{l 0}, e_{L_{j 0} 0}$. Of course we may disregard all the rotations $R$, etc. We find that $\left\{\eta_{j k}\right\}$ is a tight frame for $(I-P) \mathcal{H}$, where

$$
\begin{equation*}
\eta_{j k}(y)=\sqrt{\lambda_{j k}} \sum_{l=|r|}^{\infty} b\left(\frac{\sqrt{e_{l r}}}{B^{j}}\right) \sum_{m} Y_{l m}(y) \bar{Y}_{l m}\left(\xi_{j k}\right) . \tag{46}
\end{equation*}
$$

The last step in constructing mixed needlets is to change the variable name $r$ to $s$ in (46), then to note that $\mathcal{H}^{s}$ is unitarily equivalent to $L_{s}^{2}$, by means of the unitary equivalence $U$, where $U\left(Y_{l m}\right)=Y_{l m, s}$ (for $\left.l \geqslant|s|\right)$. Thus, if $P$ is now, once again, as in the previous section, so that $P L_{s}^{2}=\mathcal{H}_{|s|, s}$, then the mixed spin needlets (or mixed needlets for short) $\left\{\psi_{j k, s, \mathcal{M}}\right\}$ are a tight frame for $(I-P) L_{s}^{2}$, where

$$
\begin{equation*}
\psi_{j k, s \mathcal{M}}(y)=\sqrt{\lambda_{j k}} \sum_{l=|s|}^{\infty} b\left(\frac{\sqrt{e_{l s}}}{B^{j}}\right) \sum_{m} Y_{l m, s}(y) \bar{Y}_{l m}\left(\xi_{j k}\right) . \tag{47}
\end{equation*}
$$

For $F=\sum_{l} \sum_{m} a_{l m, s} Y_{l m, s} \in L_{s}^{2}$, we also define its spin needlet coefficients by

$$
\begin{equation*}
\beta_{j k, s \mathcal{M}}=\left\langle F, \psi_{j k, s \mathcal{M}}\right\rangle=\sqrt{\lambda_{j k}} \sum_{l} b\left(\frac{\sqrt{e_{l s}}}{B^{j}}\right) \sum_{m} a_{l m, s} Y_{l m}\left(\xi_{j k}\right) . \tag{48}
\end{equation*}
$$

By general frame theory, if $F \in(I-P) L_{s}^{2}$, we have the reconstruction formula

$$
\begin{equation*}
F=\sum_{j} \sum_{k} \beta_{j k, s \mathcal{M}} \psi_{j k, s \mathcal{M}} \tag{49}
\end{equation*}
$$

In this "mixed" situation, we set $Q_{j \mathcal{M}}=U b\left(\frac{\sqrt{\tilde{T}}}{B_{j}}\right)^{2} U^{-1}$. Then "mixed" analogues of (40)-(45) hold; it is only necessary to replace $Q_{j}$ by $Q_{j \mathcal{M}}$ and $\psi_{j k, s}$ by $\psi_{j k, s \mathcal{M}}$ in those equations.

But in fact we also have:

## Lemma 4.

$$
Q_{j \mathcal{M}}=Q_{j}
$$

Proof. It suffices to show that these bounded operators agree on the orthonormal basis elements $Y_{l m, s}$. But

$$
Q_{j \mathcal{M}} Y_{l m, s}=U b\left(\frac{\sqrt{\tilde{T}}}{B^{j}}\right)^{2} U^{-1} Y_{l m, s}=U b\left(\frac{\sqrt{\tilde{T}}}{B^{j}}\right)^{2} Y_{l m}=U b\left(\frac{\sqrt{e_{l s}}}{B^{j}}\right)^{2} Y_{l m}=b\left(\frac{\sqrt{e_{l s}}}{B^{j}}\right)^{2} Y_{l m, s}=Q_{j} Y_{l m, s}
$$

as desired.
In brief, the construction of spin needlets proceeds by applying the methods of (25)-(34) to $b\left(\frac{\sqrt{T}}{B^{j}}\right)^{2}$, while the construction of mixed needlets proceeds by applying the same methods to the unitarily equivalent operator $b\left(\frac{\sqrt{\tilde{T}}}{B^{j}}\right)^{2}$, then invoking the unitary equivalence. Of course this unitary equivalence is only effective on $L^{2}$, so there is no evident reason to think that there would be an effective theory in other function spaces for mixed needlets. However, the main point of this article is that mixed needlets do have nice mathematical properties beyond the $L_{s}^{2}$ theory, as well as useful astrophysical applications. In particular, as we shall show in the next section, they satisfy the usual needlet near-diagonal localization property, in the same sense as spin needlets were shown to in [14].

To understand better the meaning of mixed needlets and its relationship with the existing literature, we start from (22), and introduce the notation

$$
\begin{equation*}
f_{E}(x):=U^{-1} F^{E}=\sum_{l m} a_{l m ; E} Y_{l m}(x), \quad f_{M}(x):=U^{-1} F^{M}=\sum_{l m} a_{l m ; M} Y_{l m}(x) \tag{50}
\end{equation*}
$$

Clearly $f_{E}$ and $f_{M}$ are well-defined scalar functions which are uniquely identified from $F_{s}$; as recalled above, in the $s=2$ of interest for the physical literature they are labelled the electric and magnetic components of the spin field (in the physical literature, $f_{M}$ is rather written $f_{B}$, but we already devoted the letter $B$ for another purpose). Of course, it is possible to implement a standard (scalar) needlet construction on these spaces, enjoying the well-known properties of needlets (and indeed the same argument could be considered for other spherical wavelets). The interesting question to address is clearly what are the properties of such a procedure when viewed as acting on the original spin space $L_{s}$.

More precisely, a direct idea to implement a wavelet transform on a spin random field would be as follows. Start by to evaluating the spin transforms

$$
\int_{\mathbb{S}^{2}} F_{s}(x) \bar{Y}_{l m, s} d x=a_{l m, s}, \quad \int_{\mathbb{S}^{2}} F_{s}(x) \bar{Y}_{l m, s} d x=a_{l m, s}
$$

where

$$
a_{l m, s}=\frac{1}{2}\left\{a_{l m, s}+\bar{a}_{l,-m, s}\right\}+\frac{1}{2}\left\{a_{l m, s}-\bar{a}_{l,-m, s}\right\}=a_{l m ; E}+i a_{l m ; M}
$$

Note however that it is not true that $\operatorname{Re}\left(a_{l m ; s}\right)=a_{l m ; E}, \operatorname{Im}\left(a_{l m ; s}\right)=a_{l m ; M}$, indeed $a_{l m ; E}, a_{l m ; M}$ are complex-valued, and we have

$$
\begin{aligned}
& a_{l m ; E}=\frac{1}{2}\left\{a_{l m, s}+\bar{a}_{l,-m, s}\right\}=\frac{1}{2}\left\{a_{l m ; E}+i a_{l m ; M}+\bar{a}_{l,-m, E}-i \bar{a}_{l,-m, M}\right\} \\
& a_{l m ; M}=-\frac{i}{2}\left\{a_{l m, s}-\bar{a}_{l,-m, s}\right\}=-\frac{i}{2}\left\{a_{l m ; E}+i a_{l m ; M}-\bar{a}_{l,-m, E}+i a_{l m ; M}\right\}
\end{aligned}
$$

where we use the (involutive) property $a_{l m ; E}=\bar{a}_{l,-m ; E}, a_{l m ; M}=\bar{a}_{l,-m ; M}$. This property uniquely identifies the spherical coefficients $a_{l m ; E}, a_{l m ; M}$. Note that $a_{l m, s}$ is involutive if and only if the $M$ component is identically null, while if and only if the $E$ component vanishes $i a_{l m, s}$ is involutive.

It is then readily seen that

$$
\begin{align*}
\beta_{j k, \mathcal{M}} & :=\int_{\mathbb{S}^{2}} F_{s}(x) \bar{\psi}_{j k, s \mathcal{M}}(x) d x=\sqrt{\lambda_{j k}} \sum_{l m} b\left(\frac{\sqrt{e_{l s}}}{B^{j}}\right) a_{l m, s} Y_{l m}\left(\xi_{j k}\right) \\
& =\sqrt{\lambda_{j k}} \sum_{l m} b\left(\frac{\sqrt{e_{l s}}}{B^{j}}\right)\left\{a_{l m ; E}+i a_{l m ; M}\right\} Y_{l m}\left(\xi_{j k}\right) \\
& =\beta_{j k ; E}+i \beta_{j k ; M} \tag{51}
\end{align*}
$$

where $\beta_{j k ; E}, \beta_{j k ; M}$ are real, i.e. $\beta_{j k ; E}=\operatorname{Re}\left(\beta_{j k, \mathcal{M}}\right), \beta_{j k ; M}=\operatorname{Im}\left(\beta_{j k, \mathcal{M}}\right)$, because $a_{l m ; E}, a_{l m ; M}$ and $Y_{l m}\left(\xi_{j k}\right)$ are involutive. It is immediate to verify that $\beta_{j k ; E}, \beta_{j k, M}$ are exactly the coefficients we would obtain by evaluating a very slightly modified standard (scalar) needlet transform on the scalar functions $f_{E}, f_{M}$. (It is slightly modified because one would be using the $\eta_{j k}$ of (46) with $r=s$ instead of the usual needlets, for which one would use $b\left(l / B^{j}\right)$ instead of $b\left(\sqrt{e_{l S}} / B^{j}\right)$ in (46).)

