Regularity of solutions of initial–boundary value problems for parabolic equations in domains with conical points

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Abstract

The purpose of this paper is to establish the well-posedness and the regularity of solutions of the initial–boundary value problems for general higher order parabolic equations in infinite cylinders with the bases containing conical points.

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1. Introduction

We are concerned with initial–boundary value problems for parabolic equations in nonsmooth domains. These problems with Dirichlet boundary condition in domains containing conical points have been investigated in [6,7]. The problems with Neumann boundary condition in domains with edges have been dealt with for the classical heat equation in [11] and for general second-order parabolic equations in [2]. In the present paper, we consider such problems for higher order linear parabolic equations with more general boundary conditions, provided they enable us to reduce the problems to ones of the variational form. Such boundary conditions have been considered for elliptic equations in [10].

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The main goal of this paper is to obtain the regularity of the solutions of the problems. There are some approaches to this issue. For parabolic equations of second order in smooth domains it were established in both Hölder and Sobolev spaces in [8] by the method in which a regularizer was constructed and exact estimates of solutions in terms of the data of the problems were dealt with. Such ideas were also used in [2] with some modifications for the case of domains with edges. For the equation dealt with in [11], whose coefficients are independent of the time variable, one used Fourier transform to reduce the problem to an elliptic one with a parameter. In the present paper, for a general higher order linear parabolic equation in domains containing conical points we modify the approach suggested in [3,6,7]. First, we study the unique solvability and the regularity with respect to the time variable for generalized solutions in the Sobolev space $H^{m,1}(Q)$ by Galerkin’s approximate method. By modifying the arguments used in [6,7], we can weaken the restrictions on the data at the initial time $t = 0$ imposed therein. After that, we take the term containing the derivative in time of the unknown function to the right-hand side of the equation such that the problem can be considered as an elliptic one. With the help of some auxiliary results we can apply the results for elliptic boundary value problems and our previous ones to deal with the regularity with respect to both of time and spatial variables of the solutions.

Our paper is organized as follows. In Section 2, we introduce some notations and the formulation of the problem. The main results, Theorems 3.1 and 3.2, are stated in Section 3. The proof of Theorem 3.1 is given in Section 4. In Section 5, we present some auxiliary results and the proof of Theorem 3.2.

2. Notation and formulation of the problem

Let $G$ be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with the boundary $\partial G$. We suppose that $\Gamma = \partial G \setminus \{0\}$ is a smooth manifold and $G$ in a neighborhood of the origin 0 coincides with the cone $K = \{x : x/|x| \in \Omega\}$ where $\Omega$ is a smooth domain on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. Set $Q_t = G \times (0, t)$ for each $t \in (0, +\infty)$, $Q = Q_\infty = G \times (0, +\infty)$, and $S = \Gamma \times [0, +\infty)$. For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, set $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_xj = -i\partial/\partial x_j$.

Let $l$ be a nonnegative integer. We denote by $H^l(G)$ the usual Sobolev space of functions defined in $G$ with the norm

$$
\|u\|_{H^l(G)} = \left( \int_G \sum_{|\alpha| \leq m} |D^\alpha u|^2 \, dx \right)^{\frac{1}{2}},
$$

and by $H^{l-\frac{1}{2}}(\Gamma)$ the space of traces of functions from $H^l(G)$ on $\Gamma$ with the norm

$$
\|u\|_{H^{l-\frac{1}{2}}(\Gamma)} = \inf \{ \|v\|_{H^l(G)} : v \in H^l(G), \, v|\Gamma = u \}.
$$

Let

$$
L = L(x, t, D) = \sum_{|\alpha|, |\beta| = 0}^m D^\alpha (a_{\alpha\beta}(x, t) D^\beta)
$$
be a differential operator of order $2m$ defined in $Q$ with coefficients infinitely differentiable up to the boundary, and let

$$
B_j = B_j(x, t, D) = \sum_{|\alpha| \leq m_j} b_{j,\alpha}(x, t) D^\alpha, \quad j = 1, \ldots, m,
$$

be a system of boundary operators on $S$ with coefficients infinitely differentiable in a neighborhood of $\partial G \times [0, +\infty)$, or $\text{ord} B_j \leq m_j$ for $j = 1, \ldots, \Lambda$, $m \leq \text{ord} B_j = m_j \leq 2m - 1$ for $j = \Lambda + 1, \ldots, m$. Suppose that for each $t \in [0, +\infty)$, ${B_j(x, t, D)}_{j=1}^m$ is a normal system on $\Gamma$ (for the definition, see [4, Definition 3.1.4]), and coefficients of $B_j$ are independent of $t$ if $\text{ord} B_j < m$.

We assume further that coefficients of $L$ and $B_j$ together with all derivatives are bounded in $Q$, $\partial G \times [0, +\infty)$, respectively, and $a_{\alpha\beta} = a_{\beta\alpha}$ for $|\alpha|, |\beta| \leq m$.

We assume that Green’s formula

$$
B(t, u, v) = \int_G L u v dx + \sum_{j=1}^{\Lambda} \int_{\Gamma} \Phi_j u \overline{B_j} v ds + \sum_{j=\Lambda+1}^{m} \int_{\Gamma} B_j u \overline{\Phi_j} v ds (2.1)
$$

is valid for all $u, v \in C^\infty_0(G \setminus \{0\})$ and a.e. $t \in [0, +\infty)$, where $\Phi_j$, $j = 1, \ldots, m$, are boundary operators on $S$, and

$$
B(t, u, v) = \sum_{|\alpha|, |\beta| = 0}^m \int_G a_{\alpha\beta}(., t) D^\beta u \overline{D^\alpha v} dx, \quad t \in [0, +\infty).
$$

We also suppose that the form $B(t, ., .)$ is $H^m(G)$-elliptic uniformly with respect to $t \in [0, +\infty)$, i.e. the inequality

$$
B(t, u, u) \geq \mu \|u\|^2_{H^m(G)} (2.2)
$$

is valid for all $u \in H^m(G)$ and all $t \in [0, +\infty)$, where $\mu$ is a positive constant independent of $u$ and $t$.

We proceed to introduce some functional spaces. We denote

$$
\mathcal{H}^m(G) = \left\{ u \in H^m(G) : B_j u = 0 \text{ on } \Gamma \text{ for } j = 1, \ldots, \Lambda \right\}
$$

with the same norm in $H^m(G)$. By $\mathcal{H}^{-m}(G)$ we denote the dual space to $\mathcal{H}^m(G)$.

