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Coprimely Packed Rings

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An ideal I of a commutative ring R with identity is said to be coprimely packed by prime ideals of R if whenever I is coprime to each element of a family of prime ideals of R, I is not contained in the union of prime ideals of the family. We say that R is coprimely packed if every ideal of R is coprimely packed. It is shown that in a Noetherian arithmetical ring R every prime ideal is coprimely packed if and only if a positive power of every ideal of R is principal. Consequently for a Dedekind domain this is equivalent to the ideal class group being a torsion group (see also Theorem 2.2 of C. M. Reis and T. M. Viswanathan [A compactness property for prime ideals in Noetherian rings, *Proc. Amer. Math. Soc.* **25** (1970), 353–356]). We also show that every compactly packed ring R is coprimely packed. For characterizations of coprimely packed Prüfer domains from a different point of view see V. Erdoğdu [Modules with locally linearly ordered distributive hulls, *J. Pure Appl. Algebra* **47** (1987), 119–130].

In [4] a ring R is defined to be compactly packed by prime ideals if whenever an ideal I of R is contained in the union of a family of prime ideals of R, I is actually contained in one of the prime ideals of the family. In [5] it is shown that this property is equivalent to the condition that every prime ideal is the radical of a principal ideal. Here we generalize the notion of a compactly packed ring to a coprimely packed ring and examine the properties of such rings. It turns out that every compactly packed ring is coprimely packed. However, the converse is not true in general. But if R is an integral domain of Krull dimension one, then both conditions are equivalent.

An ideal I of a ring R is said to be coprimely packed if $I+P_s=R$ where P_s $(s\in S)$ are prime ideals of R; then $I\not\subseteq \bigcup_{s\in S} P_s$. A non-empty subset X of the set of prime ideals of R is said to be coprimely packed if whenever an element P of X is coprime to each element of a subset Y of X, then $P\not\subseteq \bigcup_{Q\in Y} Q$. If every ideal of R is coprimely packed then R is a coprimely packed ring.

Throughout R will denote a commutative ring with identity. Spec R will denote the set of prime ideals of R.

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1

A ring is said to be arithmetical if the lattice of ideals of R is distributive. We recall from [2, 3] the following known facts.

LEMMA 1.1. (i) *A Noetherian ring is arithmetical if and only if it is a direct sum of Dedekind domains and Artin principal ideal rings.*

(ii) *In a Noetherian arithmetical ring R every non-zero ideal of R is of the form $P_1^{v_1} P_2^{v_2} \cdots P_n^{v_n}$ where $P_1, P_2, ..., P_n$ are uniquely determined comaximal prime ideals of R and $v_1, v_2, ..., v_n$ are positive integers, and* $P_i^{v_i} \neq 0$.

THEOREM 1.2. *Let R be a Noetherian arithmetical ring. Then* Spec R *is coprimely packed if and only if a positive power of every ideal of R is principal.*

Proof. To prove that a positive power of every ideal of R is principal, in view of the above lemma it is enough to show that a positive power of every prime ideal of R is principal. Let P be a prime ideal of R and suppose that no positive power of P is principal. Let a be any non-zero element of P. Then $Ra = P_1^{v_1} P_2^{v_2} \cdots P_n^{v_n}$, for some prime ideals $P_1, P_2, ..., P_n$ of R and positive integers $v_1, v_2, ..., v_n$. We have two cases.

Case 1. $n = 1$. Then $Ra = P_1^{v_1} \subsetneqq P$ which implies that $P_1 \subsetneqq P$. That is, P is a maximal ideal of R and contains the non-zero prime ideal P_1. Now if $R = D_1 \oplus D_2 \oplus \cdots \oplus D_m \oplus R_{m+1} \oplus \cdots \oplus R_n$, where $D_1, D_2, ..., D_m$ are Dedekind domains and $R_{m+1}, ..., R_n$ are Artin principal ideal rings, then $P = D_1 \oplus D_2 \oplus \cdots \oplus D_{i-1} \oplus M_i \oplus D_{i+1} \oplus \cdots \oplus D_m \oplus R_{m+1} \oplus \cdots \oplus R_n$ and $P_1 = D_1 \oplus D_2 \oplus \cdots \oplus D_{i-1} \oplus 0 \oplus D_{i+1} \oplus \cdots \oplus D_m \oplus R_{m+1} \oplus \cdots \oplus R_n$, where M_i is a maximal ideal of D_i and 0 is the zero ideal of D_i. Since no power of P is principal, it follows that D_i cannot be a local (or a semilocal) domain. Thus D_i has a maximal ideal M_i' and $M_i' \neq M_i$ in D_i. But then $P' = D_1 \oplus D_2 \oplus \cdots \oplus D_{i-1} \oplus M_i' \oplus D_{i+1} \oplus \cdots \oplus D_m \oplus R_{m+1} \oplus \cdots \oplus R_n$ is a maximal ideal of R such that $Ra = P_1^{v_1} \subseteq P_1 \subsetneqq P'$ and $P + P' = R$.

Case 2. $n > 1$. Then from $Ra = P_1^{v_1} P_2^{v_2} \cdots P_n^{v_n} \subsetneqq P$ it follows that one of the P_i, say P_1 is contained in P and hence $P + P_j = R$, for all $j = 2, 3, ..., n$. Moreover we have $a \in P_i$, for all $i = 1, 2, ..., n$.