### 4.1. Localization

We fix an integer s. Because of (36), localization properties of the spin needlets $\psi_{j k, s}$ may be derived from localization properties of the kernel $\Lambda_{j}(x, y, R)$, defined in (27). Often we will only need information about its absolute value $\left|\Lambda_{j}(x, y)\right|$ (which we write as shorthand for $\left|\Lambda_{j}(x, y, R)\right|$ for any $R$ ).

For a full understanding, we need to consider the kernel

$$
\begin{equation*}
\Lambda\left(x, y, t, R, R^{\prime}, g\right)=\sum_{l} \sum_{m} g\left(t \sqrt{e_{l s}}\right) Y_{l m, s R}(x) \bar{Y}_{l m, s R^{\prime}}(y), \tag{52}
\end{equation*}
$$

for $t>0, R, R^{\prime} \in S O(3), g \in C_{c}^{\infty}(\mathbb{R})$. In particular

$$
\begin{equation*}
\left|\Lambda_{j}(x, y)\right|=\left|\Lambda\left(x, y, B^{-j}, R, R^{\prime}, b\right)\right| \tag{53}
\end{equation*}
$$

for any $R, R^{\prime}$.
When $g$ is fixed and understood we will write $\Lambda\left(x, y, t, R, R^{\prime}\right)$ for $\Lambda\left(x, y, t, R, R^{\prime}, g\right)$. In the case $s=0, \Lambda$ does not depend on $R, R^{\prime}$, and we simply write $\Lambda(x, y, t)$ for $\Lambda\left(x, y, t, R, R^{\prime}\right)$.

In the case $s=0$, localization properties of a variant of this kernel (where $e_{l 0}$ in (52) is replaced by $l^{2}$ ) were derived in [40,41]. For the kernel as it stands in (27), as well as analogous kernels on smooth compact Riemannian manifolds, localization results (including results where $b$ need not have compact support away from 0 ) were proved in [15].

For general $s$, the localization properties of $\Lambda$ were proved in [14] and [18].
For the purposes of this article, the relevant localization results are:
Lemma 5. Say $s=0$.
(a) Suppose $g \in C_{c}^{\infty}(0, \infty)$. Then for every pair of $C^{\infty}$ differential operators $X$ (in $x$ ) and $Y$ (in $y$ ) on $\mathbb{S}^{2}$, and for every integer $\tau \geqslant 0$, there exists $C>0$ as follows. Suppose $\operatorname{deg} X=j$ and $\operatorname{deg} Y=k$. Then

$$
\begin{equation*}
|X Y \Lambda(x, y, t)| \leqslant \frac{C t^{-2-j-k}}{\left\{1+\frac{d(x, y)}{t}\right\}^{\tau}} \tag{54}
\end{equation*}
$$

for all $t>0$ and all $x, y \in \mathbb{S}^{2}$.
(b) If $A>0$ is fixed, and $g \in C_{c}^{\infty}(\mathbb{R})$ is even, then the conclusion of (a) remains true as long as $0<t<A$.

Remarks. 1. In relation to the function $f$ of [15], our $g(\xi)=f\left(\xi^{2}\right)$.
2. Part (b) was shown in [15] for $0<t<1$ - see the last paragraph of Section 4 of that article. For $t \in[1, A]$, the estimate (54) is trivial, for then the right side is uniformly bounded below by a positive constant, and the left side is uniformly bounded above by a positive constant, as is apparent from (52).
3. Say $[p, q] \subseteq(0, \infty)$. Using the remarks preceding Theorem 6.1 in [14], one sees that the constant $C$ appearing in (54) may be taken to be uniform for $g$ ranging over any bounded subset of the Fréchet space $C_{c}^{\infty}([p, q])$. Indeed, this follows easily from Lemma 5 and the closed graph theorem.

Lemma 6. Say s is a fixed integer.
(a) (See [14].) Suppose $g \in C_{c}^{\infty}(0, \infty)$. Say $R, R^{\prime} \in S O(3)$. Then for every pair of compact sets $\mathcal{F}_{R} \subseteq U_{R}$ and $\mathcal{F}_{R^{\prime}} \subseteq U_{R^{\prime}}$, every pair of $C^{\infty}$ differential operators $X$ (in $x$ ) on $U_{R}$ and $Y$ (in $y$ ) on $U_{R^{\prime}}$, and for every integer $\tau \geqslant 0$, there exists $C>0$ as follows. Suppose $\operatorname{deg} X=j$ and $\operatorname{deg} Y=k$. Then

$$
\begin{equation*}
\left|X Y \Lambda\left(x, y, t, R, R^{\prime}\right)\right| \leqslant \frac{C t^{-2-j-k}}{\left\{1+\frac{d(x, y)}{t}\right\}^{\tau}} \tag{55}
\end{equation*}
$$

for all $t>0$, all $x \in \mathcal{F}_{R}$ and all $y \in \mathcal{F}_{R^{\prime}}$.
(b) (See [18].) If $A>0$ is fixed, and $g \in C_{c}^{\infty}(\mathbb{R})$ is even, then the conclusion of (a) remains true as long as $0<t<A$.

Remark. Note that if $X=Y=$ the identity operator, $\left|X Y \Lambda\left(x, y, t, R, R^{\prime}\right)\right|$ is independent of $R, R^{\prime}$, and then (55) holds for all $x, y \in \mathbb{S}^{2}$.

We are now going to show that a similar localization result to Lemma 6(a) holds for the mixed needlets. Consider then the "mixed needlet kernel"

$$
\Lambda_{\mathcal{M}}(x, y, t, R)=\sum_{l m} b\left(t \sqrt{e_{l s}}\right) Y_{l m, s R}(x) \bar{Y}_{l m}(y)
$$

It should be observed that

$$
\begin{equation*}
\Lambda_{\mathcal{M}}(x, x, t) \equiv 0 \quad \text { for all } x \in \mathbb{S}^{2}, t>0 \tag{56}
\end{equation*}
$$

because

$$
\left|\sum_{l m} b\left(t \sqrt{e_{l s}}\right) Y_{l m, s R}(x) \bar{Y}_{l m}(x)\right|=\left|\sum_{l} b\left(t \sqrt{e_{l s}}\right) \sum_{m} D_{m s R}^{l}(x) \overline{D_{m 0}^{l}}(x)\right|=0,
$$

by the unitarity properties of the Wigner's $D$ matrices. In contrast, in the unmixed situation $|\Lambda(x, x, t)|=\frac{2 l+1}{4 \pi}\left|\sum_{l} b\left(t \sqrt{e_{l S}}\right)\right|$, by (14). If $0 \neq b \geqslant 0$, as is usually the case in applications, this is non-zero. Qualitatively, then, $\Lambda_{\mathcal{M}}$ is different from $\Lambda$; (56) might even lead one to suspect that $\Lambda_{\mathcal{M}}$ might not be well-localized. However, the methods of our article [14] show that it is:

Lemma 7. Say s is a fixed integer.
Suppose $b \in C_{c}^{\infty}(0, \infty)$. Say $R, R^{\prime} \in S O(3)$. Then for every pair of compact sets $\mathcal{F}_{R} \subseteq U_{R}$ and $\mathcal{F}_{R^{\prime}} \subseteq U_{R^{\prime}}$, every pair of $C^{\infty}$ differential operators $X$ (in $x$ ) on $U_{R}$ and $Y$ (in $y$ ) on $\mathbb{S}^{2}$, and for every integer $\tau \geqslant 0$, there exists $C>0$ as follows. Suppose $\operatorname{deg} X=j$ and $\operatorname{deg} Y=k$. Then

$$
\begin{equation*}
\left|X Y \Lambda_{\mathcal{M}}(x, y, t, R)\right| \leqslant \frac{C_{\tau} t^{-2-j-k}}{\left\{1+\frac{d(x, y)}{t}\right\}^{\tau}} \tag{57}
\end{equation*}
$$

for all $t>0$, all $x \in \mathcal{F}_{R}$ and all $y \in \mathbb{S}^{2}$.
Proof. We shall modify the proof of Theorem 6.1 of [14]. To make that easier, we set $f\left(\xi^{2}\right)=b(\xi)$, so that $f \in C_{c}^{\infty}(0, \infty)$; say that in fact supp $f \subseteq[p, q]$, where $p>0$. We assume $s>0$; similar arguments will apply when $s<0$. Of course the case $s=0$ is handled by Lemma 5 .