We define the space $V^l_{2,\gamma}(G)$ as the closure of $C^\infty_0(G \setminus \{0\})$ with respect to the norm

$$
\|u\|_{V^l_{2,\gamma}(G)}^2 = \left( \sum_{|\alpha| \leq l} \int_G r^{2(\gamma + |\alpha| - l)} |D^\alpha u|^2 dx \right)^{\frac{1}{2}},
$$

where $r = |x| = (\sum_{k=1}^n x_k^2)^{\frac{1}{2}}$, and the space $H^l_{\gamma}(G)$ ($\gamma \in \mathbb{R}$) as the set of all functions in $G$ such that $r^\gamma D^\alpha u \in L^2(G)$ for $|\alpha| \leq l$ with the norm

$$
\|u\|_{H^l_{\gamma}(G)}^2 = \left( \sum_{|\alpha| \leq l} \int_G r^{2(\gamma + |\alpha| - l)} |D^\alpha u|^2 dx \right)^{\frac{1}{2}}.
$$
\[ \|u\|_{H^l_\gamma (G)} = \left( \sum_{|\alpha| \leq l} \int_G r^{2\gamma} |D^\alpha u|^2 \, dx \right)^{1/2}. \]

If \( l \geq 1 \), then \( V^l_{\gamma} (\Gamma) \), \( H^l_{\gamma} (\Gamma) \) denote the spaces consisting of traces of functions from respective spaces \( V^l_{2,\gamma} (G), H^l_\gamma (G) \) on the boundary \( \Gamma \) with the respective norms

\[ \|u\|_{V^l_{\gamma} (\Gamma)} = \inf \{ \|v\|_{V^l_{2,\gamma} (G)} : v \in V^l_{2,\gamma} (G), v|_\Gamma = u \}, \]
\[ \|u\|_{H^l_{\gamma} (\Gamma)} = \inf \{ \|v\|_{H^l_\gamma (G)} : v \in H^l_\gamma (G), v|_\Gamma = u \}. \]

Let \( X, Y \) be Banach spaces. We denote by \( L^2 (0, T; X) \) \((0 < T \leq +\infty)\) the space consisting of all measurable functions \( u : (0, T) \to X \) with the norm

\[ \|u\|_{L^2 (0, T; X)} = \left( \int_0^T \|u(t)\|^2_X \, dt \right)^{1/2}, \]

and by \( H^1 (0, T; X, Y) \) the space consisting of all functions \( u \in L^2 (0, T; X) \) such that the generalized derivative \( u_t = u' \) exists and belongs to \( L^2 (0, T; Y) \). The norm in \( H^1 (0, T; X, Y) \) is defined by

\[ \|u\|_{H^1 (0, T; X, Y)} = \left( \|u\|_{L^2 (0, T; X)}^2 + \|u_t\|_{L^2 (0, T; Y)}^2 \right)^{1/2}. \]

For shortness we set

\[ H^{l,0} (Q_T) = L^2 (0, T; H^l (G)), \quad H^{l,1} (Q_T) = H^1 (0, T; H^l (G), L^2 (G)), \]
\[ \mathcal{H}^{m,0} (Q_T) = L^2 (0, T; \mathcal{H}^m (G)), \quad \mathcal{H}^{m,1} (Q_T) = H^1 (0, T; \mathcal{H}^m (G), L^2 (G)), \]
\[ H^{-m,0} (Q_T) = L^2 (0, T; H^{-m} (G)), \quad H^{-m,1} (Q_T) = H^1 (0, T; H^{-m} (G), \mathcal{H}^{-m} (G)), \]
\[ V^l_{2,\gamma} (Q_T) = L^2 (0, T; V^l_{2,\gamma} (G)), \quad H^l_\gamma (Q_T) = L^2 (0, T; H^l_\gamma (G)), \]
\[ W^{m,1} (Q_T) = H^1 (0, T; \mathcal{H}^m (G), \mathcal{H}^{-m} (G)). \]

Finally, we define the weighted Sobolev space \( H^{2ml}_{\gamma} (Q) \) \((\gamma \in \mathbb{R})\) as the set of all functions defined in \( Q \) such that

\[ \|u\|_{H^{2ml}_{\gamma} (Q)} = \left( \int_Q \left( r^{2\gamma} \sum_{|\alpha| + 2mk \leq 2ml} |D^\alpha u_{t_k}|^2 + \sum_{k=0}^l |u_{t_k}|^2 \right) \, dx \, dt \right)^{1/2} < +\infty, \]

where \( u_{t_k} = \partial^k u / \partial t^k \).
We write \((\cdot, \cdot)\) to denote the pairing between \(H^m(G)\) and \(H^{-m}(G)\), and \((\cdot, \cdot)\) to denote the inner product in \(L^2(G)\). By identifying \(L^2(G)\) with its dual, we have the continuous imbeddings
\[ H^m(G) \hookrightarrow L^2(G) \hookrightarrow H^{-m}(G) \]
with the equation
\[ (f, v) = \langle f, v \rangle \text{ for } f \in L^2(G) \subset H^{-m}(G), v \in H^m(G). \]

In this paper we consider the following problem
\[ u_t + Lu = f \quad \text{in } Q, \]
\[ B_j u = 0 \quad \text{on } S, \ j = 1, \ldots, m, \]
\[ u = \varphi \quad \text{on } G, \]
where \(f : Q \to \mathbb{C}, \varphi : G \to \mathbb{C}\) are given functions.

Let \(f \in \mathcal{H}^{-m,0}(Q), \varphi \in L^2(G)\). A function \(u \in W^{m,1}(Q)\) is called a generalized solution of the problem (2.3)–(2.5) iff \(u(., 0) = \varphi\) and the equality
\[ (u_t, \bar{v}) + B(t, u, v) = \langle f(t), \bar{v} \rangle \]
holds for a.e. \(t \in (0, +\infty)\) and all \(v \in \mathcal{H}^m(G)\).

3. Formulation of the main results

To establish the regularity of the solution of the problem (2.3)–(2.5), one needs to impose the compatibility conditions on the known functions which we formulate as follows:

Let \(\varphi \in H^{(2h+1)m}_{y}(G), f \in H^{2hm}_{y}(Q)\), where \(h\) is a positive integer, \(y \leq m\). We set
\[ \varphi_0 = \varphi, \quad \varphi_1 = f(., 0) - L(x, 0, D)\varphi_0, \ldots, \]
\[ \varphi_h = f_{h-1}(., 0) - \sum_{k=0}^{h-1} \binom{h-1}{k} L_{i^{h-1}-k}(x, 0, D)\varphi_k, \] (3.1)

and
\[ \mathcal{H}^{2m}_{y}(G) = \{ u \in H^{2m}_{y}(G): B_j(x, 0, D)u = 0 \text{ on } \Gamma \text{ for } j = 1, \ldots, m \}, \]
where
\[ L_{i^k} = L_{i^k}(x, t, D) = \sum_{|\alpha|, |\beta| = 0}^{m} D^\alpha \left( \frac{\partial^k a_{\alpha\beta}(x, t)}{\partial t^k} D^\beta \right). \]

We say that the \(h\)th-order compatibility conditions are fulfilled if
\[ \varphi_0, \varphi_1, \ldots, \varphi_{h-1} \in \mathcal{H}^{2m}_{y}(G), \] (3.2)
and, in addition,
\[ \varphi_h \in \mathcal{H}^m(G), \quad f_{th} \in L_2(Q). \]