Thus in either case, there is a prime ideal Q of R such that $a \in Q$ and $Q + P = R$.

So by repeating this argument for each element of P we get a set X of prime ideals of R such that $P + Q = R$, for all $Q \in X$ and $P \subseteq \bigcup_{Q \in X} Q$. This is a contradiction to the fact that R is coprimely packed. Therefore a positive power of every prime ideal of R is principal.

Conversely, take P to be a prime ideal of R and suppose that there is a non-empty subset X of Spec R such that $P+Q=R$, for all $Q\in X$. Then $P\not\subseteq\bigcup_{Q\in X}Q$. [Since otherwise the condition implies that $P^v=Ra\subseteq P\subseteq\bigcup_{Q\in X}Q$ ($a\in P$, v is a positive integer). But then it follows that $P\subseteq Q$ for some $Q\in X$, a contradiction.] Therefore Spec R is coprimely packed.

We shall now characterize coprimely packed integral domains of Krull dimension 1.

PROPOSITION 1.3. *Let R be an integral domain of Krull dimension* 1. *Then the set of maximal ideals of R is coprimely packed if, and only if, for every non-empty subset X of* Max Spec R *and for every* $Q\in(\text{Max Spec } R)-X$, $R_Q\otimes_R(\bigcap_{P\in X}R_P)=K$ (= the field of fractions of R).

Proof. Let X be a non-empty subset of Max Spec R and let Q be an element of Max Spec R which is not in X. Then clearly for each $P\in X$, we have $P+Q=R$. Therefore $Q\not\subseteq\bigcup_{P\in X}P$ (since Max Spec R is coprimely packed). Hence $Q\cap(R-\bigcup_{P\in X}P)$ is non-empty. But then it follows that $R_Q\otimes_R(\bigcap_{P\in X}R_P)$ has no proper prime ideal and consequently $R_Q\otimes_R(\bigcap_{P\in X}R_P)=K$.

The converse is obvious.

THEOREM 1.4. *Let R be a Dedekind domain. Then the following statements are equivalent.*

(i) Max Spec R *is coprimely packed.*

(ii) *The ideal class group of R is torsion.*

(iii) *For each non-empty subset X of* Max Spec R *and for each* $Q\in(\text{Max Spec } R)-X$, $R_Q\otimes_R(\bigcap_{P\in X}R_P)$ *is the field of fractions of R.*

Proof. (i) $\Leftrightarrow$ (ii). Follows from Theorem 1.2.
(i) $\Leftrightarrow$ (iii). Follows from Proposition 1.3.

2

We shall now relate the concepts of compactly packed rings and coprimely packed rings.

PROPOSITION 2.1. *Every compactly packed ring is coprimely packed.*

Proof. Let R be a compactly packed ring and suppose that R is not coprimely packed. Then there is a non-zero ideal I of R and a non-empty subset X of Spec R such that $I+P=R$ for all $P \in X$ and $I \subseteq \bigcup_{P \in X} P$. Since R is compactly packed, $I \subseteq \bigcup_{P \in X} P$ implies that $I \subset P$ for some $P \in X$. This is a contradiction. Therefore R is coprimely packed.

PROPOSITION 2.2. *Let R be an integral domain of Krull dimension* 1. *Then R is compactly packed if and only if it is coprimely packed.*

Proof. It remains to show that if R is coprimely packed then it is compactly packed. Let I be a non-zero ideal of R and X a non-empty subset of Spec R. Suppose that $I \subseteq \bigcup_{P \in X} P$. Since R is coprimely packed, $I+P \neq R$, for some $P \in X$. Hence there is a maximal ideal M of R such that $I+P \subseteq M$. (Now in the subset X of Spec R, we may as well assume that $0 \notin X$, which clearly does not affect the assumption that $I \subseteq \bigcup_{P \in X} P$.) But R is an integral domain of Krull dimension 1. Therefore it follows that $P=M$ and $I \subseteq P$. Hence R is compactly packed.

THEOREM 2.3 (cf. [1, Theorem 3.1]). *Every semilocal ring is coprimely packed.*

Remarks. (1) It is not the case that every coprimely packed ring is compactly packed, for as a counter example take R to be the Noetherian local ring $K[[X_1, X_2, ..., X_n]]$, where K is a field, $n \geqslant 2$, and $X_1, X_2, ..., X_n$ are indeterminants. Then R is coprimely packed by the above Theorem. But R is not compactly packed, because by [4, Corollary 1.3] a compactly packed Noetherian ring is of Krull dimension $\leqslant 1$ and here R is of Krull dimension $n \geqslant 2$.

(2) If R is coprimely packed, then clearly every quotient ring R/J is coprimely packed. In particular, if a product $\prod_{i=1}^n R_i$ is coprimely packed, then each factor is coprimely packed.

(3) Let $\prod_{i=1}^n R_i$ be a product of domains where each factor is of dimension 1. If $\prod_{i=1}^n R_i$ is coprimely packed, so is each R_i, and hence each R_i is compactly packed (Proposition 2.2). Hence each prime in R_i is a radical of a principal ideal, and hence that is also true of $\prod_{i=1}^n R_i$. Consequently $\prod_{i=1}^n R_i$ is compactly packed.

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