We have

$$
\Lambda_{\mathcal{M}}(x, y, t, R)=\sum_{l \geqslant s} f\left(t^{2} e_{l s}\right) \mathcal{K}_{\mathcal{M} R}^{l}(x, y),
$$

where

$$
\mathcal{K}_{\mathcal{M} R}^{l}(x, y)=\sum_{m} Y_{l m, s R}(x) \bar{Y}_{l m}(y)
$$

Thus

$$
\Lambda_{\mathcal{M}}(x, y, t, R)=\Varangle_{R x}^{[s]} \sum_{l \geqslant s} f\left(t^{2} e_{l s}\right) \sqrt{\frac{(l-s)!}{(l+s)!}} \mathcal{K}_{0}^{l}(x, y),
$$

where

$$
\mathcal{K}_{0}^{l}(x, y)=\sum_{m} Y_{l m}(x) \bar{Y}_{l m}(y)
$$

and here

$$
\check{\partial}_{R x}^{[s]}=\partial_{s-1, R} \circ \cdots \circ \partial_{0, R}
$$

in the $x$ variable.
As in the proof of Theorem 6.1 of [14], we note that

$$
\begin{equation*}
\frac{(l+s)!}{(l-s)!}=\prod_{k=1}^{s}\left[e_{l 0}-\gamma_{k}\right] \tag{58}
\end{equation*}
$$

where $\gamma_{k}:=k(k-1)$. As in [14] we choose $T_{0}, T_{1}>0$ with $\gamma_{s} T_{0}^{2}<p / 2, e_{s+1, s} T_{1}^{2}>q$. We note that $\Lambda_{\mathcal{M}}(x, y, t, R) \equiv 0$ for $t \geqslant T_{1}$, and that (57) is trivial for $t$ in the compact interval $\left[T_{0}, T_{1}\right] \subset(0, \infty)$. (Indeed, there the right side of (57) is
uniformly bounded below by a positive constant, and the left side is uniformly bounded above by a positive constant.) It is then enough to focus on $t \in\left(0, T_{0}\right]$, we now define

$$
f_{t}(u):=\frac{f\left(u-s(s+1) t^{2}\right)}{\sqrt{\prod_{k=1}^{s}\left[u-\gamma_{k} t^{2}\right]}}
$$

supported in the fixed compact interval $\left[p, q_{1}\right]:=\left[p, q+s(s+1) T_{0}^{2}\right]$; we note that the denominator does not vanish in the interval. Then, using (8) and (58), we write

$$
\sum_{l \geqslant s} f\left(t^{2} e_{l s}\right) \sqrt{\frac{(l-s)!}{(l+s)!}} \mathcal{K}_{0}^{l}(x, y)=t^{s} \sum_{l \geqslant s} f_{t}\left(t^{2} e_{l 0}\right) \mathcal{K}_{0}^{l}(x, y)=t^{s} \Lambda_{[t]}(x, y),
$$

for

$$
\Lambda_{[t]}(x, y)=\sum_{l \geqslant s} f_{t}\left(t^{2} e_{l 0}\right) \mathcal{K}_{0}^{l}(x, y)
$$

Now note that the functions $f_{t}$ for $t \in\left(0, T_{0}\right]$ form a bounded subset of $C_{c}^{\infty}\left(\left[p, q_{1}\right]\right)$, and recall Remark 3, after Lemma 5 . Choose smooth differential operators on $\mathbb{S}^{2}$, in $x$, of degree $s$ and $j$, which agrees with $\partial_{R x}^{[s]}$ and $X$ respectively, in a neighborhood of $\mathcal{F}_{R}$. As in the proof of Theorem 6.1 of [14], we find

$$
\left|X Y \Lambda_{\mathcal{M}}(x, y, t, R)\right|=t^{s}\left|X Y{\underset{\nabla}{R x}}_{[s]} \Lambda_{[t]}(x, y)\right| \leqslant c t^{s} \frac{C t^{-2-j-s-k}}{\{1+d(x, y) / t\}^{\tau}}=\frac{C t^{-2-j-k}}{\{1+d(x, y) / t\}^{\tau}}
$$

as desired.

Remark 8. For astrophysical applications, it is very common to observe spin random fields only in a subset of the sphere, i.e. $\mathbb{S}^{2} \backslash G$, say, where $G \subset \mathbb{S}^{2}$ is a region contaminated by foreground emission, for instance the Milky Way radiation. Lemma 7 implies that, for $\tau=1,2, \ldots$,

$$
\begin{aligned}
\left|\beta_{j k ; E}-\operatorname{Re}\left\{\int_{\mathbb{S}^{2} \backslash G} F_{S}(x) \bar{\psi}_{j k, s \mathcal{M}} d x\right\}\right| & =\left|\operatorname{Re}\left\{\beta_{j k ; E}-\int_{\mathbb{S}^{2} \backslash G} F_{S}(x) \bar{\psi}_{j k, s \mathcal{M}} d x\right\}\right| \leqslant\left|\int_{G} F_{S}(x) \bar{\psi}_{j k, s \mathcal{M}} d x\right| \\
& \leqslant \int_{G}\left|F_{s}(x)\right|\left|\psi_{j k, s \mathcal{M}}\right| d x \leqslant\left\{\sup _{x \in G}\left|F_{S}(x)\right|\right\} \frac{C_{\tau} \mu(G)}{\left\{1+B^{j} d\left(\xi_{j k}, G\right)\right\}^{\tau}}, \\
\left|\beta_{j k ; M}-\operatorname{Im}\left\{\int_{\mathbb{S}^{2} \backslash G} F_{S}(x) \bar{\psi}_{j k, s \mathcal{M}} d x\right\}\right| & =\left|\operatorname{Im}\left\{i \beta_{j k ; M}-\int_{\mathbb{S}^{2} \backslash G} F_{S}(x) \bar{\psi}_{j k, s \mathcal{M}} d x\right\}\right| \leqslant\left|\int_{G} F_{S}(x) \bar{\psi}_{j k, s \mathcal{M}} d x\right| \\
& \leqslant\left\{\sup _{x \in G}\left|F_{S}(x)\right|\right\} \frac{C_{\tau} \mu(G)}{\left\{1+B^{j} d\left(\xi_{j k}, G\right)\right\}^{\tau}},
\end{aligned}
$$

where $\mu($.$) denotes Lebesgue measure on \mathbb{S}^{2}$. In other words, for all the coefficients corresponding to locations $\xi_{j k} \in \mathbb{S}^{2} \backslash G_{\varepsilon}$ (where $G_{\varepsilon}:=\xi \in \mathbb{S}^{2}: d(\xi, G)>\varepsilon$ ) the mixed needlet coefficients are asymptotically unaffected by the presence of unobserved regions. This is clearly a property of the greatest importance for cosmological applications.

We now derive the following corollaries from Lemmas 6 and 7, which will be essential for the characterizations of Besov spaces in the next section.

## Corollary 9.

(a) In (a) or (b) of Lemma 6, we have that for some $C>0$,

$$
\int|\Lambda(x, y, t)| d x \leqslant C, \quad \int|\Lambda(x, y, t)| d y \leqslant C
$$

where the integrals are over $\mathbb{S}^{2}$.
(b) In Lemma 7, we have that for some $C>0$,

$$
\int\left|\Lambda_{\mathcal{M}}(x, y, t)\right| d x \leqslant C, \quad \int\left|\Lambda_{\mathcal{M}}(x, y, t)\right| d y \leqslant C
$$

where the integrals are over $\mathbb{S}^{2}$.

Proof. This follows at once from the fact that $\int[1+d(x, y) / t]^{-N} d x \leqslant C_{N} t^{n}$ for any $N>n$ (see for example, the third bulleted point after Proposition 3.1 of [15]).

Corollary 10. Say $1 \leqslant p \leqslant \infty$.
(a) For each $j, Q_{j}, P_{j}: C_{s}^{\infty} \rightarrow L_{s}^{p}$. If $p \geqslant 2$, the restriction of $Q_{j}, P_{j}$ to $L_{s}^{p} \subseteq L_{s}^{2}$ is bounded on $L_{s}^{p}$. If $p<2, Q_{j}, P_{j}$ may be extended from $C_{s}^{\infty}$ to be bounded operators on $L_{s}^{p}$.
Further, the operators $Q_{j}, P_{j}$ are uniformly bounded on $L_{s}^{p}$ for $-\infty<j<\infty$.
(b) $\left\|\psi_{j k, s}\right\|_{1},\left\|\psi_{j k, s \mathcal{M}}\right\|_{1} \leqslant C 2^{-j}$.

Proof. For (a), recall from (39) that $Q_{j}=b\left(\frac{\sqrt{T}}{B^{j}}\right)^{2}$ on $L_{s}^{2}$. Take $\Lambda$ as in (52) for $g=b^{2}$. Then for $x \in U_{R}, y \in U_{R^{\prime}}$, we have

$$
\begin{equation*}
\left(Q_{j} F\right)_{R}(x)=\int \Lambda\left(x, y, t, R, R^{\prime}\right) F_{R^{\prime}}(y) d y \tag{59}
\end{equation*}
$$

so that

$$
\left|\left(Q_{j} F\right)(x)\right| \leqslant \int|\Lambda(x, y, t)||F(y)| d y
$$

for $F \in L_{s}^{2}$. Part (a) for $Q_{j}$ is apparent from this and from Corollary 9(a). (Note also that (59) continues to hold for all $F \in L_{s}^{p}$ (one uses a density argument if $p<2$ ).) Similarly, part (a) for $P_{j}$ also follows from Corollary 9(a) (where one references part (b) of Lemma 6), because $P_{j}=\phi\left(\frac{\sqrt{T}}{B^{j+1}}\right)$ on $L_{s}^{2}$ by (23). (Note that $\phi$ equals 1 near 0 , and so has an even extension to a $C_{c}^{\infty}$ function.)

Finally, part (b) follows from Corollary 9(b), once we observe that

$$
\begin{equation*}
\left|\psi_{j k, s \mathcal{M}}(x)\right|=\sqrt{\lambda_{j k}}\left|\Lambda_{\mathcal{M}}\left(x, \xi_{j k}, B^{-j}\right)\right|, \quad \lambda_{j k} \sim B^{-2 j} \tag{60}
\end{equation*}
$$

This completes the proof.

