A globally convergent method based on Fischer–Burmeister operators for solving second-order cone constrained variational inequality problems

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ABSTRACT

The Karush–Kuhn–Tucker system of a second-order cone constrained variational inequality problem is transformed into a semismooth system of equations with the help of Fischer–Burmeister operators over second-order cones. The Clarke generalized differential of the semismooth mapping is presented. A modified Newton method with Armijo line search is proved to have global convergence with local superlinear rate of convergence under certain assumptions on the variational inequality problem. An illustrative example is given to show how the globally convergent method works.

1. Introduction

Fischer–Burmeister (FB) function, as one of the complementarity functions, has been widely and deeply studied for dealing nonlinear complementarity problems and variational inequality problems with polyhedral cone constraints, see the famous book by Facchinei and Pang [1]. Recently, some researchers have studied a lot of methods for complementarity problems, variational inequality problems and nonsmooth equations, see [2,3,1,4–8] and extended Fischer–Burmeister (FB) function to Fischer–Burmeister operator over the second-order cone so that second-order cone complementarity problems and second-order cone constrained optimization problems can be solved as in the cases of polyhedral cone constraints, see for instances [2,1,5,9,10]. However, because of the nondifferentiability of FB function, most researchers cannot obtain the nonsingularity of the Clarke generalized Jacobian of the Fischer–Burmeister (FB) function when removing the strict complementarity condition.

In this paper, we use the Fischer–Burmeister operator over the second-order cone to deal with second-order cone constrained variational inequality (SOCCI) problems. The Karush–Kuhn–Tucker system of a second-order cone constrained variational inequality problem is transformed into a semismooth system of equations (namely \( \Phi_{FB}(x, \mu, \lambda) = 0 \)) with the help of Fischer–Burmeister operators over second-order cones. The differentiability of the mapping \( \Phi_{FB} \) at the Karush–Kuhn–Tucker point is guaranteed when the strict complementarity condition holds, whereas a modified mapping \( \hat{\Phi}_{FB} \) of \( \Phi_{FB} \) is proved to have the differentiability property at the Karush–Kuhn–Tucker point without the strict complementarity condition. Furthermore, the formula for the Clarke generalized differential of the semismooth mapping \( \Phi_{FB} \) at the

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The second-order cone (SOC) in \( \mathbb{R}^n \) (\( n \geq 1 \)), also called the Lorentz cone or the ice-cream cone, is defined as
\[
\mathcal{K}^n = \{(x_1; x_2) \mid x_1 \in \mathbb{R}, x_2 \in \mathbb{R}^{n-1} \text{ and } x_1 \geq \|x_2\|\}.
\]
If \( n = 1 \), \( \mathcal{K}^n \) is the set of nonnegative reals \( \mathbb{R}_+ \). Here and below, \( \| \cdot \| \) is the \( l_2 \)-norm. For any \( x = (x_1; x_2), \ y = (y_1; y_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \), we define their Jordan product as
\[
x \cdot y = (x^T y; y_1 x_2 + x_1 y_2).
\]
Denote \( x^2 = x \cdot x \) and \( |x| = \sqrt{x^2} \), where for any \( y \in \mathcal{K}^n \), \( \sqrt{y} \) is the unique vector in \( \mathcal{K}^n \) such that \( y = \sqrt{y} \cdot \sqrt{y} \).

A mapping \( \phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) is called a complementarity operator over the second-order cone if
\[
\phi(x, y) = 0 \iff \mathcal{K}^n \ni x \perp y \in \mathcal{K}^n,
\]
where \( x \perp y \iff x \cdot y = 0 \). Fischer–Burmeister operator has the following expression
\[
\phi_{FB}(x, y) = x + y - \sqrt{x^2 + y^2},
\]
which is a complementarity operator over the second-order cone. From [9], we know that \( \phi_{FB} \) is strongly semismooth.

In this paper, we are interested in solving the variational inequality whose constraints involve the Cartesian product of second-order cones (SOCs). The problem is to find \( x \in C \) satisfying
\[
\langle f(x), y - x \rangle \geq 0, \quad \forall y \in C,
\]
where the set \( C \) is finitely representable as
\[
C = \{x \in \mathbb{R}^n : h(x) = 0, \ -g(x) \in \mathcal{K}^m\}.
\]
\( \langle \cdot, \cdot \rangle \) denotes the Euclidean inner product, \( h : \mathbb{R}^n \to \mathbb{R}^l \) and \( g : \mathbb{R}^n \to \mathbb{R}^m \) are continuously differentiable functions and
\[
\mathcal{K}^m = \mathcal{K}^{m_1} \times \mathcal{K}^{m_2} \times \cdots \times \mathcal{K}^{m_p}
\]
with \( l \geq 0, m_1, m_2, \ldots, m_p \geq 1 \) and \( m_1 + m_2 + \cdots + m_p = m \). We will refer to (1.1)–(1.3) as the second-order cone constrained variational inequality (SOCVI) problem.

An important special case of the SOCVI corresponds to the Karush–Kuhn–Tucker (KKT) conditions of the convex second-order cone program (CSOCP):
\[
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad Ax = b, \\
& \quad g(x) \in -\mathcal{K}^m,
\end{align*}
\]
where \( A \in \mathbb{R}^{l \times n} \) has full row rank, \( b \in \mathbb{R}^l, g : \mathbb{R}^n \to \mathbb{R}^m \) and \( f : \mathbb{R}^n \to \mathbb{R} \). When \( f \) is a convex twice continuously differentiable function, problem (1.4) is equivalent to the following SOCVI problem: Find \( x \in C \) satisfying
\[
\langle \nabla f(x), y - x \rangle \geq 0, \quad \forall y \in C,
\]
where the set \( C \) is finitely representable as
\[
C = \{x \in \mathbb{R}^n : Ax - b = 0, \ -g(x) \in \mathcal{K}^m\}.
\]