Let \( L_0(x, t, D), B_{0j}(x, t, D) \) be the principal homogeneous parts of \( L(x, t, D), B_j(x, t, D) \). We can write \( L_0(0, t, D), B_{0j}(0, t, D) \) in the form
\begin{align}
L_0(0, t, D) &= r^{-2m} P(\omega, t, D_\omega, rD_r), \quad (3.3) \\
B_{0j}(0, t, D) &= r^{-mj} P_j(\omega, t, D_\omega, rD_r), \quad (3.4)
\end{align}
where \( r = |x| \), \( \omega \) is an arbitrary local coordinate system on \( \mathbb{S}^{n-1} \), \( D_r = -i \partial / \partial r \). We denote by \( U(\lambda, t)(\lambda \in \mathbb{C}, t \in (0, +\infty)) \) the operator of the parameter-depending boundary problem
\begin{align}
P(\omega, t, D_\omega, \lambda) &= f \quad \text{in} \quad \Omega, \quad (3.5) \\
P_j(\omega, t, D_\omega, \lambda) &= g_j \quad \text{on} \quad \partial \Omega, \quad j = 1, \ldots, m. \quad (3.6)
\end{align}
For every fixed \( \lambda \in \mathbb{C} \) this operator continuously maps
\[ H^l(\Omega) \to H^{l-2m}(\Omega) \times \prod_{j=1}^m H^{l-mj-\frac{1}{2}}(\partial \Omega) \quad (l \geq 2m). \]

For each \( t \in (0, +\infty) \) we have the operator pencil \( U(\lambda, t) \) which has the spectrum being an enumerable set of eigenvalues (see [4, Theorem 5.2.1]).

Now let us give the main results of the present paper:

**Theorem 3.1.** Let \( h \) be a nonnegative integer and \( \gamma \) be a real number, \( \gamma \leq m \). Assume that \( \varphi \in H^{2(h+1)m}(G) \), \( f \in H^{2hm}(Q) \) and \( h \)th-order compatibility conditions are fulfilled if \( h \geq 1 \). Then the problem (2.3)–(2.5) has a unique generalized solution \( u \in W^{m,1}(Q) \), moreover,
\[ u_{tk} \in \mathcal{H}^{m,1}(Q) \quad \text{for} \quad k = 0, \ldots, h, \quad (3.7) \]
and
\[ \sum_{k=0}^h \|u_{tk}\|^2_{\mathcal{H}^{m,1}(Q)} \leq C(\|f\|^2_{H^{2hm}(Q)} + \|\varphi\|^2_{H^{2(h+1)m}(G)}), \quad (3.8) \]
where \( C \) is the constant independent of \( u, f, \varphi \).

**Theorem 3.2.** Suppose that the assumptions of Theorem 3.1 hold. Assume further that \( 0 \leq \gamma \leq m \) and the strip \( \gamma - 2hm - 2m + \frac{n}{2} \leq \text{Im} \lambda \leq -m + \frac{n}{2} \) does not contain any eigenvalue of \( U(\lambda, t) \) for all \( t \in (0, +\infty) \). Then \( u \in H^{2(h+1)m}(Q) \) and
\[ \|u\|^2_{H^{2(h+1)m}(Q)} \leq C(\|f\|^2_{H^{2hm}(Q)} + \|\varphi\|^2_{H^{2(h+1)m}(G)}), \quad (3.9) \]
where \( C \) is the constant independent of \( u, f, \varphi \).
4. Proof of Theorem 3.1

For simplicity in the following we will write \( v(t) \) instead of \( v(., t) \) for functions \( v(x, t) \) defined on \( Q \). For integer \( k \geq 0 \), \( u, v \in H^m,0 (Q_T) \), \( t \in [0, +\infty) \) we set

\[
B_{t^k}(t, u, v) = \sum_{|\alpha|, |\beta| \leq m} \int G \partial^k a_{\alpha\beta}(x, t) D^\beta u(x, t) D^\alpha v(x, t) dx,
\]

\[
B_{t^k}^T(u, v) = \int_0^T B_{t^k}(t, u, v) dt, \quad B^T(u, v) = B_{t^0}^T(u, v).
\]

Lemma 4.1. Let \( F(t, ., .) \) be a bilinear form on \( H^m(G) \times H^m(G) \) satisfying

\[
|F(t, v, w)| \leq C \|v\|_{H^m(G)} \|w\|_{H^m(G)} \quad (C = \text{const}) \tag{4.1}
\]

for all \( t \in [0, +\infty) \) and all \( v, w \in H^m(G) \), and \( F(., v, w) \) is measurable on \([0, +\infty)\) for each pair \( v, w \in H^m(G) \). Assume that \( u \in W^{m,1}(Q) \) satisfies \( u(0) \equiv 0 \) and

\[
[u_t(t), v] + B(t, u(t), v) = \int_0^t F(\tau, u(\tau), v) d\tau \tag{4.2}
\]

for a.e. \( t \in [0, +\infty) \) and all \( v \in H^m(G) \). Then \( u \equiv 0 \) on \( Q \).

Proof. Substituting \( v := u(t) \) into (4.2), then integrating both sides of the obtained equality with respect to \( t \) from 0 to \( b \) (\( b > 0 \)), after all using the assumptions (2.2), (4.1), we arrive at

\[
\frac{1}{2} \|u(b)\|_{L^2(G)}^2 + \mu \|u\|_{H^m,0(Q_b)}^2 \leq C \int_0^b \int_0^t \|u(t)\|_{H^m(G)} \|u(\tau)\|_{H^m(G)} d\tau dt
\]

\[
\leq \frac{1}{2} C \int_0^b \int_0^t (\|u(t)\|_{H^m(G)}^2 + \|u(\tau)\|_{H^m(G)}^2) d\tau dt
\]

\[
\leq bC \|u\|_{H^m,0(Q_b)}^2.
\]

Choosing \( b \leq \frac{\mu}{2C} \), we have \( \frac{1}{2} \|u(b)\|_{L^2(G)}^2 + \mu \|u\|_{H^m,0(Q_b)}^2 \leq 0 \). This implies \( u \equiv 0 \) on \([0, \frac{\mu}{2C}]\).

Repeating this argument we can show that \( u \equiv 0 \) on intervals \([ \frac{\mu}{2C}, \frac{\mu}{C}], [ \frac{\mu}{C}, \frac{3\mu}{2C}], \ldots \), and, therefore, \( u \equiv 0 \) on \( Q \). \( \square \)

Lemma 4.2. If \( f \in H^{-m,0}(Q) \), \( \varphi \in L^2(G) \), then there exists a unique generalized solution \( u \in W^{m,1}(Q) \) of the problem (2.3)–(2.5).
Proof. The uniqueness of the solution follows directly from Lemma 4.1. We will prove its existence. By the assumption $a_{\alpha \beta} = a_{\beta \alpha}$ for $|\alpha|, |\beta| \leq m$, $L$ is a formally self-adjoint operator. Moreover, $H^m(G)$ is compact imbedded in $L^2(G)$. Thus, the operator $L(x, 0, D)$ possesses a set $\{\psi_k\}_{k=1}^\infty$ consisting of all its eigenfunctions, which is not only an orthogonal basis of $H^m(G)$ but also an orthonormal basis of $L^2(G)$. For each positive integer $N$, we consider the function $u^N(x, t) = \sum_{k=1}^N C_k^N(t)\psi_k(x)$, where $\{C_k^N(t)\}_{k=1}^N$ is the solution of the ordinary differential system

$$
\begin{align*}
(u_t^N, \psi_l) + B(t, u^N, \psi_l) &= \langle f, \psi_l \rangle, \quad l = 1, \ldots, N, \\
C_k^N(0) &= C_k, \quad k = 1, \ldots, N.
\end{align*}
$$