For notational simplicity, we take $B=2$ for the rest of this article; the results would easily generalize to general $B$.
Using Lemma 7, we obtain the following estimates on the $L_{s}^{p}$ norms of the $\psi_{j k, s \mathcal{M}}$.
Lemma 11. For $1 \leqslant p \leqslant \infty$, we have

$$
\begin{equation*}
\left\|\psi_{j k, s \mathcal{M}}\right\|_{p} \sim 2^{j\left(1-\frac{2}{p}\right)} \tag{61}
\end{equation*}
$$

Proof. Let us call the estimate $\left\|\psi_{j k, s \mathcal{M}}\right\|_{p} \leqslant C 2^{j\left(1-\frac{2}{p}\right)}$ the majorization for this value of $p$, and call the reverse inequality the minorization.

First we do the cases $p=1,2, \infty$. For $p=\infty$, the majorization (by $C 2^{j}$ ) follows at once from (60) and Lemma 7. For $p=1$, the majorization (by $C 2^{-j}$ ) is Corollary $10(\mathrm{~b})$. For $p=2$, the majorization (by a constant) follows at once from the tight frame property.

For the remaining estimates, we adapt arguments from [1] and [3]. Fix $c>0,0<v<1$ such that $b^{2}>c$ on the interval $[\nu, 1]$. Using the orthonormality of the spin spherical harmonics, one obtains the minorization for $p=2$ from

$$
\left\|\psi_{j k, s \mathcal{M}}\right\|_{2}^{2}=\sum_{l m} \lambda_{j k}\left|Y_{l m}\left(\xi_{j k}\right)\right|^{2} b^{2}\left(\frac{\sqrt{e_{l s}}}{2^{j}}\right) \geqslant c 2^{-2 j} \sum_{\left\{l: v^{2} \leqslant e_{l s} / 4^{j} \leqslant 1\right\}} \frac{2 l+1}{4 \pi} c \geqslant c^{\prime}>0 .
$$

The minorizations for $p=1, \infty$ now follow at once from the simple general inequality

$$
\begin{equation*}
\|f\|_{2}^{2} \leqslant\|f\|_{1}\|f\|_{\infty} \tag{62}
\end{equation*}
$$

and the majorizations for $p=1, \infty$.
Thus we may assume $1<p<\infty, p \neq 2$. The majorization follows from the general inequality

$$
\begin{equation*}
\|f\|_{p}^{p} \leqslant\|f\|_{1}\|f\|_{\infty}^{p-1} \tag{63}
\end{equation*}
$$

and the majorizations for 1 and $\infty$.
For the minorization, we note that if $q<2<r$, and if $0<\theta<1$ is the number with $1 / 2=\theta / q+(1-\theta) / r$, then one has the general inequality

$$
\begin{equation*}
\|f\|_{2} \leqslant\|f\|_{q}^{\theta}\|f\|_{r}^{1-\theta} \tag{64}
\end{equation*}
$$

If $p>2$, the minorization follows, after a brief computation, from (64) in the case $q=1, r=p$, and the minorizations for 2 and 1 . If $p<2$, the minorization follows, after a briefer computation, from (64) in the case $q=p, r=\infty$, and the minorizations for 2 and $\infty$. This completes the proof.

## 5. Spin Besov spaces and their characterization

The purpose of this section is the characterization of functional spaces by mixed needlets. This issue was already addressed by [1], where the characterizations of Besov spaces by the asymptotic behavior of spin needlet coefficients is addressed; here we aim at an analogous goal by focussing on mixed needlet coefficients. Most of our notations and of the arguments to follow are classical and close to those provided by [1]. We start by recalling that (compare (15)) if $2^{j} \geqslant|s|$,

$$
V_{2^{j}, s}=\bigoplus_{l=|s|}^{2^{j}} \mathcal{H}_{l s},
$$

the space of spin functions spanned by spin spherical harmonics of degree up to $2^{j}$. We let

$$
\sigma_{j}\left(F_{s} ; p\right):=\inf _{G_{s} \in V_{2^{j}, s}}\left\|F_{s}-G_{s}\right\|_{L_{s}^{p}},
$$

the error from the best approximations in that same space. The definition of Besov spaces is then natural.
Definition 12 (Spin Besov space). (See [1].) We say that the spin function $F_{s} \in L_{S}^{p}$ belongs to the Besov space of order $\{p, q, r ; s\}$ (written $F_{s} \in B_{r ; s}^{p q}$ ) if and only if

$$
\sigma_{j}\left(F_{s} ; p\right)=\varepsilon_{j} 2^{-j r},
$$

where $\left\{\varepsilon_{j}\right\} \in \ell^{q}$ and $p \geqslant 1, q, r>0, s \in \mathbf{N}$. The associated norm is

$$
\left\|F_{s}\right\|_{B_{r ; s}^{p q}}^{p q}:=\left\|F_{S}\right\|_{L_{s}^{p}}+\left\|\varepsilon_{j}\right\|_{\ell q} .
$$

Remark. Recall that $L_{j}$ is the largest integer with $e_{L_{j} s} \leqslant 2^{2(j+1)}$, and that by (30), $L_{j} \sim 2^{j}$ as $j \rightarrow \infty$. In particular, there is a $c>0$ such that $2^{j-c} \leqslant L_{j} \leqslant 2^{j+c}$ for all $j$. Thus, if $2^{j-c} \geqslant|s|$,

$$
V_{2^{j-c}, s} \subseteq V_{L_{j}, s} \subseteq V_{2^{j}, s}
$$

Thus if we set

$$
\tilde{\sigma}_{j}\left(F_{s} ; p\right):=\inf _{G_{s} \in V_{L_{j}, s}}\left\|F_{s}-G_{s}\right\|_{L_{s}^{p}}^{p}
$$

it is evident that we could use the $\tilde{\sigma}_{j}$ in place of the $\sigma_{j}$ in Definition 12 to define the same spaces with an equivalent norm.
The following characterization is provided by [1] and extends to spin fiber bundles classical results on approximation spaces.

Theorem 13. (See [1].) If $F_{s} \in L_{s}^{p}$, the following conditions are equivalent to $F_{s} \in B_{r ; s}^{p q}$ :

1. $\quad\left\|P_{j} F_{s}-F_{s}\right\|_{L_{s}^{p}}=\varepsilon_{1 j} 2^{-j r}$;
2. $\left\|P_{j} F_{s}-P_{j-1} F_{s}\right\|_{L_{s}^{p}}=\left\|Q_{j} F_{S}\right\|_{L_{s}^{p}}=\varepsilon_{2 j} 2^{-j r}$;
3. $\left\{\sum_{k}\left|\beta_{j k, s}\right|^{p}\left\|\psi_{j k, s}\right\|_{L_{s}^{p}}^{p}\right\}^{1 / p}=\varepsilon_{3 j} 2^{-j r}$,
where $\left\{\varepsilon_{1 j}\right\},\left\{\varepsilon_{2 j}\right\},\left\{\varepsilon_{3 j}\right\} \in \ell^{q}$, and $c_{p}, C_{p}>0$.
The fact that (1) implies (2) is easy, while the converse is standard from Hardy's inequality. The fact that $F_{s} \in B_{r ; s}^{p q}$ implies (1) follows at once from using the $\tilde{\sigma}_{j}$ in Definition 12, while the converse follows from Corollary 10(a) for the $P_{j}$, once one
notes that for any $G_{s} \in V_{L_{j-1} s}, P_{j} G_{s}=\phi\left(\frac{\sqrt{T}}{2^{j+1}}\right) G_{s}=G_{s}$, so

$$
\left\|P_{j} F_{s}-F_{s}\right\|_{L_{s}^{p}} \leqslant\left\|P_{j}\right\|\left\|F_{S}-G_{s}\right\|_{L_{s}^{p}}+\left\|G_{S}-F_{s}\right\|_{L_{s}^{p}} .
$$

The equivalence of (2) with (3) is established by first showing that for any $F \in L_{s}^{p}$,

$$
\begin{equation*}
c_{p}\left\|Q_{j} F_{s}\right\|_{L_{s}^{p}} \leqslant\left\{\sum_{k}\left|\beta_{j k, s}\right|^{p}\left\|\psi_{j k, s}\right\|_{L_{s}^{p}}^{p}\right\}^{1 / p} \leqslant C_{p}\left\{\left\|Q_{j-1} F_{s}\right\|_{L_{s}^{p}}+\left\|Q_{j} F_{s}\right\|_{L_{s}^{p}}+\left\|Q_{j+1} F_{s}\right\|_{L_{s}^{p}}\right\} . \tag{65}
\end{equation*}
$$