In what follows, \( I \) represents an identity matrix of suitable dimension, \( \mathbb{R}^n \) denotes the space of \( n \)-dimensional real column vectors, and \( \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m} \) is identified with \( \mathbb{R}^{n_1+n_2+\cdots+n_m} \). For a closed convex cone \( \mathcal{K}^n \), we denote \( \text{int}((\mathcal{K}^n)), \ bd(\mathcal{K}^n) \) and \( bd^*(\mathcal{K}^n) \) by the interior, the boundary and the boundary excluding the origin, respectively. For any \( x, y \in \mathcal{K}^n \), we write \( x \geq y \) or \( x \leq y \) (respectively, \( x > y \) or \( x < y \)) if \( x - y \) or \( y - x \) is in \( \mathcal{K}^m \) (respectively, \( \text{int}(\mathcal{K}) \)). For any two square matrices \( A, B \in \mathbb{R}^{m \times m} \), we write \( A > B \) (respectively, \( A \geq B \)) if the symmetric part of \( A - B \), namely \( (A - B + A^T - B^T)/2 \), is positive definite (respectively, positive semidefinite). For any Fréchet-differentiable mapping \( F : \mathbb{R}^n \to \mathbb{R}^m \), we denote its Jacobian at \( x \in \mathbb{R}^n \) by \( JF(x) \in \mathbb{R}^{m \times n} \), i.e., \( JF(x) = (F(x + u) - F(x))/\|u\| \to 0 \) as \( u \to 0 \). Let \( F \) be locally Lipschitz at \( x \in \mathbb{R}^n \), if \( F \) is not differentiable at \( \tilde{x} \in \mathbb{R}^n \), then the Clarke generalized Jacobian of \( F \) at \( \tilde{x} \) is:
\[
\partial F(\tilde{x}) \equiv \text{convjacF}(\tilde{x}) = \text{conv}JF(\tilde{x}) = \text{conv}\partial F(\tilde{x}) = \text{conv}\{H \in \mathbb{R}^{m \times n} : H = \lim_{k \to \infty} JF(\tilde{x})^k\},
\]
for some sequence \( \{\tilde{x}^k\} \to \tilde{x} \cdot x \in N_F \), where we denote by \( N_F \) the negligible set of point at which \( F \) is not differentiable.
2. Preliminaries

In this section, we recall some background materials and preliminary results that will be used later. For each \( x = (x_1; x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \), the determinant and the trace of \( x \) are defined by

\[
\det(x) = x_1^2 - \|x_2\|^2 \quad \text{and} \quad \text{tr}(x) = 2x_1,
\]

respectively. Unlike matrices, we have in general \( \det(x \cdot y) \neq \det(x)\det(y) \) unless \( x = \alpha y \) for some \( \alpha \in \mathbb{R} \). A vector \( x = (x_1; x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \) is said to be invertible if \( \det(x) \neq 0 \), and its inverse is denoted by \( x^{-1} \), satisfying \( x \cdot x^{-1} = e \), where \( e = (1, 0, \ldots, 0)^T \in \mathbb{R}^n \).

Direct calculation yields \( x^{-1} = \frac{\det(x) - x_1 x_2^T}{\det(x)} \). Clearly, \( x \in \text{int}(\mathcal{K}^n) \) if and only if \( x^{-1} \in \text{int}(\mathcal{K}^n) \). For any \( x = (x_1; x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \), we define the following symmetric matrix

\[
L_x = \begin{bmatrix} x_1 & x_2^T \\ x_2 & x_1 \end{bmatrix}
\]

which can be viewed as a linear mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^n \). It is easily verified that \( L_x y = x \cdot y, L_{x+y} = L_x + L_y \) for all \( y \in \mathbb{R}^n \), and \( L_x \) is positive definite (and hence invertible) if and only if \( x \in \text{int}(\mathcal{K}^n) \). If \( x \in \text{int}(\mathcal{K}^n) \), then \( L_x \) is invertible with

\[
L_x^{-1} = \frac{1}{\det(x)} \begin{bmatrix} x_1 & -x_2^T \\ -x_2 & \det(x)I - x_2x_2^T \\ \end{bmatrix}.
\]

It follows from [11] that each \( x = (x_1; x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \) admits a spectral factorization, associated with \( \mathcal{K}^n \), of the form

\[
x = \rho_1 u^{(1)} + \rho_2 u^{(2)},
\]

where \( \rho_1, \rho_2 \) and \( u^{(1)}, u^{(2)} \) are the spectral values and the associated spectral vectors of \( x \) given by

\[
\rho_i = x_1 + (-1)^i \|x_2\|,
\]

\[
u^{(i)} = \begin{cases} \frac{1}{2} \begin{pmatrix} 1; (-1)^i \frac{x_2}{\|x_2\|} \end{pmatrix}, & \text{if } x_2 \neq 0, \\ \frac{1}{2} \begin{pmatrix} 1; (-1)^i w \end{pmatrix}, & \text{if } x_2 = 0. \end{cases}
\]

for \( i = 1, 2 \), with \( w \) being any vector in \( \mathbb{R}^{n-1} \) satisfying \( \|w\| = 1 \). If \( x_2 \neq 0 \), the factorization is unique. The spectral deposition along with the Jordan algebra associated with second-order cone has some basic properties as below, whose proofs can be found in [12,11].

**Proposition 2.1.** For any \( x = (x_1; x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \) with the spectral values \( \rho_1, \rho_2 \) and spectral vectors \( u^{(1)}, u^{(2)} \) given as above, we have that

(a) \( u^{(1)} \) and \( u^{(2)} \) are orthogonal under Jordan product and have length \( \frac{1}{\sqrt{2}} \), that is

\[
u^{(1)} \cdot u^{(2)} = 0, \quad \|u^{(1)}\| = \|u^{(2)}\| = \frac{1}{\sqrt{2}}.
\]

(b) \( u^{(1)} \) and \( u^{(2)} \) are idempotent under Jordan product, that is \( u^{(i)} u^{(i)} = u^{(i)} \) for \( i = 1, 2 \).

(c) \( x \in \mathcal{K}^n \) if and only if \( \rho_1 \geq 0 \), and \( x \in \text{int}(\mathcal{K}^n) \) if and only if \( \rho_1 > 0 \).

(d) \( x^2 = \rho_1^2 u^{(1)} + \rho_2^2 u^{(2)} \in \mathcal{K}^n \).

(e) \( x^{1/2} = \sqrt{\rho_1} u^{(1)} + \sqrt{\rho_2} u^{(2)} \in \mathcal{K}^n \) if \( x \in \mathcal{K}^n \).

(f) \( \det(x) = \rho_1 \rho_2, \text{tr}(x) = \rho_1 + \rho_2 \) and \( \|x\| = [\rho_1^2 + \rho_2^2]/2 \).

(g) \( \rho_1, \rho_2 \) are the eigenvalues of the \( n \times n \) matrix \( L_x \) with \( u^{(1)}, u^{(2)} \) being the corresponding eigenvectors. The remaining \( n-2 \) eigenvalues of this matrix are identically \( x_1 \), with corresponding eigenvectors of the form \( (0, v) \), where \( v \) lies the subspace of \( \mathbb{R}^{n-1} \) orthogonal to \( x_2 \).