(4.3)

Here $C_k = (\varphi, \psi_k)$, $k = 1, 2, \ldots$. After multiplying both sides of (4.3) by $C_l^N(t)$, taking sum with respect to $l$ from 1 to $N$, and integrating with respect to $t$ from 0 to $T$ ($T > 0$), we arrive at

$$
\int_0^T (u_t^N, u^N) dt + B^T(u^N, u^N) = \int_0^T \langle f, u^N \rangle dt.
$$

(4.5)

Adding (4.5) with its complex conjugate, we obtain

$$
\|u^N(T)\|^2_{L^2(G)} + 2B^T(u^N, u^N) = \|u^N(0)\|^2_{L^2(G)} + 2\text{Re} \int_0^T \langle f, u^N \rangle dt.
$$

(4.6)

Noting that $\|u^N(0)\|^2_{L^2(G)} = \|\sum_{k=1}^N (\varphi, \psi_k)\psi_k\|^2_{L^2(G)} \leq \|\varphi\|^2_{L^2(G)}$ and

$$
\left| 2\text{Re} \int_0^T \langle f, u^N \rangle dt \right| \leq 2 \int_0^T \|f\|_{H^{-m}(G)} \|u^N\|_{H^m(G)} dt
$$

$$
\leq \epsilon \|u^N\|^2_{H^{m,0}(Q_T)} + \frac{1}{\epsilon} \|f\|^2_{H^{-m,0}(Q_T)}
$$

(0 $< \epsilon < 2\mu$), and using the assumption (2.2), we have from (4.6) that

$$
\|u^N\|^2_{H^{m,0}(Q_T)} \leq C(\|\varphi\|^2_{L^2(G)} + \|f\|^2_{H^{-m,0}(Q_T)}).
$$

(4.7)

Sending $T \to +\infty$, we obtain

$$
\|u^N\|^2_{H^{m,0}(Q)} \leq C(\|\varphi\|^2_{L^2(G)} + \|f\|^2_{H^{-m,0}(Q)}).
$$

(4.7)

Now fix any $v \in H^m(G)$ with $\|v\|_{H^m(G)} \leq 1$, and write $v = v_1 + v_2$, where $v_1 \in \text{span}\{\psi_l\}_{l=1}^N$, $v_2, \psi_l)_{L^2(G)} = 0$, $l = 1, \ldots, N$. Since the functions $\{\psi_l\}_{l=1}^N$ are orthogonal in $H^m(G)$, $\|v_1\|_{H^m(G)} \leq \|v\|_{H^m(G)} \leq 1$. We obtain from (4.3) that

$$
(u_t^N, v_1) + B(t, u^N, v_1) = \langle f, v_1 \rangle.
$$
Therefore,
\[ \langle u^N_t, v \rangle = \langle u^N_t, v_1 \rangle = \langle f, v_1 \rangle - B(t, u^N, v_1). \]

Hence, we get
\[ \| \langle u^N_t, v \rangle \| \leq C(\| f \|_{\mathcal{H}^{-m}(G)} + \| u^N \|_{\mathcal{H}^m(G)}) \]
since \( \| v_1 \|_{\mathcal{H}^m(G)} \leq 1 \). Thus,
\[ \| u^N_t \|_{\mathcal{H}^{-m}(G)} \leq C(\| f \|_{\mathcal{H}^{-m}(G)} + \| u^N \|_{\mathcal{H}^m(G)}), \]
and therefore, by (4.7),
\[ \| u^N_t \|_{\mathcal{H}^{-m,0}(Q)}^2 \leq C(\| f \|_{\mathcal{H}^{-m,0}(Q)}^2 + \| u^N \|_{\mathcal{H}^m(G)}^2) \]
\[ \leq C(\| \varphi \|_{L^2(G)}^2 + \| f \|_{\mathcal{H}^{-m,0}(Q)}^2). \quad (4.8) \]

Combining (4.7) and (4.8), we get
\[ \| u^N_t \|_{W^{m,1}(Q)}^2 \leq C(\| \varphi \|_{L^2(G)}^2 + \| f \|_{\mathcal{H}^{-m,0}(Q)}^2), \quad (4.9) \]
where \( C \) is a constant independent of \( \varphi, f \) and \( N \). From this estimate, by the same arguments as in [1, Chapter 7, Theorem 3], we conclude that there exists a subsequence of \( \{ u^N \} \) which weakly converges to a generalized solution \( u \in W^{m,1}(Q) \) of the problem (2.3)–(2.5).

**Lemma 4.3.** Let \( \varphi \in \mathcal{H}^m(G) \) and \( f \in L^2(Q) \) or \( f \in \mathcal{H}^{-m,1}(Q) \). Then the generalized solution \( u \in W^{m,1}(Q) \) of the problem (2.3)–(2.5) in fact belongs to \( \mathcal{H}^{m,1}(Q) \) and the following estimate
\[ \| u \|_{\mathcal{H}^{m,1}(Q)}^2 \leq C(\| \varphi \|_{L^2(G)}^2 + \| f \|_{X}^2) \quad (4.10) \]
holds with the constant \( C \) independent of \( g, f \), and \( u \). Here \( X = L^2(Q) \) or \( \mathcal{H}^{-m,1}(Q) \) which \( f \) belongs to.

**Proof.** (i) Let us consider first the case \( f \in L^2(Q) \). Let \( u^N \) be the functions defined as in the proof of Theorem 4.2 with \( C_k = (\varphi, \psi_k) \) \((k = 1, 2, \ldots)\) replaced by
\[ C_k = \| \psi_k \|_{\mathcal{H}^m(G)}^2 (\varphi, \psi_k)_{\mathcal{H}^m(G)}, \]
where \((\ldots)_{\mathcal{H}^m(G)}\) denotes the inner product in \( \mathcal{H}^m(G) \). Multiplying both sides of (4.3) by \( \frac{dC_N}{dt} \), then taking sum with respect to \( l \) from 1 to \( N \), after that integrating with respect to \( t \) from 0 to \( T \) \((0 < T < +\infty)\), and adding the attained equality with its complex conjugate, we arrive at
\[ 2\| u^N_t \|_{L^2(Q_T)}^2 + \sum_{|\alpha|,|\beta|\leq 0}^m \int_{Q_T} a_{\alpha\beta} \frac{\partial}{\partial t} (D^\beta u^N D^\alpha u^N) \, dx \, dt = 2 \text{Re} \int_0^T (f, u^N_t) \, dt. \]
By the integration by parts, we get

\[ 2\|u_t^N\|_{L^2(Q_T)}^2 + B(T, u^N, u^N) = B(0, u^N, u^N) + B_T^T(u^N, u^N) + 2 \text{Re} \int_0^T (f, u_t^N) \, dt. \] (4.11)