We will prove a "mixed" analogue of (65) in Theorem 15 below, by a proof which is very close to the proof in [1].
Remark 14. In the previous theorem, the crucial result is of course provided by (3), which provides the characterizations of Besov classes by means of the decay of spin needlet coefficients. This feature could be provided by many alternative formulations; in particular, as in the mixed case, one has (see [1]) that

$$
\left\|\psi_{j k, s}\right\|_{p} \sim 2^{j\left(1-\frac{2}{p}\right)}
$$

The previous result can hence be formulated as follows: The measurable spin function $F_{s}$ belongs to the Besov space of order $\{p, q, r ; s\}$ if and only if

$$
\left[\int_{\mathbb{S}^{2}}\left|F_{S}(x)\right|^{p} d x\right]^{1 / p}+\left[\sum_{j} 2^{q j\left\{r+2\left(\frac{1}{2}-\frac{1}{p}\right)\right\}}\left\{\sum_{k}\left|\beta_{j k, s}\right|^{p}\right\}^{q / p}\right]^{1 / q}<\infty
$$

Our main result in this section is to show that the mixed needlet coefficients can play exactly the same role as the spin coefficients in the characterization of functional spaces, despite their different mathematical features. More precisely, we have the following alternative characterization of Besov spaces:

Theorem 15. The function $F_{s} \in L_{s}^{p}$ belongs to the spin Besov space $B_{r ; s}^{p q}$ if and only if

$$
\begin{equation*}
\left\{\sum_{k}\left|\beta_{j k, s \mathcal{M}}\right|^{p}\left\|\psi_{j k, s \mathcal{M}}\right\|_{L_{s}^{p}}^{p}\right\}^{1 / p}=\varepsilon_{4 j} 2^{-j r} \tag{66}
\end{equation*}
$$

where $\left\{\varepsilon_{4 j}\right\} \in \ell^{q}$, and $c_{p}, C_{p}>0$. Equivalently, $F_{s} \in B_{r ; s}^{p q}$ if and only if

$$
\left[\int_{\mathbb{S}^{2}}\left|F_{S}(x)\right|^{p} d x\right]^{1 / p}+\left[\sum_{j} 2^{j q\left\{r+2\left(\frac{1}{2}-\frac{1}{p}\right)\right\}}\left\{\sum_{k}\left|\beta_{j k, s \mathcal{M}}\right|^{p}\right\}^{q / p}\right]^{1 / q}<\infty
$$

Proof. Given Theorem 13 and the results established in the previous section, the proof is rather standard and very close, for instance, to the arguments in [1].

By Theorem 13, it suffices to establish that for all $F \in L_{s}^{p}$,

$$
\begin{equation*}
c_{p}\left\|Q_{j} F_{s}\right\|_{L_{s}^{p}} \leqslant\left\{\sum_{k}\left|\left\langle F_{s}, \psi_{j k, s \mathcal{M}}\right\rangle\right|^{p}\left\|\psi_{j k, s \mathcal{M}}\right\|_{L_{s}^{p}}^{p}\right\}^{1 / p} \leqslant C_{p}\left\{\left\|Q_{j-1} F_{s}\right\|_{L_{s}^{p}}+\left\|Q_{j} F_{s}\right\|_{L_{s}^{p}}+\left\|Q_{j+1} F_{s}\right\|_{L_{s}^{p}}\right\} . \tag{67}
\end{equation*}
$$

In addition to Lemma 11, we will need the inequality

$$
\begin{equation*}
\sum_{k}\left|\psi_{j k, s \mathcal{M}}(x)\right| \leqslant C_{M} \sum_{k} \frac{C_{M} 2^{j}}{\left\{1+2^{j} d\left(x, \xi_{j k}\right)\right\}^{M}} \leqslant C_{M} 2^{j} \tag{68}
\end{equation*}
$$

which follows from the properties of $\epsilon$-nets (see [3] or [1]).
To establish the rightmost inequality of (67), we note first that, in view of the mixed analogue of (45),

$$
\sum_{k}\left|\left\langle F_{s}, \psi_{j k, s \mathcal{M}}\right\rangle\right|^{p}\left\|\psi_{j k, s \mathcal{M}}\right\|_{L_{s}^{p}}^{p}=\sum_{k}\left|\left\langle Q_{j-1} F_{S}+Q_{j} F_{s}+Q_{j+1} F_{s}, \psi_{j k, s \mathcal{M}}\right\rangle\right|^{p}\left\|\psi_{j k, s \mathcal{M}}\right\|_{L_{s}^{p}}^{p}
$$

(In fact, the sums are clearly termwise equal at least for $F_{s} \in L_{s}^{2}$; the equality for general $F_{s} \in L_{s}^{p}$ follows by a density argument.)

The result will then follow if we can prove that, for all $G_{s} \in L_{s}^{p}$,

$$
\sum_{k}\left|\left\langle G_{s}, \psi_{j k, s \mathcal{M}}\right\rangle\right|^{p}\left\|\psi_{j k, s \mathcal{M}}\right\|_{L_{s}^{p}}^{p} \leqslant C^{p}\left\|G_{s}\right\|_{L_{s}^{p}}^{p} .
$$

In view of Lemma 11 and (68), the result can be established along exactly the same lines as in Lemma 14 of [1]; more precisely, by Holder's inequality

$$
\left|\left\langle G_{s}, \psi_{j k, s \mathcal{M}}\right\rangle\right| \leqslant\left\{\int_{\mathbb{S}^{2}}\left|G_{s}(x)\right|^{p}\left|\psi_{j k, s \mathcal{M}}(x)\right| d x\right\}^{1 / p}\left\{\int_{\mathbb{S}^{2}}\left|\psi_{j k, s \mathcal{M}}(x)\right| d x\right\}^{1-1 / p}
$$

whence

$$
\sum_{k}\left|\left\langle G_{s}, \psi_{j k, s \mathcal{M}}\right\rangle\right|^{p}\left\|\psi_{j k, s \mathcal{M}}\right\|_{L_{s}^{p}}^{p} \leqslant\left(\int_{\mathbb{S}^{2}}\left|G_{s}(x)\right|^{p} \sum_{k}\left|\psi_{j k, s \mathcal{M}}(x)\right| d x\right)\left\|\psi_{j k, s \mathcal{M}}\right\|_{L_{s}^{1}}^{p-1}\left\|\psi_{j k, s \mathcal{M}}\right\|_{L_{s}^{p}}^{p} \leqslant C\left\|G_{s}\right\|_{L_{s}^{p}}^{p},
$$

as desired, by Lemma 11 and (68). The proof of the rightmost inequality is hence completed.

As far as the leftmost inequality of (67) is concerned, again our arguments are very close to [1], Lemmas 15 and 16. Note first that, by (41) and Lemma 4,

$$
Q_{j} F_{s}=\sum_{k}\left\langle F_{s}, \psi_{j k, s \mathcal{M}}\right\rangle \psi_{j k, s \mathcal{M}}
$$

at least for $F_{s} \in L_{s}^{2}$. Now using Holder's inequality

$$
\begin{aligned}
\left\|\sum_{k}\left\langle F_{s}, \psi_{j k, s \mathcal{M}}\right\rangle \psi_{j k, s \mathcal{M}}\right\|_{L_{s}^{p}}^{p} & \leqslant \int_{\mathbb{S}^{2}}\left\{\sum_{k}\left|\left\langle F_{s}, \psi_{j k, s \mathcal{M}}\right\rangle\right|\left|\psi_{j k, s \mathcal{M}}(x)\right|^{1 / p}\left|\psi_{j k, s \mathcal{M}}(x)\right|^{1-1 / p}\right\}^{p} d x \\
& \leqslant \int_{\mathbb{S}^{2}} \sum_{k}\left|\left\langle F_{s}, \psi_{j k, s \mathcal{M}}\right\rangle\right|^{p}\left|\psi_{j k, s \mathcal{M}}(x)\right|\left\{\sum_{k}\left|\psi_{j k, s \mathcal{M}}(x)\right|\right\}^{p-1} d x \\
& \leqslant C 2^{j(p-1)} \sum_{k}\left|\left\langle F_{s}, \psi_{j k, s \mathcal{M}}\right\rangle\right|^{p}\left\|\psi_{j k, s \mathcal{M}}\right\|_{L_{s}^{1}} \leqslant C \sum_{k}\left|\left\langle F_{s}, \psi_{j k, s \mathcal{M}}\right\rangle\right|^{p}\left\|\psi_{j k, s \mathcal{M}}\right\|_{L_{s}^{p}}^{p,}
\end{aligned}
$$

again by (68) and Lemma 11. This gives the leftmost inequality of (67) for $F_{s} \in L_{s}^{2}$, and hence for $F_{s} \in L_{s}^{p}$ if $p \geqslant 2$. If instead $p<2$, and $F_{s} \in L_{s}^{p}$ is general, we take a sequence $F_{s}^{m} \in C_{s}^{\infty}$ approaching $F_{s}$ in $L_{s}^{p}$, consider that inequality with $F_{s}^{m}$ in place of $F_{s}$, and take the limsup of both sides in $m$, to obtain the inequality for $F_{s}$, as desired.