Throughout the paper, \( F, g \) and \( h \) are continuously differentiable.

3. Equation reformulation for the KKT conditions

The KKT conditions of the SOCCVI are

\[
F(x) + Jh(x)^T \mu + Jg(x)^T \lambda = 0,
\]

\[
h(x) = 0, \quad 0 \geq g(x) \perp \lambda \geq 0.
\]

(3.1)
In what follows, we write \( x = (x_{m_1}; \ldots; x_{m_p}) \in \mathbb{R}^m \) to implicitly mean \( x_{m_i} \in \mathbb{R}^{m_i}, i = 1, \ldots, p. \) Then using the direct product structure of (1.3), \( g(x) \perp \lambda \) can be written equivalently as

\[
-g_{m_i}(x) \perp \lambda_{m_i}, \quad -g_{m_i}(x) \in \mathcal{K}^{m_i}, \quad \lambda_{m_i} \in \mathcal{K}^{m_i}, \quad i = 1, \ldots, p.
\]

Let

\[
\Phi_{FB}(x, \mu, \lambda) = \begin{cases} \frac{L(x, \mu, \lambda)}{\rho_{21}^2}, & \text{if } z_2 = 0, \\ \frac{\rho_{21}^2}{\sqrt{\rho_{21}^2}} \begin{bmatrix} \frac{cz_z^T}{\|z_2\|} \\ aL(x, \mu, \lambda) \end{bmatrix}, & \text{if } z_2 \neq 0, \end{cases}
\]

where

\[
L(x, \mu, \lambda) \equiv F(x) + fh(x)^T \mu + fg(x)^T \lambda
\]

is the variational inequality Lagrangian function. Then (3.1) is equivalent to

\[
\Phi_{FB}(x, \mu, \lambda) = 0.
\]

**Proposition 3.1.** Let \( f : \mathcal{K}^n \to \mathcal{K}^n \) be defined by \( f(x) = \sqrt{x}, \forall x \in \text{int}(\mathcal{K}^n). \) Then its Jacobian at \( z = (z_1; z_2) \in \mathcal{K}^n \) and \( z \neq 0 \) is given by

\[
Jf(z) = \frac{1}{2} L_{w}^{-1} = \begin{bmatrix} 1 \\ b \end{bmatrix} \begin{bmatrix} \frac{cz_z^T}{\|z_2\|} \\ aL(x, \mu, \lambda) \end{bmatrix}
\]

where \( w = \sqrt{z}, a = \frac{\rho_{21}^2}{\rho_1^2 - \rho_{21}^2}, b = \frac{1}{4} \left( \frac{1}{\sqrt{z_2}} + \frac{1}{\sqrt{z_1}} \right), c = \frac{1}{4} \left( \frac{1}{\sqrt{z_2}} - \frac{1}{\sqrt{z_1}} \right) \) with \( \rho_2 = z_1 + (-1)^i \|z_2\|, \ i = 1, 2, \) and \( Jf(z) \) is positive definite for all \( z \in \mathcal{K}^n. \)

**Proof.** By the Proposition 5.2 and Corollary 5.2 in [11], we can easily get the formula. \( \square \)

Using Proposition 3.1 and following the same argument as [5], we have Proposition 3.2.

**Proposition 3.2.** Given a general point \((x, y) \in \mathbb{R}^n \times \mathbb{R}^n, \) each element \( V \in \partial_{\theta} \Phi_{FB}(x, y) \) has the following representation:

(a) If \( x^2 + y^2 \in \text{int}(\mathcal{K}^n), \) \( \Phi_{FB}(x, y) \) is continuously differentiable at \((x, y)\) and

\[
V = J\Phi_{FB}(x, y) = [I - L_w^{-1} L_y],
\]

where \( w = \sqrt{z}, a = \frac{\rho_{21}^2}{\rho_1^2 - \rho_{21}^2}, b = \frac{1}{4} \left( \frac{1}{\sqrt{z_2}} + \frac{1}{\sqrt{z_1}} \right), c = \frac{1}{4} \left( \frac{1}{\sqrt{z_2}} - \frac{1}{\sqrt{z_1}} \right) \) with \( \rho_2 = z_1 + (-1)^i \|z_2\|, \ i = 1, 2, \) and \( Jf(z) \) is positive definite for all \( z \in \mathcal{K}^n. \)

(b) If \( x^2 + y^2 \in bd^+ (\mathcal{K}^n), \) then \( V \in \{U(x, y), \} \), where

\[
U(x, y) = \begin{bmatrix} [I, I] - \frac{1}{2} \begin{bmatrix} \frac{z_1}{z_2} & 4I - 3\bar{z}_2 z_2^T \end{bmatrix} \\ -\frac{1}{2} \begin{bmatrix} 4I - 3\bar{z}_2 z_2^T & \bar{z}_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \bar{z}_2 \\ \bar{z}_2 \end{bmatrix}
\]

for some vectors \( u = (u_1; u_2), \ v = (v_1; v_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \) satisfying \( |u_1| \leq \|u_2\| \leq 1 \) and \( |v_1| \leq \|v_2\| \leq 1 \) and \( \bar{z}_2 = \frac{z_2}{\|z_2\|}. \)

(c) If \((x, y) = (0, 0), \) then \( V \in \{U \} \cup \{[L_0 - I, L_0 - I] \}, \) where

\[
U = [I, I] - \frac{1}{2} \begin{bmatrix} U_1, U_2 \end{bmatrix},
\]

\[
U_1 = \begin{bmatrix} \xi_1 + u_1 & (\xi_1 - u_1) \bar{z}_2^T + 4s_2^T (I - \bar{z}_2 \bar{z}_2^T) \\ \xi_2 + u_2 & (\xi_2 - u_2) \bar{z}_2^T + 4s_1^T (I - \bar{z}_2 \bar{z}_2^T) \end{bmatrix}
\]

and

\[
U_2 = \begin{bmatrix} \eta_1 + v_1 & (\eta_1 - v_1) \bar{z}_2^T + 4s_2^T (I - \bar{z}_2 \bar{z}_2^T) \\ \eta_2 + v_2 & (\eta_2 - v_2) \bar{z}_2^T + 4s_1^T (I - \bar{z}_2 \bar{z}_2^T) \end{bmatrix}
\]

for some vectors \( \xi = (\xi_1; \xi_2), \ \eta = (\eta_1; \eta_2), \ u = (u_1; u_2), \ v = (v_1; v_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \) satisfying \( |\xi_1| \leq |\xi_2| \leq 1, \ |\eta_1| \leq |\eta_2| \leq 1, \ |u_1| \leq \|u_2\| \leq 1 \) and \( |v_2| \leq \|v_2\| \leq 1. \)