Since \(a_{\alpha\beta}, \frac{\partial a_{\alpha\beta}}{\partial t}\) are bounded on \(\overline{Q}\), using Cauchy’s inequality, we get

\[ |B(0, u^N, u^N)| \leq C\|u^N(0)\|_{H^m(G)}^2 \leq C\|\varphi\|_{H^m(G)}^2, \]
\[ |B_T^T(u^N, u^N)| \leq C\|u^N\|_{H^{m,0}(Q_T)}^2, \]
\[ |2 \text{Re} \int_0^T (f, u_t^N) \, dt| \leq \epsilon \|u_t^N\|^2_{L^2(Q_T)} + \frac{1}{4\epsilon} \|f\|^2_{L^2(Q_T)} \quad (0 < \epsilon < 2). \]

Hence, it follows from (4.7) and (4.11) that

\[ \|u_t^N\|^2_{L^2(Q_T)} \leq C(\|\varphi\|_{H^m(G)}^2 + \|f\|^2_{L^2(Q_T)}). \]

Sending \(T \to +\infty\), we obtain

\[ \|u_t^N\|^2_{L^2(Q)} \leq (\|\varphi\|^2_{H^m(G)} + \|f\|^2_{L^2(Q)}). \] (4.12)

Combining (4.7) and (4.12), we have

\[ \|u^N\|^2_{H^{m,1}(Q)} \leq C(\|\varphi\|^2_{H^m(G)} + \|f\|^2_{L^2(Q)}). \] (4.13)

This implies that the sequence \(\{u^N\}\) contains a subsequence which weakly converges to a function \(v \in H^{m,1}(Q)\). Passing to the limit of the subsequence, we can see that \(v\) is a generalized solution of the problem (2.3)–(2.5). Thus, \(u = v \in H^{m,1}(Q)\). The estimate (4.10) with \(X = L^2(Q_T)\) follows from (4.13).

(ii) Now let \(f \in H^{-m,1}(Q)\). Then \(f\) is continuous on \([0, +\infty)\) and has the representation \(f(t) = f(s) + \int_s^t f_1(\tau) \, d\tau\) for all \(s, t \in [0, +\infty)\) (see [1, Section 5.9, Theorem 2]). This implies

\[ \|f(t)\|^2_{H^{-m}(G)} \leq 2\|f(s)\|^2_{H^{-m}(G)} + 2 \int_J \|f_1(\tau)\|^2_{H^{-m}(G)} \, d\tau, \] (4.14)

where \(J = [a, b] \subset [0, +\infty)\) such that \(a \leq s, t \leq b\) and \(b - a = 1\). Integrating both sides of (4.14) with respect to \(s\) on \(J\), we obtain

\[ \|f(t)\|^2_{H^{-m}(G)} \leq 2\|f\|^2_{H^{-m,1}(Q)} \quad (t \in [0, +\infty)). \] (4.15)

Now by the same way to get (4.11), we have

\[ 2\|u_t^N\|_{L^2(Q_T)} + B(T, u^N, u^N) = B(0, u^N, u^N) + B_T^T(u^N, u^N) + 2 \text{Re} \int_0^T (f, u_t^N) \, dt. \] (4.16)
Noting that \( \int_0^T \langle f, u_t^N \rangle \, dt = - \int_0^T \langle f_1, u^N \rangle \, dt + \langle f, u^N \rangle \big|_0^T \), and using (4.15), we obtain
\[
\left| \int_0^T \langle f, u_t^N \rangle \, dt \right| \leq \| f_t \|_{\mathcal{H}^{-m,0}(Q)} \| u^N \|_{\mathcal{H}^m,0(Q)} + \| f(T) \|_{\mathcal{H}^{-m}(G)} \| u^N(T) \|_{\mathcal{H}^m(G)}
\]
\[
+ \| f(0) \|_{\mathcal{H}^{-m}(G)} \| u^N(0) \|_{\mathcal{H}^m(G)}
\]
\[
\leq C(\epsilon) \| f \|^2_{\mathcal{H}^{-m,1}(Q)}
\]
\[
+ \epsilon \left( \| u^N \|^2_{\mathcal{H}^m,0(Q_T)} + \| u^N(T) \|^2_{\mathcal{H}^m(G)} + \| u^N(0) \|^2_{\mathcal{H}^m(G)} \right).
\] (4.17)

Using (4.7), (2.2) and (4.17) for \( 0 < \epsilon < \mu \), we get from (4.16) that
\[
\| u_t^N \|^2_{L^2(Q_T)} \leq C \left( \| \varphi \|^2_{\mathcal{H}^m(G)} + \| f \|^2_{\mathcal{H}^{-m,1}(Q)} \right).
\] (4.18)

Sending \( T \to +\infty \), we can see
\[
\| u_t^N \|^2_{L^2(Q)} \leq C \left( \| \varphi \|^2_{\mathcal{H}^m(G)} + \| f \|^2_{\mathcal{H}^{-m,1}(Q)} \right).
\] (4.19)

From this, by the same argument as in the part (i) above, we obtain the assertion of the lemma for the case \( f \in \mathcal{H}^{-m,1}(Q) \). \( \square \)

**Remark.** It follows from the proof of Lemma 4.3 that if \( \varphi \in \mathcal{H}^m(G) \) and \( f = f_1 + f_2 \), where \( f_1 \in L^2(Q) \), \( f_2 \in \mathcal{H}^{-m,1}(Q) \) then the generalized solution \( u \in \mathcal{W}^m,1(Q) \) of the problem (2.3)–(2.5) belongs to \( \mathcal{H}^m,1(Q) \) and the estimate (4.10) holds with \( \| f \|^2_{X} \) replaced by \( \| f_1 \|^2_{L^2(Q)} + \| f_2 \|^2_{\mathcal{H}^{-m,1}(Q)} \).

By Hardy’s inequality, we have (see, e.g., [4, Lemma 7.1.1])
\[
\| u \|^2_{\mathcal{H}^m(G)} \leq C \| u \|^2_{\mathcal{H}^{2m,0}(G)} \leq C \| u \|^2_{\mathcal{H}^{2m}(G)}
\] (4.20)
for all \( u \in \mathcal{H}_\gamma^m(G) \), where \( \gamma \leq m \), \( C \) is a constant independent of \( u \). Thus, if \( \varphi, f \) satisfy the \( h \)th-order compatibility conditions, we have
\[
\varphi_k \in \mathcal{H}^m(G), \quad f_{t,k} \in L^2(Q), \quad k = 0, \ldots, h.
\] (4.21)

**Proof of Theorem 3.1.** We will show by induction that not only the assertions (3.7), (3.8) but also the following equalities hold:
\[
u_{t,k}(0) = \varphi_k, \quad k = 1, \ldots, h.
\] (4.22)
and
\[
(u_{h+1}, \eta) + \sum_{k=0}^{h} \binom{h}{k} B_{h-k} (t, u_{tk}, \eta) = (f_{h}, \eta) \quad \text{for all } \eta \in \mathcal{H}^m(G). \tag{4.23}
\]

The case \( h = 0 \) follows from Lemmas 4.2, 4.3. Assuming now that they hold for \( h - 1 \), we will prove them for \( h (h \geq 1) \). We consider first the following problem: find a function \( v \in \mathcal{W}^{m,1}(Q) \) satisfying
\[
\langle v_t, \bar{\eta} \rangle + B(t, v, \eta) = (f_{h}, \eta) - \sum_{k=0}^{h-1} \binom{h}{k} B_{h-k} (t, u_{tk}, \eta) \tag{4.24}
\]
for all \( \eta \in \mathcal{H}^m(G) \) and a.e. \( t \in (0, +\infty) \).