Theorem 15 could be formulated more directly as: The section $F_{s}$ belongs to the spin Besov space $B_{r ; s}^{p q}$ if and only if there exists $\left\{\varepsilon_{j}\right\} \in \ell^{q}$ such that

$$
\sum_{k}\left|\beta_{j k, s \mathcal{M}}\right|^{p}=\varepsilon_{j} 2^{-j\left\{r+2\left(\frac{1}{2}-\frac{1}{p}\right)\right\}}
$$

Combining Theorems 15 and 13, we have the bounds

$$
c \varepsilon_{2, j} 2^{-j r+2 j\left(\frac{1}{2}-\frac{1}{p}\right)} \leqslant\left\{\sum_{k}\left|\beta_{j k, s \mathcal{M}}\right|^{p}\right\}^{1 / p}, \quad\left\{\sum_{k}\left|\beta_{j k, s}\right|^{p}\right\}^{1 / p} \leqslant C\left\{\frac{\varepsilon_{2, j-1}}{2}+\varepsilon_{2, j}+2 \varepsilon_{2, j+1}\right\} 2^{-j r+2 j\left(\frac{1}{2}-\frac{1}{p}\right)},
$$

where the $\ell^{q}$ sequence $\left\{\varepsilon_{2 j}\right\}$ is such that

$$
\left\|Q_{j} F_{S}\right\|_{L_{s}^{p}}=\varepsilon_{2 j} 2^{-j r} .
$$

More explicitly, the asymptotic behavior of the norms of needlet coefficients is of the same order for the spin and mixed spin case, despite the fact that the coefficients in the two cases have a rather different nature (the $\left\{\beta_{j k, s}\right\}$ are spin-valued, while $\left\{\beta_{j k, s \mathcal{M}}\right\}$ are complex valued scalars). An alternative way to formulate this conclusion is the following. Define $B_{r}^{p q}(\mathbb{C})$ as the Besov space of complex-valued functions on the sphere. Then:

The spin function $F_{s}$ belongs to the spin Besov space $B_{r ; s}^{p q}$ if and only if the scalar complex-valued function $f=\left(f_{E}+i f_{M}\right)$ belongs to $B_{r}^{p q}(\mathbb{C})$, i.e.

$$
\left\{F_{s} \in B_{r ; s}^{p q}\right\} \quad \Leftrightarrow \quad\left\{\left(f_{E}+i f_{M}\right) \in B_{r}^{p q}(\mathbb{C})\right\} .
$$

Note that the complex-valued function $f$ does not correspond to the function $g$ introduced in (21), indeed for a given array of coefficients $\left\{a_{l m, s}=a_{l m}^{E}+i a_{l m}^{M}\right\}_{l m}$ we can write

$$
F_{S}(x)=\sum_{l m} a_{l m, s} Y_{l m, s}(x)=\sum_{l m} a_{l m, s} \sqrt{\frac{(l-s)!}{(l+s)!}}(ð)^{s} Y_{l m}=(ð)^{s} g(x),
$$

whence

$$
g(x)=\sum_{l m} \sqrt{\frac{(l-s)!}{(l+s)!}} a_{l m, s} Y_{l m}(x) \neq f(x)=\sum_{l m} a_{l m, s} Y_{l m}(x)
$$

## 6. Statistical applications

### 6.1. Estimation of angular power spectra and cross-spectra

A major asset explaining the success of needlets for the analysis of cosmological data refers to their uncorrelation properties. More precisely, it was shown in [2], that for isotropic random fields, needlet coefficients are asymptotically uncorrelated at any fixed angular distance as the frequency $j$ diverges. This result was extended to the Mexican needlet case by [31,38], and motivated many applications to astrophysical data, for instance (cross-)angular power spectrum estimation (see [43,11]), detection of asymmetries (see [44]), bispectrum estimation (see $[30,46,45]$ ) and many others. An analogous property was established for spin needlets in [14]; statistical techniques were then developed in [13], while applications to CMB polarization data were detailed in [12]. In this section, we shall show how mixed needlets allow for further applications which have great physical interest and are not feasible by the pure spin approach, such as, for instance, estimation of cross-spectra between scalar and spin fields.

To this aim, we shall focus on zero-mean, isotropic spin Gaussian random fields. As discussed by $[14,13,32,33]$, the latter can be characterized by assuming that $\left\{a_{l m}^{E}, a_{l m}^{M}\right\}$ are complex-valued Gaussian random sequences satisfying

$$
E a_{l m}^{E}=E a_{l m}^{M}=0, \quad E a_{l m}^{E} \bar{a}_{l^{\prime} m^{\prime}}^{E}=\delta_{l}^{l^{\prime}} \delta_{m}^{m^{\prime}} C_{l}^{E}, \quad E a_{l m}^{M} \bar{a}_{l^{\prime} m^{\prime}}^{M}=\delta_{l}^{l^{\prime}} \delta_{m}^{m^{\prime}} C_{l}^{M}, \quad m=-l, \ldots, l
$$

and

$$
a_{l m}^{E}=\bar{a}_{l,-m}^{E}, \quad a_{l m}^{M}=\bar{a}_{l,-m}^{M}
$$

For $m=0,\left\{a_{l 0}^{E}, a_{l 0}^{M}\right\}$ are real-valued Gaussian with the same moments. In the cosmological literature, the sequences $\left\{C_{l}^{E}, C_{l}^{M}\right\}$ are known as the angular power spectra of the $E$ and $M$ modes; clearly in the Gaussian case they encode the full information on the dependence structure of the random field. In these area of applications, data are collected also on a standard scalar field (the so-called temperature of CMB radiation), which is again assumed to be Gaussian and isotropic with angular power spectrum

$$
E a_{l m}^{T} \bar{a}_{l^{\prime} m^{\prime}}^{T}=\delta_{l}^{l^{\prime}} \delta_{m}^{m^{\prime}} C_{l}^{T}, \quad E a_{l m}^{T} \bar{a}_{l^{\prime} m^{\prime}}^{E}=\delta_{l}^{l^{\prime}} \delta_{m}^{m^{\prime}} C_{l}^{T E}, \quad E a_{l m}^{T} \bar{a}_{l^{\prime} m^{\prime}}^{M}=\delta_{l}^{l^{\prime}} \delta_{m}^{m^{\prime}} C_{l}^{T M}
$$

The cross-spectra $\left\{C_{l}^{T E}, C_{l}^{T M}\right\}$ are themselves of great physical relevance. The former is used to constrain cosmological parameters, in particular the so-called reionization epoch, while a detection of a non-zero value for the latter would entail an (unexpected) violation of parity invariance at the cosmological scales. Note that, for jointly isotropic random fields $T, E$, we have

$$
\begin{aligned}
C_{l}^{T E} & =E a_{l m}^{T} \bar{a}_{l m}^{E}=E\left\{\left[\operatorname{Re}\left(a_{l m}^{T}\right)+i \operatorname{Im}\left(a_{l m}^{T}\right)\right]\left[\operatorname{Re}\left(a_{l m}^{E}\right)-i \operatorname{Im}\left(a_{l m}^{E}\right)\right]\right\} \\
& =E\left\{\operatorname{Re}\left(a_{l m}^{T}\right) \operatorname{Re}\left(a_{l m}^{E}\right)\right\}+E\left\{\operatorname{Im}\left(a_{l m}^{T}\right) \operatorname{Im}\left(a_{l m}^{E}\right)\right\}+i E\left\{\operatorname{Re}\left(a_{l m}^{T}\right) \operatorname{Im}\left(a_{l m}^{E}\right)\right\}-i E\left\{\operatorname{Im}\left(a_{l m}^{T}\right) \operatorname{Re}\left(a_{l m}^{E}\right)\right\} \\
& =E\left\{\operatorname{Re}\left(a_{l m}^{T}\right) \operatorname{Re}\left(a_{l m}^{E}\right)\right\}+E\left\{\operatorname{Im}\left(a_{l m}^{T}\right) \operatorname{Im}\left(a_{l m}^{E}\right)\right\}=C_{l}^{E T},
\end{aligned}
$$

because $\left\{\operatorname{Re}\left(a_{l m}^{T}\right), \operatorname{Im}\left(a_{l m}^{E}\right)\right\} \stackrel{d}{=}\left\{\operatorname{Im}\left(a_{l m}^{T}\right), \operatorname{Re}\left(a_{l m}^{E}\right)\right\}$ by isotropy, where $\stackrel{d}{=}$ denotes equality in distribution; the latter property follows (as in [5] and [37], compare Theorem 7.2 in [14]) from

$$
\binom{D_{m m^{\prime}}^{l}(R) a_{l m}^{T}}{D_{m m^{\prime}}^{l}(R) a_{l m}^{E}} \stackrel{d}{=}\binom{a_{l m}^{T}}{a_{l m}^{E}}, \quad \text { for all } R \in S O(3)
$$

where $\left\{D_{m_{1} m_{2}}^{l}(R)\right\}_{m_{1}, m_{2}}$ denotes as usual the family of irreducible unitary representations of $S O(3)$ by means of Wigner's matrices (see $[50,49]$ ). It follows, in particular, that the cross-power spectrum is always real-valued.

We shall now provide an uncorrelation result that generalizes [2] and [14].