\( \bar{z}_2 \in \mathbb{R}^{n-1} \) satisfying \( \|\bar{z}_2\| = 1, \) and \( s = (s_1; s_2), \ \omega = (\omega_1; \omega_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \) such that \( |s_1| \leq \|s_2\| \leq 1 \) and \( |\omega_1| \leq \omega_2 \leq 1, \) and \( \bar{u} = (\bar{u}_1; \bar{u}_2), \ \bar{v} = (\bar{v}_1; \bar{v}_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \) such that \( \|\bar{u}_1\| + \|\bar{u}_2\| \leq 1 \) and \( |\bar{v}_1| + \|\bar{v}_2\| \leq 1. \)
By the semismoothness of $\Phi_{FB}$, we know that $\Phi_{FB}$ is semismooth. The following result identifies the structure of the matrices in the generalized Jacobian $\partial \Phi_{FB}(x, \mu, \lambda)$.

**Proposition 3.3.** The generalized Jacobian $\partial \Phi_{FB}(x, \mu, \lambda)$ is contained in the following family of matrices:

$$J(x, \mu, \lambda) = \begin{cases} J_L(x, \mu, \lambda) & J_h(x)^T & J_g(x)^T \\ -\text{diag}(\partial_x \Phi_{FB}(-g_i(x), \lambda_i))_{i=1}^p & 0 & 0 \\ -\text{diag}(\partial_x \Phi_{FB}(-g_i(x), \lambda_i))_{i=1}^p & 0 & \text{diag}(\partial_x \Phi_{FB}(-g_i(x), \lambda_i))_{i=1}^p \end{cases},$$

where $\partial_x \Phi_{FB}(-g_i(x), \lambda_i)$ and $\partial_x \Phi_{FB}(-g_i(x), \lambda_i)$ are given as in Proposition 3.2 with $x, y$ replaced by $-g_i(x)$ and $\lambda_i$, respectively.

**Proof.** The containment of $\partial \Phi_{FB}(x, \mu, \lambda)$ in $J(x, \mu, \lambda)$ follows easily from Propositions 7.1.14 and 7.1.11 in [1].

Now we study the nonsingularity of differential $\partial \Phi_{FB}(x, \mu, \lambda)$ at a point. We need Lemma 3.1 to get our goal.

**Lemma 3.1 (Lemma 3.1 in [1]).** Let $x \in \mathbb{K}^n, y \in \mathbb{K}^n$ and $w = \sqrt{x^2 + y^2}$, then we have

$$(L_w - L_x)(L_w - L_y) \succeq 0, \quad L_w - L_x \succeq 0, \quad L_w - L_y \succeq 0.$$

**Theorem 3.1.** Assume that $-g_i(x) + \lambda_i \in \text{int}(\mathbb{K}^m)$ holds for all block components $i = 1, 2, \ldots, p$. Let $(x, \mu, \lambda) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$ be any given triple. If

(a) The gradients $\{\nabla h_j(x) : j = 1, \ldots, l\} \cup \{\nabla g_i(x) : i = 1, \ldots, m\}$ are linear independent;
(b) $J_L(x, \mu, \lambda)$ is positive definite on the null space of the gradients $\{\nabla h_j(x) : j = 1, \ldots, l\}$.

then the Jacobian $J \Phi_{FB}(x, \mu, \lambda)$ is nonsingular.

**Proof.** By Propositions 3.2 and 3.3, $J \Phi_{FB}(x, \mu, \lambda)$ has the following structure:

$$J \Phi_{FB}(x, \mu, \lambda) = \begin{pmatrix} J_L(x, \mu, \lambda) & J_h(x)^T & J_g(x)^T \\ -\text{diag}(I + L^{-1}_w L_{g_i})_{i=1}^p & 0 & \text{diag}(I - L^{-1}_w L_{g_i})_{i=1}^p \\ -(I + L^{-1}_w L_{g_i}) & J_h(x)^T & J_g(x)^T \end{pmatrix},$$

where $L_w = \text{diag}(L_{g_{1w}}, L_{g_{2w}}, \ldots, L_{g_{mw}})$, $L_x = \text{diag}(L_{g_{1x}}, L_{g_{2x}}, \ldots, L_{g_{mx}})$ and $L_{g_i} = \text{diag}(L_{g_{1w}}, L_{g_{2w}}, \ldots, L_{g_{mw}})$.

We know that $J \Phi_{FB}$ is nonsingular if and only if the following matrix

$$J = \begin{pmatrix} J_L(x, \mu, \lambda) & J_h(x)^T & J_g(x)^T \\ -(I + L^{-1}_w L_{g_i}) & J_h(x)^T & J_g(x)^T \end{pmatrix}^T$$

is nonsingular.

Let $(u, v, t) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$ be a vector in the null space of $J$. We will show that $u = 0, v = 0$ and $t = 0$. By (3.3) we have that

$$j_L u - (j_h)^T v - (j_g)^T (L_w + L_g) t = 0,$$

$$j_h u = 0$$

and

$$(j_g) u + (L_w - L_x) t = 0.$$  (3.6)

From (3.4) and (3.5), we deduce that

$$u^T (j_L) u - u^T (j_g^T) (L_w + L_g) t = 0.$$  (3.7)

From (3.6), we have that

$$t^T (L_w + L_g) (j_g) u + t^T (L_w + L_g) (L_w - L_x) t = 0.$$  (3.8)

Lemma 3.1 and (3.8) imply that

$$t^T (L_w + L_g) (j_g) u = u^T (j_g^T) (L_w + L_g) t \leq 0.$$  (3.9)
It follows (3.7) and (3.9) that
\[ u^T (J_L) u \leq 0. \]
By (b), \( u^T (J_L) u = 0 \) and hence \( u = 0 \). Therefore, (3.4) and (3.6) become
\[ (Jh^T) v + (Jg^T) (L_w + L_g) t = 0, \]
and
\[ (L_w - L_A) t = 0. \] (3.11)
By (a) and (3.10), we get easily that
\[ v = 0, \quad (L_w + L_g) t = 0. \] (3.12)
From (3.11) and (3.12), we deduce that
\[ L_A t = L_{-g} t. \]
We know that \( L_A L_{-g} = 0 \) and \( \lambda \) and \(-g\) are strict complementarity, which implies that \( t = 0 \). This completes the proof. \( \square \)

Note that the FB function is not differentiable at the boundary points of \( \mathcal{K}_{i} = \{ x \mid g_i(x) + \lambda_i \in \text{int}(\mathcal{K}_m), \lambda_i = 0 \} \). This means we cannot obtain the result of Theorem 3.1 when removing the strict complementarity condition \(-g_i(x) + \lambda_i \in \text{int}(\mathcal{K}_m), \lambda_i = 0\). However, by analyzing some properties of complementarity problem, we can give another reformulation of KKT system (3.1) and give the nonsingularity theorem without the strict complementarity condition.