Let \( F(t), t \in [0, +\infty) \), be functionals defined by
\[
\langle F(t), \eta \rangle = (f_{h}, \eta) - \sum_{k=0}^{h-1} \binom{h}{k} B_{h-k} (t, u_{tk}, \eta), \quad \eta \in \mathcal{H}^m(G). \tag{4.25}
\]

Then \( F \in \mathcal{H}^{-m,0}(Q) \) by the inductive assumption. Hence, according to Lemma 4.2, the problem (4.24) has a solution \( v \in \mathcal{W}^{m,1}(Q) \). We put now
\[
w(x,t) = \phi_{h-1}(x) + \int_0^t v(x, \tau) d\tau, \quad x \in G, \ t \in [0, +\infty).
\]
Then we have \( w(0) = \phi_{h-1}, w_t = v, w_t(0) = \phi_h \). It follows from (4.24) that
\[
\langle w_{tt}, \bar{\eta} \rangle + \frac{\partial}{\partial t} B(t, w, \eta) = (f_{h}, \eta) + B_t(t, w - u_{h-1}, \eta) - \sum_{k=0}^{h-2} \binom{h-1}{k} B_{h-1-k} (t, u_{tk}, \eta). \tag{4.26}
\]

It follows from equality (2.1) that
\[
B_{tk} (t, \omega, \eta) = \int_G L_{tk} (x, t, D) \omega \eta \, dx
\]
for all \( \omega \in \mathcal{H}^{2m}(G), \eta \in \mathcal{H}^m(G) \) and all \( t \in [0, +\infty), k \) is an arbitrary nonnegative integer. Thus, we have from (3.1) and (3.2) that
\[
(\phi_h, \eta) = (f_{h-1}(0), \eta) - \sum_{k=0}^{h-1} \binom{h-1}{k} B_{h-1-k} (0, \phi_k, \eta). \tag{4.27}
\]

Now integrating equality (4.26) with respect to \( t \) from 0 to \( t \) and using (4.27), we arrive at...
\begin{align}
\langle w_t, \eta \rangle + B(t, w, \eta) = & \left(f_t, \eta \right) + \int_0^t B_t(\tau, w - u_{t,h-1}, \eta) \, d\tau - \sum_{k=0}^{h-1} \binom{h-1}{k} B_{t,h-1-k}(t, u_{t,k}, \eta). \\
& \tag{4.28}
\end{align}

Put \( z = w - u_{t,h-1} \). Then \( z(0) = 0 \) since \( u(0) = \varphi_{h-1} \). It follows from the inductive assumption (4.23) with \( h \) replaced by \( h - 1 \) and (4.28) that
\begin{align}
\langle z_t(t), \eta \rangle + B(t, z(t), \eta) = & \int_0^t B_t(\tau, z(\tau), \eta) \, d\tau. \\
& \tag{4.29}
\end{align}

Applying Lemma 4.1, we can see from (4.29) that \( z \equiv 0 \) on \( Q \). Therefore, \( u_{t,h} = w_t = v \in \mathcal{W}^{m,1}(Q) \).

Now we show that in fact \( u_{t,h} \in \mathcal{H}^{m,1}(Q) \). We rewrite (4.24) in the form
\begin{align}
\langle v_t, \eta \rangle + B(t, v, \eta) = & \left(f_t, \eta \right) \quad \text{with } \hat{F}(t) \text{ defined by}
\langle \hat{F}(t), \eta \rangle = - \sum_{k=0}^{h-1} \binom{h}{k} B_{t,h-k}(t, u_{t,k}, \eta), \quad \eta \in \mathcal{H}^m(G). \\
& \tag{4.31}
\end{align}

Since \( u_{t,k} \in \mathcal{H}^{m,0}(Q) \) for \( k = 0, \ldots, h \), we can see from (4.31) that \( \hat{F}_t \in \mathcal{H}^{-m,0}(G) \) and
\begin{align}
\langle \hat{F}_t(t), \eta \rangle = & - \sum_{k=0}^{h-1} \binom{h+1}{k} B_{t,h+1-k}(t, u_{t,k}, \eta) - h B_t(t, u_{t,h}, \eta), \quad \eta \in \mathcal{H}^m(G). \\
& \tag{4.32}
\end{align}

Then, according to the remark below Lemma 4.3, we obtain from (4.30) that \( u_{t,h} \in \mathcal{H}^{m,1}(Q) \). The desired estimate holds since \( \| f_t \|_{L^2(Q)} \) and \( \| \hat{F}_t \|_{\mathcal{H}^{-m,1}(Q)} \) can be estimated by the right-hand side of (3.8). The proof is completed. □

5. Proof of Theorem 3.2

The following lemma can be proved similarly to Theorems 4.2, 4.2’ of [9].

**Lemma 5.1.** For every fixed \( t_0 \in [0, +\infty) \) let \( u \in H^{l+2m}_{loc}(G \setminus \{0\}) \cap V^0_{2,\gamma-1-2m}(G) \) be a solution of the problem
\begin{align}
L(x, t_0, D)u = f \quad & \text{in } G, \\
B_j(x, t_0, D)u = g_j \quad & \text{on } \Gamma, \quad j = 1, \ldots, m, \\
& \tag{5.1}
\end{align}

where \( f \in V^l_{2,\gamma}(G) \), \( g_j \in V^{l+2m-m_j-\frac{1}{2}}_{2,\gamma}(\Gamma) \), \( l \) is a nonnegative integer. Then \( u \in V^{l+2m}_{2,\gamma}(G) \) and the following estimate
\[ \|u\|_{V_{2,\gamma}^{l+2m}(G)}^2 \leq C \left( \|f\|_{V_{2,\gamma}^l(G)}^2 + \sum_{j=1}^{m} \|g_j\|_{V_{2,\gamma}^{l+2m-m_j-\frac{1}{2}}(G)}^2 + \|u\|_{V_{0,\gamma-I-2m}^0(G)}^2 \right) \]  \tag{5.3}

holds with the constant \( C \) independent of \( u, f, g_j \) and \( t_0 \).