Theorem 16. Let $F_{S}$ and $T$ be jointly isotropic spin and scalar random fields (respectively), with angular power spectra such that

$$
\begin{array}{ll}
C_{l}^{E}=g_{E}(l) l^{-\alpha_{E}}, & C_{l}^{M}=g_{M}(l) l^{-\alpha_{M}}, \quad C_{l}^{T}=g_{T}(l) l^{-\alpha_{T}}, \\
\alpha_{E}, \alpha_{M}, \alpha_{T}>2, & \left|g_{E}^{(i)}(u)\right|,\left|g_{M}^{(i)}(u)\right|,\left|g_{T}^{(i)}(u)\right| \leqslant c_{i} u^{-i}, \quad c_{i}>0, i=0,1,2, \ldots
\end{array}
$$

Assume also that for sufficiently large l,

$$
\left|g_{E}(l)\right|,\left|g_{M}(l)\right|,\left|g_{T}(l)\right|>c>0
$$

Then for all $\tau>0$ there exists $C_{\tau}>0$ such that

$$
\left|\operatorname{Corr}\left(\beta_{j k_{1} ; E}, \beta_{j k_{2} ; E}\right)\right|,\left|\operatorname{Corr}\left(\beta_{j k_{1} ; M}, \beta_{j k_{2} ; M}\right)\right|,\left|\operatorname{Corr}\left(\beta_{j k_{1} ; T}, \beta_{j k_{2} ; T}\right)\right| \leqslant \frac{C_{\tau}}{\left\{1+2^{j} d\left(\xi_{j k_{1}}, \xi_{j k_{2}}\right)\right\}^{\tau}}
$$

Proof. For the coefficients of the scalar random field $T$, the proof was provided in [2] (see [14] for the extension to the spin case). In view of the expressions provided in Section 4 for the needlet coefficients ( $\beta_{j k ; E}, \beta_{j k ; M}$ ) and the discussion following Eq. (51), the proof is identical to the argument for the scalar case, and it is hence omitted for brevity's sake.

As we mentioned, mixed needlets allow for statistical procedures which were unfeasible in the scalar and pure spin cases. Consider for instance the issue of testing for a non-zero value of the cross-spectrum $C_{l}^{T E}$, which is one of the main objectives of the ESA satellite mission Planck. Let us introduce the estimators

$$
\widehat{\Gamma}_{j}^{T E}=\operatorname{Re}\left\{\sum_{k} \beta_{j k ; E} \beta_{j k ; T}\right\}, \quad \widehat{\Gamma}_{j}^{T M}=\operatorname{Re}\left\{\sum_{k} \beta_{j k_{1} ; M} \beta_{j k ; T}\right\} .
$$

We have easily, for $A=E, M$,

$$
\begin{aligned}
E \widehat{\Gamma}_{j}^{T A} & =E \operatorname{Re}\left\{\sum_{l_{1} m_{1}} \sum_{l_{2} m_{2}} b\left(\frac{\sqrt{e_{l_{1} s}}}{B^{j}}\right) b\left(\frac{\sqrt{e_{l_{2} s}}}{B^{j}}\right) a_{l_{1} m_{1} ; T} \bar{a}_{l_{2} m_{2} ; E} \sum_{k} Y_{l_{1} m_{1}}\left(\xi_{j k}\right) \bar{Y}_{l_{2} m_{2}}\left(\xi_{j k}\right) \lambda_{j k}\right\} \\
& =\sum_{l} b^{2}\left(\frac{\sqrt{e_{l_{1} s}}}{B^{j}}\right) \frac{2 l+1}{4 \pi} C_{l}^{T A}
\end{aligned}
$$

Likewise

$$
\operatorname{Var}\left\{\widehat{\Gamma}_{j}^{T A}\right\}=\left\{\sum_{l} b^{4}\left(\frac{\sqrt{e_{l s}}}{B^{j}}\right) \frac{2 l+1}{4 \pi} C_{l}^{T} C_{l}^{A}\right\}+\left\{\sum_{l} b^{4}\left(\frac{\sqrt{e_{l s}}}{B^{j}}\right) \frac{2 l+1}{4 \pi}\left(C_{l}^{T A}\right)^{2}\right\}
$$

The following result is straightforward:
Lemma 17. As $j \rightarrow \infty$, for $A=E, M$,

$$
\frac{\widehat{\Gamma}_{j}^{T A}-E \widehat{\Gamma}_{j}^{T A}}{\sqrt{\operatorname{Var}\left\{\widehat{\Gamma}_{j}^{T A}\right\}}} \rightarrow_{d} N(0,1)
$$

Proof. It suffices to note that

$$
\widehat{\Gamma}_{j}^{T A}=\sum_{l} b^{2}\left(\frac{\sqrt{e_{l_{1} s}}}{B^{j}}\right) \frac{1}{4 \pi}\left[a_{l 0}^{A} a_{l 0}^{T}+2 \sum_{m=1}^{l}\left(\operatorname{Re}\left(a_{l m}^{A}\right) \operatorname{Im}\left(a_{l m}^{T}\right)+\operatorname{Re}\left(a_{l m}^{T}\right) \operatorname{Im}\left(a_{l m}^{A}\right)\right)\right]
$$

and the summands satisfy all the assumptions of the classical Lindeberg-Levy Central Limit Theorem, see [14] for an analogous argument.

Remark 18. As in [13], it is indeed possible to prove stronger results than Lemma 17, namely, it can be shown that the same limiting result holds, if the estimator is constructed by using only the coefficients belonging to a connected subset of $\mathbb{S}^{2}$. This result is important for astrophysical applications, where observations are available only on subsets of the sphere, due to various forms of astrophysical contamination.

Remark 19. It would be straightforward to exploit the properties of mixed needlet coefficients for many other statistical applications. To provide an example, it is possible to advocate estimation of the joint bispectrum of scalar and spin random fields, along the lines of the procedures advocated for the scalar case by [30] (see also [34,35,37]). While these extensions are straightforward from the mathematical point of view, and hence omitted here for brevity's sake, they are certainly of great practical importance for applications to CMB datasets, as we discussed in the Introduction.

### 6.2. Spin nonparametric regression

While the uncorrelation properties of needlet coefficients have already been widely exploited in statistical inference, the characterization provided for spin Besov spaces entails even richer statistical opportunities which are still almost completely open for research (see also [23,25]) for classical results on adaptive nonparametric regression, [29,27] for optimal spherical deconvolution methods, [4] and [26] for some results on needlet-based shrinkage estimation for densities on the sphere, and [28] for adaptive nonparametric regression on vector bundles.

We envisage, in particular, applications to spin nonparametric regression by means of mixed-needlets shrinkage, in the following sense. Consider the regression model

$$
Y_{s}\left(X_{i}\right)=F_{s}\left(X_{i}\right)+\varepsilon_{i}, \quad i=1,2, \ldots, n,
$$

where $\left\{X_{i}\right\}_{i=1, \ldots, n}$ are (deterministic or stochastic) locations on the sphere $X_{i} \in \mathbb{S}^{2}, F_{s} \in B_{r ; s}^{p q}$ is a deterministic section of the spin fiber bundle and $\left\{\varepsilon_{i}\right\}_{i=1, \ldots, n}$ is a sequence of random observational errors, themselves spin $s$ variables. From the point of view of applications, we have in mind measurements of so-called weak gravitational lensing effects (see for instance [6]); here, $F_{s}$ is the shear induced by gravitational effects on the image of distant galaxies, and $\left\{\varepsilon_{k}\right\}$ are observational errors, due for instance to the intrinsic variability in the shape of the galaxies. The aim is to reconstruct $F_{S}$ upon observations on $\left\{Y_{s}\left(X_{i}\right)\right\}$; this is the object of a number of ongoing challenges, detailed for instance in [6]. Assume $\left\{X_{i}\right\}_{i=1, \ldots, n}$ make up an (approximate) sequence of cubature points; we suggest to estimate $F_{S}$ by means of the shrinkage procedure

$$
\tilde{F}_{s}(x):=\sum_{j k} \tilde{\beta}_{j k, s \mathcal{M}}^{*} \psi_{j k, s \mathcal{M}}(x)
$$

where

$$
\tilde{\beta}_{j k, s \mathcal{M}}^{*}=\tilde{\beta}_{j k, s \mathcal{M}} \mathbb{I}\left(\left|\tilde{\beta}_{j k, s \mathcal{M}}\right|>c t_{n}\right), \quad t_{n} \rightarrow 0 \text { as } n \rightarrow \infty \text { and } c>0,
$$

and

$$
\tilde{\beta}_{j k, s \mathcal{M}}:=\frac{4 \pi}{n} \sum_{i} Y_{s}\left(X_{i}\right) \bar{\psi}_{j k, s \mathcal{M}}\left(X_{i}\right) \simeq \int_{\mathbb{S}^{2}} F_{s}(x) \bar{\psi}_{j k, s \mathcal{M}}(x) d x
$$

The Besov space characterization opens the possibility of investigating optimality properties (in the minimax sense) of the shrinkage estimator $\tilde{F}_{s}$ over Besov balls $B_{r ; s}^{p q}(Q)$, where $Q<\infty$, defined as

$$
F_{s}: \quad\left\|F_{s}\right\|_{B_{r ; s}^{p q}}<Q
$$

A full investigation of these issues will be reported elsewhere.