We introduce three index sets associated with a given pair \((x, \lambda)\) in \( \mathbb{R}^{n+m} \). All these index sets are subsets of \( I \equiv \{ 1, 2, \ldots, p \} \). Specifically, let
\[
\mathcal{A}(x) \equiv \{ i \in I : g_m(x) = 0, \lambda_m \in \text{bd}(\mathcal{K}_m) \} \\
\mathcal{B}(x) \equiv \{ i \in I : -g_m(x) \in \text{bd}^+(\mathcal{K}_m), \lambda_m = 0 \} \\
\mathcal{C}(x) \equiv I \setminus (\mathcal{A}(x) \cup \mathcal{B}(x)).
\]
Note that \( i \in \mathcal{C}(x) \) if and only if \(-g_m(x) + \lambda_m \in \text{int}(\mathcal{K}_m)\). If \(-g(x) + \lambda \in \text{bd}(\mathcal{K}_m)\) and \(0 \geq g_m(x) \perp \lambda_m \geq 0\), then \( i \in \mathcal{A}(x) \cup \mathcal{B}(x)\).

For a mapping \( G(x) = (G_{m_1}(x); G_{m_2}(x); \ldots; G_{m_p}(x)) : \mathbb{R}^n \rightarrow \mathbb{R}^m \) with \( G_m(x) \in \mathbb{R}_m, I(x) \subseteq I \), we denote \{(G_{m_1}(x); \ldots; G_{m_j}(x)) | m_i \in I(x), s = 1, \ldots, j\} by \( G_I(x) \). Let \( (x, \mu, \lambda) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \) be a triple and
\[
\hat{\Phi}_{FB}(x, \mu, \lambda) = \begin{pmatrix}
L(x, \mu, \lambda) \\
-\frac{h(x)}{\hat{g}_{\mathcal{A}}(x)} \\
\hat{g}_{\mathcal{B}}(-g_{\mathcal{C}}(x), \lambda_{\mathcal{C}})
\end{pmatrix},
\]
where
\[
L(x, \mu, \lambda) \equiv F(x) + Jh(x)^T \mu + Jg(x)^T \lambda
\]
and
\[
\hat{g}_{\mathcal{B}}(-g_{\mathcal{C}}(x), \lambda_{\mathcal{C}}) = \begin{pmatrix}
\hat{g}_{\mathcal{B}}(-g_{\mathcal{C}}(x), \lambda_{\mathcal{C}}) \\
\vdots \\
\hat{g}_{\mathcal{B}}(-g_{\mathcal{C}}(x), \lambda_{\mathcal{C}})
\end{pmatrix}
\]
with \( \{ \mathcal{C}_1(x), \mathcal{C}_2(x), \ldots, \mathcal{C}_q(x) \} = \mathcal{C}(x) \).

Based on the above analysis, another reformulation of the system (3.1) is:
\[
\hat{\Phi}_{FB}(x, \mu, \lambda) = 0.
\]
It follows from Proposition 3.2 (a) that \( \hat{\Phi}_{FB} \) is continuously differentiable. Hence, we can give its Jacobian matrix as follows:
\[
j \hat{\Phi}_{FB}(x, \mu, \lambda) = \begin{pmatrix}
J_L(x, \mu, \lambda) & Jh(x)^T & Jg_{\mathcal{A}}(x)^T & Jg_{\mathcal{B}}(x)^T & Jg_{\mathcal{C}}(x)^T \\
-Jh(x) & 0 & 0 & 0 & 0 \\
-Jg_{\mathcal{A}}(x)^T & 0 & 0 & 0 & 0 \\
-(I + L_{-g}^{-1} L_g) Jg_{\mathcal{C}}(x) & 0 & 0 & I - L_{-g}^{-1} L_{\lambda_{\mathcal{C}}} & 0 \\
0 & 0 & 0 & 0 & I
\end{pmatrix}, \] (3.13)
where

\[ L_{w'c} = \text{diag}\{L_{w'e_1}, L_{w'e_2}, \ldots, L_{w'e_q}\}, \]
\[ L_{g'c} = \text{diag}\{L_{g'e_1}, L_{g'e_2}, \ldots, L_{g'e_q}\}, \]
\[ L_{\lambda'c} = \text{diag}\{L_{\lambda'e_1}, L_{\lambda'e_2}, \ldots, L_{\lambda'e_q}\}. \]

Now, we give the nonsingularity theorem without the strict complementarity condition.

**Theorem 3.2.** Let \((x, \mu, \lambda) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m\) be any given triple. If the assumption (a) and (b) of **Theorem 3.1** hold, then the Jacobian \(J\Phi_{FB}(x, \mu, \lambda)\) is nonsingular.

**Proof.** For simplicity, we suppress the Jacobian \(J\Phi_{FB}(x, \mu, \lambda)\) \((3.13)\) as follows:

\[
J\Phi_{FB} = \begin{pmatrix}
J_L & J_h^T & J_{g_{s,f}}^T & J_{g_{e'}}^T & J_{g_{s,e}}^T \\
-\frac{J_h}{J_f} & 0 & 0 & 0 & 0 \\
-(I + L_{w'e})g_{e'} & 0 & 0 & I - L_{w'e}^{-1}L_{\lambda'e} & 0 \\
0 & 0 & 0 & 0 & I
\end{pmatrix}.
\]  

(3.14)

Clearly, \(J\Phi_{FB}\) is nonsingular if and only if the following matrix

\[
\bar{J} = \begin{pmatrix}
J_L & J_h^T & J_{g_{s,f}}^T & J_{g_{e'}}^T \\
-\frac{J_h}{J_f} & 0 & 0 & 0 \\
-(I + L_{w'e})g_{e'} & 0 & 0 & L_{w'e} - L_{\lambda'e}
\end{pmatrix}^T
\]  

(3.15)

is nonsingular.