Let \( \varepsilon \) be an arbitrary positive number. We introduce the following integral operator

\[ (Kw)(r) = \xi(r) \int_{\frac{1}{2}}^{1} w(tr) \psi(t) \, dt \quad \text{for } 0 < r < \varepsilon, \]  \tag{5.4}

where \( \xi \) is a cut-off function on \([0, +\infty)\) equal to one in \([0, \varepsilon)\) and to zero outside \([0, \varepsilon)\), and \( \psi \in C_0^\infty((\frac{1}{2}, 1)) \) satisfying the condition \( \int_{\frac{1}{2}}^{1} \psi(t) \, dt = 1 \). For \( r > \varepsilon \) we set \((Kw)(r) = 0.\)

It is known (see [4, Lemma 7.3.3]) that \( K \) is a continuous mapping

\[ H_{\frac{1}{2}}^1((0, \varepsilon)) \rightarrow H_{-\frac{1}{2}}^1((0, +\infty)) \]  \tag{5.5}

for arbitrary integer \( l \geq 1 \), where \( H_{\frac{1}{2}}^1((0, \varepsilon)) \) is the space of all functions defined on \((0, \varepsilon)\) with the finite norm

\[ \|u\|_{H_{\frac{1}{2}}^1((0, \varepsilon))} = \left( \|u\|_{L_2((0, \varepsilon))}^2 + \int_0^\varepsilon \int_0^\varepsilon \frac{|u(r) - u(\rho)|^2}{r - \rho} \, dr \, d\rho \right)^{\frac{1}{2}}. \]

It is obvious that \( V_{2,\gamma}^l(G) \) is continuously imbedded in \( H_{\gamma}^l(G) \). We have continuous imbeddings (see [4, p. 192])

\[ V_{2,\gamma}^l(G) \subset V_{2,\gamma-k}^{l-k}(G) \quad \text{for } 0 \leq k \leq l, \]

and

\[ V_{2,\gamma}^{l-\frac{1}{2}}(\Gamma) \subset V_{2,\gamma-k}^{l-k-\frac{1}{2}}(\Gamma) \quad \text{for } 0 \leq k < l. \]

In the following, by \( p_k(u) \) we mean the Taylor polynomial at the point \( x = 0 \) of degree \( k \) of the function \( u \) defined in \( G \) if it exists.

**Lemma 5.2.** Let \( u \in H_{\gamma}^l(G) \), where \( 0 < \gamma + \frac{n}{2} \leq l \). Then for an arbitrary integer \( k \geq 0 \), \( u \) admits the representation \( u = v + w \), where \( v \in V_{2,\gamma}^l(G) \) and \( w \in H_{\gamma+k}^{l+k}(G) \), moreover,

\[ \|v\|_{V_{2,\gamma}^l(G)}^2 + \|w\|_{H_{\gamma+k}^{l+k}(G)}^2 \leq C \|u\|_{H_{\gamma}^l(G)}^2 \]  \tag{5.6}

with the constant \( C \) independent of \( u \).

If in addition \( u|_\Gamma \in V_{2,\gamma-k}^{l-\frac{1}{2}}(\Gamma) \), \( q \) is an integer < \( l \), \( l \geq 1 \), then \( u|_\Gamma \in V_{2,\gamma}^{l-\frac{1}{2}}(\Gamma) \).
Proof. Let $s = [\gamma + \frac{n}{2}]$ be the greatest integer not exceeding $\gamma + \frac{n}{2}$. Denote by $\zeta$ a smooth function equal to one near the origin and to zero outside a neighborhood in which $G$ coincides with the cone $K$. By the [4, Theorems 7.2.1, 7.3.2], $u \in H^{l}_{\gamma}(G)$ can be written in the form $u = v + w$, where $v \in V^{l}_{2,\gamma}(G)$ with the norm estimated by $\|u\|_{H^{l}_{\gamma}(G)}$, and,

$$w = \zeta p_{l-s-1}(u) + \zeta \sum_{|\alpha|=l-s}^{\cdot} (Ku_{\alpha})(|x|) \frac{x^{\alpha}}{\alpha !},$$

$u_{\alpha}$ are functions from $H^{l}_{\gamma}((0, \epsilon))$ (the sum in the representation of $w$ above is absent if $\gamma + \frac{n}{2}$ is not integer). Since the coefficients of $p_{l-s-1}(u)$ are estimated by $\|u\|_{H^{l}_{\gamma}(G)}$ (see the theorems quoted above), then $\zeta p_{l-s-1}(u)$ are estimated by $\|u\|_{H^{l}_{\gamma}(G)} (C = \text{const})$.

By (5.5) and [4, Lemma 7.3.1], we also see that $\zeta \sum_{|\alpha|=l-s}^{\cdot} (Ku_{\alpha})(|x|) \frac{x^{\alpha}}{\alpha !} \in H^{l+k}_{\gamma+k}(G)$ with the norm estimated by $\|u\|_{H^{l}_{\gamma}(G)}$. Thus, we have proved the first assertion of the lemma.

We have $v \in V^{l-\frac{1}{2}}_{2,\gamma}(G) \subset V^{l-q-\frac{1}{2}}_{2,\gamma-q}(G)$. If $u|_{\gamma} \in V^{l-q-\frac{1}{2}}_{2,\gamma-q}(G)$, then $w|_{\gamma} = u|_{\gamma} - v|_{\gamma} \in V^{l-q-\frac{1}{2}}_{2,\gamma-q}(G)$. Then, according to [4, Lemma 7.1.7, 7.3.5], $p_{l-s-1}(u)|_{\gamma} \equiv 0$ and $u_{\alpha}$ can be taken to be vanishing functions. Thus, $u|_{\gamma} = v|_{\gamma} \in V^{l-\frac{1}{2}}_{2,\gamma}(G)$. The proof is completed. 

Lemma 5.3. For every fixed $t_{0} \in (0, +\infty)$ let $f \in H^{0}_{m}(G)$ and $u \in \mathcal{H}^{m}(G)$ be a generalized solution of the problem (5.1), (5.2), i.e. $u$ satisfies the identity

$$B(t_{0}, u, \eta) = (f, \eta) \quad \text{for all } \eta \in \mathcal{H}^{m}(G).$$

Then $u \in H^{2m}_{m}(G)$ and

$$\|u\|^{2}_{H^{2m}_{m}(G)} \leq C\|f\|^{2}_{H^{m}_{m}(G)} + \|u\|^{2}_{\mathcal{H}^{m}(G)}, \quad (5.7)$$

where the constant $C$ is independent of $u$, $f$ and $t_{0}$.

Proof. According to results for elliptic boundary value problem in domains with smooth boundaries, we have $u \in H^{2m}_{\text{loc}}(\overline{G} \setminus \{0\})$. If $m < \frac{n}{2}$, then $H^{m}(G) = V^{m}_{2,0}(G)$ by [4, Theorem 7.1.1]. Thus the assertion of the lemma follows from Lemma 5.1.

Let us consider the case $m \geq \frac{n}{2}$. According to Lemma 5.2, $u \in H^{m}(G)$ can be written in the form $u = v + w$, where $v \in V^{m}_{2,0}(G)$, $w \in H^{2m}_{m}(G)$, and

$$\|v\|^{2}_{V^{m}_{2,0}(G)} + \|w\|^{2}_{H^{2m}_{m}(G)} \leq C\|u\|^{2}_{H^{m}(G)}, \quad (C = \text{const}) \quad (5.8)$$

Now we rewrite (5.1), (5.2) in the form
where \( F = f - L(x, t_0, D)w \in H^0_m(G) = V^0_{2,m}(G) \), \( \psi_j = -B_j(x, t_0, D)w \in H^{2m-m_j-\frac{1}{2}}_m(\Gamma) \).