## References

[1] P. Baldi, G. Kerkyacharian, D. Marinucci, D. Picard, Besov spaces for sections of spin fiber bundles on the sphere, preprint, 2009.
[2] P. Baldi, G. Kerkyacharian, D. Marinucci, D. Picard, Asymptotics for spherical needlets, Ann. Statist. 37 (3) (2009) 1150-1171, arXiv:math.st/0606599.
[3] P. Baldi, G. Kerkyacharian, D. Marinucci, D. Picard, Subsampling needlet coefficients on the sphere, Bernoulli 15 (2009) 438-463, arXiv:0706.4169.
[4] P. Baldi, G. Kerkyacharian, D. Marinucci, D. Picard, Adaptive density estimation for directional data using needlets, Ann. Statist. 37 (6A) (2009) 33623395, arXiv:0807.5059.
[5] P. Baldi, D. Marinucci, Some characterizations of the spherical harmonics coefficients for isotropic random fields, Statist. Probab. Lett. 77 (2007) $490-$ 496, arXiv:math/0606709.
[6] S. Bridles, et al., Handbook for the GREAT08 challenge: an image analysis competition for gravitational lensing, Ann. Appl. Stat. 2 (2009) 6-37.
[7] P. Cabella, M. Kamionkowski, Theory of cosmic microwave background polarization, arXiv:astro-ph/0403392v2, 2005.
[8] J. Delabrouille, J.-F. Cardoso, M. Le Jeune, M. Betoule, G. Fay, F. Guilloux, A full sky, low foreground, high resolution CMB map from WMAP, Astron. Astrophys. 493 (3) (2008) 835-857, arXiv:0807.0773.
[9] S. Dodelson, Modern Cosmology, Academic Press, 2003.
[10] M. Eastwood, P. Tod, Edth - a differential operator on the sphere, Math. Proc. Cambridge Philos. Soc. 92 (1982) 317-330.
[11] G. Faÿ, F. Guilloux, M. Betoule, J.-F. Cardoso, J. Delabrouille, M. Le Jeune, CMB power spectrum estimation using wavelets, Phys. Rev. D 78 (2008) 083013, arXiv:0807.1113.
[12] D. Geller, F.K. Hansen, D. Marinucci, G. Kerkyacharian, D. Picard, Spin needlets for cosmic microwave background polarization data analysis, Phys. Rev. D 78 (2008) 123533, arXiv:0811.2881.
[13] D. Geller, X. Lan, D. Marinucci, Spin needlets spectral estimation, Electron. J. Stat. 3 (2009) 1497-1530, arXiv:0907.3369.
[14] D. Geller, D. Marinucci, Spin wavelets on the sphere, J. Fourier Anal. Appl., doi:10.1007/s00041-010-9128-3, in press, arXiv:0811.2835, 2008.
[15] D. Geller, A. Mayeli, Continuous wavelets on manifolds, Math. Z. 262 (2009) 895-927, arXiv:math/0602201.
[16] D. Geller, A. Mayeli, Nearly tight frames and space-frequency analysis on compact manifolds, Math. Z. 263 (2009) 235-264, arXiv:0706.3642.
[17] D. Geller, A. Mayeli, Besov spaces and frames on compact manifolds, Indiana Univ. Math. J. 58 (2009) 2003-2042, arXiv:0709.2452.
[18] D. Geller, A. Mayeli, Nearly tight frames of spin wavelets on the sphere, preprint, arXiv:0907.3164, 2009.
[19] D. Geller, I. Pesenson, Band-limited localized parseval frames and Besov spaces on compact homogeneous manifolds, arXiv:1002.3841, 2010.
[20] J.N. Goldberg, E.T. Newman, Spin-s spherical harmonics and ð, J. Math. Phys. 8 (11) (1967) 2155-2166.
[21] T. Ghosh, J. Delabrouille, M. Remazeilles, J.-F. Cardoso, T. Souradeep, Foreground maps in WMAP frequency bands, arXiv:1006.0916, 2010.
[22] F. Guilloux, G. Fay, J.-F. Cardoso, Practical wavelet design on the sphere, Appl. Comput. Harmon. Anal. 26 (2009) 143-160, arXiv:0706.2598.
[23] W. Hardle, G. Kerkyacharian, D. Picard, A. Tsybakov, Wavelets, Approximation, and Statistical Applications, Lecture Notes in Statist., vol. 129, SpringerVerlag, New York, 1998.
[24] M. Kamionkowski, A. Kosowski, A. Stebbins, Statistics of cosmic microwave background polarization, Phys. Rev. D 55 (12) (1996) $7368-7388$.
[25] G. Kerkyacharian, D. Picard, Regression in random design and warped wavelets, Bernoulli 10 (6) (2004) 1053-1105.
[26] G. Kerkyacharian, T.M. Pham Ngoc, D. Picard, Localized deconvolution on the sphere, arXiv:0908.1952, 2009.
[27] P.T. Kim, J.-Y. Koo, Optimal spherical deconvolution, J. Multivariate Anal. 80 (2002) 21-42.
[28] P.T. Kim, J.-Y. Koo, Z.-M. Luo, Weyl eigenvalue asymptotics and sharp adaptation on vector bundles, J. Multivariate Anal. 100 (2009) $1962-1978$.
[29] J.-Y. Koo, P.T. Kim, Sharp adaptation for spherical inverse problems with applications to medical imaging, J. Multivariate Anal. 99 (2008) 165-190.
[30] X. Lan, D. Marinucci, The needlets bispectrum, Electron. J. Stat. 2 (2008) 332-367, arXiv:0802.4020.
[31] X. Lan, D. Marinucci, On the dependence structure of wavelet coefficients for spherical random fields, Stochastic Process. Appl. 119 (2008) 3749-3766, arXiv:0805.4154.
[32] N. Leonenko, L. Sakhno, On spectral representations of tensor random fields on the sphere, arXiv:0912.3389, 2009.
[33] A. Malyarenko, Invariant random fields in vector bundles and applications to cosmology, arXiv:0907.4620, 2009.
[34] D. Marinucci, High-resolution asymptotics for the angular bispectrum of spherical random fields, Ann. Statist. 34 (2006) 1-41, arXiv:math/0502434.
[35] D. Marinucci, A central limit theorem and higher order results for the angular bispectrum, Probab. Theory Related Fields 141 (2008) 389-409, arXiv:math/0509430.
[36] D. Marinucci, D. Pietrobon, A. Balbi, P. Baldi, P. Cabella, G. Kerkyacharian, P. Natoli, D. Picard, N. Vittorio, Spherical needlets for CMB data analysis, Mon. Not. R. Astron. Soc. 383 (2) (2008) 539-545, arXiv:0707.0844.
[37] D. Marinucci, G. Peccati, Representations of SO(3) and angular polyspectra, J. Multivariate Anal. 101 (2010) 77-100, arXiv:0807.0687.
[38] A. Mayeli, Asymptotic uncorrelation for Mexican needlets, J. Math. Anal. Appl. 363 (1) (2010) 336-344, arXiv:0806.3009.
[39] J.D. McEwen, Fast, exact (but unstable) spin spherical harmonic transforms, arXiv:0807.4494 [astro-ph], 2008.
[40] F.J. Narcowich, P. Petrushev, J.D. Ward, Localized tight frames on spheres, SIAM J. Math. Anal. 38 (2006) 574-594.
[41] F.J. Narcowich, P. Petrushev, J.D. Ward, Decomposition of Besov and Triebel-Lizorkin spaces on the sphere, J. Funct. Anal. 238 (2) (2006) $530-564$.
[42] E.T. Newman, R. Penrose, Note on the Bondi-Metzner-Sachs group, J. Math. Phys. 7 (5) (1966) 863-870.
[43] D. Pietrobon, A. Balbi, D. Marinucci, Integrated Sachs-Wolfe effect from the cross correlation of WMAP3 year and the NRAO VLA sky survey data: New results and constraints on dark energy, Phys. Rev. D 74 (2006) 043524.
[44] D. Pietrobon, A. Amblard, A. Balbi, P. Cabella, A. Cooray, D. Marinucci, Needlet detection of features in WMAP CMB sky and the impact on anisotropies and hemispherical asymmetries, Phys. Rev. D 78 (2008) 103504, arXiv:0809.0010.
[45] D. Pietrobon, P. Cabella, A. Balbi, G. de Gasperis, N. Vittorio, Constraints on primordial non-Gaussianity from a needlet analysis of the WMAP-5 data, Mon. Not. R. Astron. Soc. 396 (3) (2009) 1682-1688, arXiv:0812.2478.
[46] O. Rudjord, F.K. Hansen, X. Lan, M. Liguori, D. Marinucci, S. Matarrese, An estimate of the primordial non-Gaussianity parameter $f_{N L}$ using the needlet bispectrum from WMAP, Astrophys. J. 701 (2009) 369-376, arXiv:0901.3154.
[47] O. Rudjord, F.K. Hansen, X. Lan, M. Liguori, D. Marinucci, S. Matarrese, Directional variations of the non-Gaussianity parameter $f_{N L}$, Astrophys. J. 708 (2) (2010) 1321-1325, arXiv:0906.3232.
[48] U. Seljak, M. Zaldarriaga, An all-sky analysis of polarization in the microwave background, Phys. Rev. D 55 (4) (1997) 1830-1840.
[49] D.A. Varshalovich, A.N. Moskalev, V.K. Khersonskii, Quantum Theory of Angular Momentum, World Scientific, Singapore, 1988.
[50] N.Ja. Vilenkin, A.U. Klimyk, Representation of Lie Groups and Special Functions, Kluwer, Dordrecht, 1991.
[51] Y. Wiaux, L. Jacques, P. Vandergheynst, Fast spin $\pm 2$ spherical harmonics and applications in cosmology, J. Comput. Phys. 226 (2007) $2359-2371$.


[^0]:    * Corresponding author.

    E-mail address: daryl@math.sunysb.edu (D. Geller).

[^1]:    ${ }^{1}$ See e.g. [3] for the definition of $\varepsilon$-net.