Let \((H_1, H_2, H_3, H_4) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{m_{s,f}(c)} \times \mathbb{R}^{m_{e}(c)}\) be a vector in the null space of \(\bar{J}\). We will show that \(H_1 = 0, H_2 = 0, H_3 = 0\) and \(H_4 = 0\). By (3.15) we have that

\[
(j_h)LH_1 - (J_h^T)H_2 - (J_{g_{s,e}}^T)H_3 - (J_{g_{e'}}^T)(L_{w'e} + L_{g'c})H_4 = 0,
\]  

(3.16)

\[
(j_h)H_1 = 0.
\]  

(3.17)

\[
(j_{g_{s,e}})H_1 = 0.
\]  

(3.18)

and

\[
(J_{g_{e'}})H_1 + (L_{w'e} - L_{\lambda'e})H_4 = 0.
\]  

(3.19)

From (3.16) and (3.17) and the positive definition of \(J_L\) in the null space of \(h\), we deduce that

\[
H_1^T(j_h)LH_1 = H_1^T(j_{g_{s,e}}^T)(L_{w'e} + L_{g'c})H_4.
\]  

(3.20)

By (3.19), we have that

\[
H_1^T(L_{w'e} + L_{g'c})(L_{g'c})H_1 + H_1^T(L_{w'e} + L_{g'c})(L_{w'e} - L_{\lambda'e})H_4 = 0.
\]  

(3.21)

It follows from Lemma 3.1 and (3.21) that

\[
H_1^T(L_{w'e} + L_{g'c})(L_{g'c})H_1 + H_1^T(L_{w'e}) = H_1^T(j_{g_{e'}}^T)(L_{w'e} + L_{g'c})H_4 \leq 0.
\]  

(3.22)

(3.20) and (3.22) tell us \(H_1 = 0\). Hence, (3.16) and (3.19) become

\[
(j_h^T)H_2 + (j_{g_{s,e}}^T)H_3 + (j_{g_{e'}}^T)(L_{w'e} + L_{g'c})H_4 = 0
\]  

(3.23)

and

\[
(L_{w'e} - L_{\lambda'e})H_4 = 0.
\]  

(3.24)

By assumption (a) and (3.16), we get that

\[
H_2 = 0,
\]

\[
H_3 = 0
\]

and

\[
(L_{w'e} + L_{g'c})H_4 = 0.
\]  

(3.25)

(3.24) and (3.25) imply that

\[
L_{\lambda'e}H_4 = -L_{g'c}H_4.
\]

We know that \(\lambda'\) and \(-g'\) are strict complementarity, which implies that \(H_4 = 0\). This completes the proof. □
4. A modified Newton method

In this section, we present an algorithm by solving the semismooth system with a nonsmooth Newton-type method. The algorithm is globalized by using the smooth merit function $\theta_{FB}$ given by

$$\theta_{FB}(x, \mu, \lambda) \equiv \frac{1}{2} \| \Phi_{FB}(x, \mu, \lambda) \|^2.$$  

The proposed algorithm is actually a counterpart in the case of second-order cone constrained VI problems of [1, Algorithm 9.1.10], which is used to solve polyhedral cone constrained VI problems. Note that although $\Phi_{FB}$ is nonsmooth, the merit function $\theta_{FB}$ is continuously differentiable if $F$ is, see [1, Proposition 1.5.3]. In the following proposition we give the relationship between the merit function and the KKT condition.

**Proposition 4.1.** Suppose that $F$, $g$ and $h$ are continuously differentiable. If every matrix in $\partial \Phi_{FB}(x, \mu, \lambda)$ is nonsingular, then every stationary point of the merit function $\theta_{FB}$ is a KKT triple of the SOCCVI.

**Proof.** By the generalized Jacobian of composite functions and the continuous differentiability of $\theta_{FB}$, we can deduce that its gradient $\nabla \theta_{FB}(x, \mu, \lambda)$ is equal to $H^T \Phi_{FB}(x, \mu, \lambda)$ for every $H$ in $\partial \Phi_{FB}(x, \mu, \lambda)$.

If $x$ is a stationary point of $\theta_{FB}$, we have that

$$H^T \Phi_{FB}(x, \mu, \lambda) = 0$$

for every $H \in \partial \Phi_{FB}(x, \mu, \lambda)$. If $\Phi_{FB}(x, \mu, \lambda) \neq 0$, then every matrix in $\partial \Phi_{FB}(x, \mu, \lambda)$ is singular. This is a contradiction. Thus, $x$ is a KKT triple of SOCCVI.

**Algorithm 4.1.** Data Given $z^0 = (x^0, \mu^0, \lambda^0) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$, $\sigma > 0$, $p > 1$ and $\gamma \in (0, 1)$.

**Step 1.** Set $k = 0$.

**Step 2.** If $z^k = (x^k, \mu^k, \lambda^k)$ is a stationary point of $\theta_{FB}$, stop.

**Step 3.** Select an element $H^k$ in $\partial \Phi_{FB}(z^k)$ and find a solution $d^k$ of the system

$$\Phi_{FB}(z^k) + H^k d = 0.$$  
(4.1)

If the system (4.1) is not solvable or if the condition

$$\nabla \theta_{FB}(z^k)^T d^k \leq -\sigma \| d^k \|^p$$  
(4.2)

is not satisfied, (re)set $d^k \equiv -\nabla \theta_{FB}(z^k)$.

**Step 4.** Find the smallest nonnegative integer $i_k$ such that, with $i = i_k$,

$$\theta_{FB}(z^k + 2^{-i}d^k) \leq \theta_{FB}(z^k) + \gamma 2^{-i} \| \nabla \theta_{FB}(z^k) \| d^k.$$  
(4.3)

set $\tau_k \equiv 2^{-i_k}$.

**Step 5.** Set $z^{k+1} \equiv z^k + \tau_k d^k$ and $k \leftarrow k + 1$; go to Step 2.

Now, we are going to give the complete description of the convergence properties of the above algorithm.

**Theorem 4.1.** Suppose that $\{z^k = (x^k, \mu^k, \lambda^k)\}$ is a sequence generated by Algorithm 4.1. Then the following three statements hold.

(i) Every limit point $z^*$ of $\{z^k\}$ satisfies $\nabla \theta_{FB}(z^*) = 0$.