We show now that

\[
\psi_j \in V^{2m-m_j-\frac{1}{2}}_{2,m}(\Gamma), \quad j = 1, \ldots, m.
\]

Indeed, if \( m_j > m - \frac{n}{2} \), then, by [4, Theorem 7.1.1], \( H^{2m-m_j}_{m}(G) = V^{2m-m_j}_{2,m}(G) \), and therefore, \( H^{2m-m_j-\frac{1}{2}}_m(\Gamma) = V^{2m-m_j-\frac{1}{2}}_{2,m}(\Gamma) \). Thus, \( \psi_j \in V^{2m-m_j-\frac{1}{2}}_{2,m}(\Gamma) \) for \( m_j > m - \frac{n}{2} \). Now let us fix \( j \in \{0, \ldots, m\} \) with \( m_j \leq m - \frac{n}{2} \), and let \( \Psi_j \in H^{2m-m_j}_{m}(G) \) be an extension of \( \psi_j \) into the domain \( G \). Then we have \( \Psi_j|_{\Gamma} = \psi_j \in V^{2m-m_j-\frac{1}{2}}_{2,0}(\Gamma) \) since \( v \in V^m_{2,0}(G) \). This implies \( \psi_j = \Psi_j|_{\Gamma} \in V^{2m-m_j-\frac{1}{2}}_{2,m}(\Gamma) \) according to Lemma 5.2. Consequently, (5.11) holds.

Now applying Lemma 5.1, we can see from (5.9), (5.10) that \( v \in V^{2m}_{2,m}(G) \). Therefore, \( u = v + w \in H^{2m}_m(G) \). The estimate (5.7) follows from (5.3) and (5.8).

**Lemma 5.4.** Let \( u \in H^{l+2m,0}_y(Q) \) be a solution of the problem

\[
L(x, t, D)u = f \quad \text{in} \quad Q,
\]

\[
B_j(x, t, D)u = g_j \quad \text{on} \quad S, \quad j = 1, \ldots, m,
\]

where \( f \in H^{0}_{\delta}(Q), g_j \in H^{k+2m-m_j-\frac{1}{2},0}(S), l, k \) are nonnegative integers, \( k - \delta > l - \gamma \). Suppose that the strip \( \delta - k - 2m + \frac{n}{2} \leq \text{Im} \lambda \leq -2m + \frac{n}{2} \) does not contain any eigenvalue of \( \mathcal{U}(\lambda, t) \) for all \( t \in (0, +\infty) \) and \( \gamma + \frac{n}{2} \) is not an integer. Then \( u \in H^{k+2m,0}_{\delta}(Q) \) and

\[
\|u\|^2_{H^{k+2m,0}_{\delta}(Q)} \leq C \left( \|f\|^2_{H^{0}_{\delta}(Q)} + \sum_{j=1}^{m} \|g_j\|^2_{H^{k+2m-m_j-\frac{1}{2},0}(S)} + \|u\|^2_{H^{l+2m,0}_y(Q)} \right)
\]

with the constant \( C \) independent of \( u, f, g_j \).

**Proof.** First, we fix \( t \in (0, +\infty) \) and consider (5.12), (5.13) as an elliptic boundary value problem. Since coefficients of \( L(x, t, D), B_j(x, t, D) \) are bounded smooth functions, as a special case of the results for elliptic boundary problems in weighted Sobolev spaces with nonhomogeneous norms (see [4, Chapter 7]) we can see from (5.12), (5.13) that \( u(t) \in H^{k+2m}_{\delta}(G) \) and

\[
\|u(t)\|^2_{H^{k+2m}_{\delta}(G)} \leq C \left( \|f(t)\|^2_{H^{0}_{\delta}(G)} + \sum_{j=1}^{m} \|g_j(t)\|^2_{H^{k+2m-m_j-\frac{1}{2},0}(\Gamma)} + \|u(t)\|^2_{H^{l+2m}_y(G)} \right),
\]

where the constant \( C \) is independent of \( u, f, g_j \) and \( t \). Now integrating both sides of (5.15) with respect to \( t \) from 0 to \( +\infty \), we get the assertion of the lemma. \( \square \)
Proof of Theorem 3.2. The proof is an induction on \( h \). Let us consider first the case \( h = 0 \). We rewrite Eq. (2.3) in the form

\[
Lu = f_1 := f - u_t \quad \text{in } Q.
\] (5.16)

According to Theorem 3.1, we have \( u_t \in L_2^2(Q) \). Thus, \( f_1 \in H_m^{1,0}(Q) \) since \( 0 \leq \gamma \leq m \). By Lemma 5.3, it follows from (5.16) that \( u(.,t) \in H_2^{2m}(G) \) and

\[
\| u(t) \|^2_{H_2^{2m}(G)} \leq C \left( \| f_1(t) \|^2_{L_2^2(Q)} + \| u(t) \|^2_{H_2^m(G)} \right)
\] (5.17)

for a.e. \( t \in (0, +\infty) \), where \( C \) is a constant independent of \( u, f, f_1 \) and \( t \). Integrating both sides of (5.17) with respect to \( t \) from 0 to \( +\infty \), we obtain

\[
u(t) \in H_2^{2m}(G) \]

Choose \( \epsilon \) such that \( 0 \leq \epsilon < \frac{1}{2} \) and \( \epsilon + \frac{n}{2} \) is not an integer. It is known that the strip \( -m + \frac{n}{2} < \Im \lambda \leq -m + \epsilon + \frac{n}{2} \) does not contain any eigenvalue of \( U_\lambda(t) \) for all \( t \in (0, +\infty) \) (see [5, p. 392]). This and the assumptions of the theorem imply that the strip \( \gamma - 2m + \frac{n}{2} < \Im \lambda \leq -m + \epsilon + \frac{n}{2} \) is free of eigenvalues of \( U_\lambda(t) \) for all \( t \in (0, +\infty) \). Hence, we have \( u \in H_\gamma^{2m,0}(Q) \) by Lemma 5.4. This and the fact that \( u_t \in L_2^2(Q) \) imply \( u \in H_\gamma^{2m,0}(Q) \). Thus, the theorem is valid for \( h = 0 \).

Assume that it is true for some nonnegative \( h - 1 \). We will prove it for \( h \). We have to show that \( u \in H_\gamma^{2(h+1)m}(Q) \). To this end, it is only needed to make clear that

\[
u_{ik} \in H_\gamma^{2(h-k+1)m,0}(Q)
\] (5.18)

for \( k \leq h + 1 \). We will also prove these by induction on \( k \). By Theorem 3.1, \( u_{h+1} \in L_2^2(Q) \). This means that (5.18) holds for \( k = h + 1 \). Assume that it holds for \( k = h + 1, h, \ldots, p + 1 \) \((0 < p < h)\). Differentiating both sides of (5.16) with respect to \( t \) \( p \) times, we have

\[
Lu_{tp} = f_{tp} - u_{tp+1} - \sum_{s=0}^{p-1} \binom{p}{s} L_{tp-s} u_{tp}.
\] (5.19)

By the supposition of the theorem and the inductive assumption, the right-hand side of (5.19) belongs to \( H_\gamma^{2(h-p)m,0}(Q) \). Hence, by Lemma 5.4, \( u_{tp} \in H_\gamma^{2(h-p+1)m,0}(Q) \). Thus, (5.18) holds for \( k \leq h + 1 \), and the proof is done. \( \Box \)

References