(ii) If $z^*$ is an isolated accumulation point of $\{z^k\}$, then the entire sequence $\{z^k\}$ converges to $z^*$.

(iii) Let $z^*$ be a limit point and $F$, $h$, $g$ satisfy the conditions of Theorem 3.1. Assume that $p > 2$ and $\gamma < 1/2$ in Algorithm 4.1.

Then, we have

(i) $z^*$ is a KKT triple of SOCCVI.

(ii) Eventually $d^k$ is always the solution of system (4.1).

(iii) The sequence $\{z^k\}$ convergence to $z^*$ Q-superlinearly.

**Proof.** The following inequality holds for all $k$:

$$\| d^k \| \leq \max \{\| \nabla \theta_{FB}(z^k) \|, \rho^{-1} \| \nabla \theta_{FB}(z^k) \|^{1/(p-1)} \}.$$  
(4.4)

To prove (a), let $\{z^k\}$ be a subsequence of $\{z^k\}$ converging to $z^*$. From (4.3), we know that

$$\theta_{FB}(z^{k+1}) - \theta_{FB}(z^k) - \gamma 2^{-i} \| \nabla \theta_{FB}(z^k) \| d^k \leq 0$$  
(4.5)

and $\theta_{FB}(z^k)$ is bounded below. This implies that

$$\lim_{k \to \infty} [\theta_{FB}(z^{k+1}) - \theta_{FB}(z^k)] = 0.$$  
(4.6)
From \( (\text{4.5}) \) and \( (\text{4.6}) \), we can deduce that
\[
\lim_{k \to \infty} -\gamma 2^{\text{-}i} \nabla \theta_{FB}(z^k)^T d^k \leq 0.
\]

It follows from \( (\text{4.2}) \) that \(-\gamma 2^{\text{-}i} \nabla \theta_{FB}(z^k)^T d^k \geq 0\). The above two inequalities yield
\[
\lim_{k \to \infty} \nabla \theta_{FB}(z^k)^T d^k = 0. \tag{4.7}
\]

If \( d^k = -\nabla \theta_{FB}(z^k) \) for infinitely many \(k\), then, \( \nabla \theta_{FB}(z^*) = 0 \); hence \( z^* \) is a stationary point of \( \theta \). If \( (\text{4.1}) \) and \( (\text{4.2}) \) hold for all but finitely many \(k\), then \( (\text{4.2}) \) implies that \( d^k \) converges to zero. Since \( \Phi_{FB}(z^k) + H^k d = 0 \) and \( H^k \) is bounded by the boundedness of \( \partial \Phi_{FB} \). It follows that \( \Phi_{FB}(z^*) = 0 \). Therefore, in either case, we have established that every accumulation of the sequence \( \{z^k\} \) is a stationary point of \( \theta_{FB} \).

Now, we prove (b). As we get the limit \( (\text{4.7}) \), we can get that
\[
\lim_{k \to \infty} \tau_k \nabla \theta_{FB}(z^k)^T d^k = 0.
\]

Since the sequence \( \{\tau_k\} \) of step size is bounded, the above limit and \( (\text{4.4}) \) easily imply
\[
\lim_{k \to \infty} \tau_k d^k = 0.
\]

It follows from \( z^{k+1} - z^k = \tau_k d_k \) that \( \lim_{k \to \infty} \|z^{k+1} - z^k\| = 0 \). Hence, \( \lim_{k \to \infty} z^k = z^* \) by \( z^* \) being a isolate limit point of \( \{z^k\} \).

From the assumption of (iii), we can get that every \( H \in \partial \Phi_{FB}(z^*) \) is nonsingularity. It follows from Proposition 4.1 and (i) that \( z^* \) is a KKT triple of SOCCVI, which means that the statement (i) in (iii) holds.

By the nonsingularity of \( H^k \) it follows that eventually the system \( (\text{4.1}) \) has a unique solution \( d^k \). We still need to show that this \( d^k \) satisfies \( (\text{4.2}) \). To this end it is sufficient to prove that \( d^k \) satisfies, for some positive \( \rho_1 \), independent of \( k \), the condition
\[
\nabla \theta_{FB}(z^k)^T d^k \leq -\rho_1 \|d^k\|^2. \tag{4.8}
\]

Since \( d^k \) is the solution of system \( (\text{4.1}) \), we have that
\[
\|d^k\| \leq c \|\Phi_{FB}(z^k)\|,
\]
where \( c \) is an upper bound on \( \|(H^k)^{-1}\| \). This show that \( \{d^k\} \) convergence to zero. By assumption, \( \nabla \theta_{FB}(z^k) \) is equal to \( (H^k)^T \Phi_{FB}(z^k) \). We have
\[
\nabla \theta_{FB}(z^k)^T d^k = -\|\Phi_{FB}(z^k)\|^2 \leq -\frac{\|d^k\|^2}{c^2}. \tag{4.9}
\]

Thus \( (\text{4.8}) \) follows from \( (\text{4.9}) \) by taking \( \rho_1 \leq 1/c^2 \). Since \( \{\|d^k\|\} \) converges to zero, \( (\text{4.8}) \) implies that eventually \( (\text{4.2}) \) holds for any \( p > 2 \) and any positive \( \rho \).

To complete the proof, we need to show that the convergence rate is superlinear. In fact, we can write
\[
\Phi_{FB}(z^*) = [\Phi_{FB}(z^k) + H^k \tau_k d^k] - [\Phi_{FB}(z^k) - \Phi_{FB}(z^*) - H^k(z^k - z^*)] - H^k(z^k + \tau_k d^k - z^*). \tag{4.10}
\]

That is
\[
z^k + \tau_k d^k - z^* = (H^k)^{-1}[(\Phi_{FB}(z^k) + H^k \tau_k d^k) - (\Phi_{FB}(z^k) - \Phi_{FB}(z^*) - H^k(z^k - z^*))]. \tag{4.11}
\]

From (ii) of (iii), we can deduce that eventually the step size determined by the Armijo text \( (\text{4.3}) \) is one. Using the boundedness of \( \{(H^k)^{-1}\}\) and the semismoothness of \( \Phi_{FB} \), we can conclude that
\[
\lim_{k \to \infty} \frac{z^k + \tau_k d^k - z^*}{\|z^k - z^*\|} = 0,
\]
that is
\[
\lim_{k \to \infty} \frac{z^{k+1} - z^*}{\|z^k - z^*\|} = 0.
\]

This completes the proof. \(\square\)
5. An illustrative numerical example

Now we report numerical results to illustrate Algorithm 4.1 for solving a SOCCVI problem. Our numerical experiments are carried out in Matlab 7.1 running on a PC Intel Pentium IV of 2.80 GHz CPU and 512 MB memory.

Example 5.1. We consider the following SOCCVI problem:

\[
\frac{1}{2}Dx, y - x \geq 0, \quad \forall y \in C,
\]

where

\[
C = \{x \in \mathbb{R}^n : Ax - a = 0, \ Bx - b \preceq 0\}.
\]

\(D\) is an \(n \times n\) symmetric matrix, \(A\) and \(B\) are \(l \times n\) and \(m \times n\) matrices, respectively, \(d\) is an \(n \times 1\) vector, \(a\) and \(b\) are \(l \times 1\) and \(m \times 1\) vectors with \(l + m \leq n\), respectively. The KKT system of the above problem is

\[
\begin{align*}
Dx + A^T \mu + B^T \lambda &= 0, \\
Ax - a &= 0, \\
(Bx - b)^T \lambda &= 0,
\end{align*}
\]

where \(\mu\) and \(\lambda\) is the dual variable of the equality and inequality constraints, respectively.

Let \(m = \sum_{i=1}^p m_i, \lambda = (\lambda_{m_1}, \ldots, \lambda_{m_p}), x = (x_{m_1}, \ldots, x_{m_p}), b = (b_{m_1}, \ldots, b_{m_p}),\) and \(B = (B_{m_1}, \ldots, B_{m_p}).\) Then to solve the above KKT system (5.1) is equivalent to find a root of

\[
\phi_{FB}(x, \mu, \lambda) = \begin{pmatrix} Dx + A^T \mu + B^T \lambda \\ -Ax + a \\ \vdots \\ \phi_{FB}(-B_{m_p} - b_{m_p}), \lambda_{m_p} \end{pmatrix}.
\]

Then

\[
J \phi_{FB}(x, \mu, \lambda) = \begin{pmatrix} J_x L & J_h^T & J_g^T \\ -Lh & 0 & 0 \\ -(I + L_w^{-1} L_g)g & 0 & I - L_w^{-1} L_g \end{pmatrix},
\]

where \(L_w = \text{diag}\{L_{w_{m_1}}, L_{w_{m_2}}, \ldots, L_{w_{m_p}}\}, \ w_i = \sqrt{(B_{m_i} - b_{m_i})^2 + \lambda_{m_i}^2}, L_{x-b} = \text{diag}\{L_{x_{m_1} - b_{m_1}}, L_{x_{m_2} - b_{m_2}}, \ldots, L_{x_{m_p} - b_{m_p}}\}\) and \(L_{\lambda} = \text{diag}\{L_{\lambda_{m_1}}, L_{\lambda_{m_2}}, \ldots, L_{\lambda_{m_p}}\}.

In fact, we can determine the data \(a, b, A, B\) and \(D\) randomly. However, to ensure that we can reproduce the results, we choose the data as follows: \(D = (D_{ij})_{n \times n},\) where

\[
D_{ij} = \begin{cases} 2, & i = j \\ 1, & |i - j| = 1 \\ 0, & \text{otherwise}, \end{cases}
\]

\[
A = (I_{l \times 1} 0_{l \times (n-l)})_{l \times n},
\]

\[
B = ((0_{m \times (n-m)} I_{m \times m})_{m \times n},
\]

\[
a = 0_{l \times 1},
\]

\[
b = (e_{m_1}, e_{m_2}, \ldots, e_{m_p}),
\]

where \(e_{m_i} = (1, 0, \ldots, 0)^T \in \mathbb{R}^m\) and \(l + m \leq n.\) Clearly, \(A\) and \(B\) are full row rank and \(\text{rank}(A^T B^T) = l + m.\)

The problem instance is solved by Algorithm 4.1 using an initial point whose element are randomly generated form the interval \([0, 1].\) In the implementation, we use the following parameters in the method,

\[
p = 3, \quad \sigma = 0.01, \quad \gamma = 0.45.
\]

Numerical results are summarized in Table 1 where \(n, l, K, \) Iter and Time represent the number of variables, the number of the row vector of \(A,\) the number of the inequality constants, the number of iterations and cpu running time in seconds, respectively.
Table 1
Numerical results of Example 5.1.

<table>
<thead>
<tr>
<th>n</th>
<th>l</th>
<th>$\mathcal{K}^m$</th>
<th>Iter</th>
<th>Time (s)</th>
<th>$|\nabla \theta_{FB}(x^k, \mu^k, \lambda^k)|$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5</td>
<td>$\mathcal{K}^5$</td>
<td>6</td>
<td>2.031250e−001</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>40</td>
<td>20</td>
<td>$\mathcal{K}^{10} \times \mathcal{K}^{10}$</td>
<td>10</td>
<td>2.500000e−001</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>100</td>
<td>50</td>
<td>$\mathcal{K}^{10} \times \mathcal{K}^{20} \times \mathcal{K}^{20}$</td>
<td>22</td>
<td>1.062500e+000</td>
<td>$10^{-6}$</td>
</tr>
</tbody>
</table>

6. Conclusions

In this paper, we use the Fischer–Burmeister operator over the second-order cone to deal with second-order cone constrained variational inequality (SOCCVI) problems. With the help of Fischer–Burmeister operators, we transform the Karush–Kuhn–Tucker system of a second-order cone constrained variational inequality problem into a semismooth system of equations and prove the nonsingularity of the Jacobian matrix of the mapping $\Phi_{FB}$ under the assumption that the strict complementarity condition hold, see Theorem 3.1. To weaken the assumptions of Theorem 3.1, we define another mapping $\hat{\Phi}_{FB}$ by introducing three index sets associated with a given pair $(x, \lambda)$ and give another reformulation of the KKT system. Without strict complementarity condition, we prove the nonsingularity of the Jacobian matrix of $\hat{\Phi}_{FB}$ at the given point, see Theorem 3.2. Based on Theorem 3.1, we propose a modified Newton method with Armijo line search and proved the global convergence with local superlinear rate of convergence under certain assumptions on the variational inequality problem. In theory, $\Phi_{FB}$ does not require the strict complementarity condition, but in practice, it is difficult to use this mapping to construct an implementable algorithm. That is why we use the mapping $\Phi_{FB}$ under the strict complementarity condition. In the future research work, we will study how to construct a smoothing Newton method to overcome the difficulties encountered by the lack of strict complementarity condition.

References