MATHEMATICS

ON n-ISOCLINIC GROUPS

BY

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The notion of isoclinism was introduced by P. Hall [2]. In [1] we have proved that monomiality is an invariant of the families of finite isoclinic groups. In this paper we consider a more general form of isoclinism, called n-isoclinism, and we prove that strong-monomiality is a family-invariant for finite n-isoclinic groups. Moreover, using a theorem of P. M. Weichsel [7] we give short proofs for results of P. Hall [2] and J. Tappe [6] on the irreducible characters of isoclinic groups. As a corollary we obtain the above mentioned result on the M-group property proved in [1].

Notations are standard and can be found in Huppert's book [4].

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1. n-ISOCLINIC GROUPS

The notion of n-isoclinism of groups is implicit in a short note of P. Hall [3] on verbal and marginal subgroups.

Let $G = K_1(G) > K_2(G) > \dots$ be the lower central series of the group G. Each term of this series, being generated by commutator words, is a verbal subgroup. An element g of G is called a period of $K_n(G)$, if for all simple commutators $[g_1, \dots, g_n] \in K_n(G)$ we have

$$[g_1, \dots, g_j g, \dots, g_n] = [g_1, \dots, g_j, \dots, g_n], \; j = 1, 2, \dots, n.$$

The set of all periods of a verbal subgroup X is called the marginal subgroup of X. The marginal subgroup of $K_i(G)$ is $Z_{i-1}(G)$, where the latter group is the $(i-1)$-th term of the upper central series of G:

$$1 = Z_0(G) < Z_1(G) = Z(G) < Z_2(G) < \dots .$$

As is well-known, the subgroups $K_i(G)$ and $Z_i(G)$ centralize each other, see [4] theorem III.2.11.

1.1. DEFINITION. Two groups G and H are n-isoclinic, $G \underset{n}{\sim} H$, if there exist isomorphisms α and β:

$$\alpha: G/Z_n(G) \to H/Z_n(H)$$

$$\beta\colon\ K_{n+1}(G) \to K_{n+1}(H),$$

such that α induces β in the following sense: if $g_i \in G$, $i=1, \ldots, n+1$ and if $h_i \in \alpha(g_i Z_n(G))$, then

$$\beta([g_1, \ldots, g_{n+1}]) = [h_1, \ldots, h_{n+1}].$$

The pair (α, β) is called an n-isoclinism between G and H. //

It will be clear from this definition that an n-isoclinism induces also an $(n+1)$-isoclinism. Hence we have for each rational integer $n \geqslant 1$ an equivalence relation with corresponding equivalence classes of groups (families). If $n=1$, then G and H are called isoclinic groups.

In the following two lemma's we state some results on n-isoclinic groups, which were outlined by P. Hall [2] for $n=1$. For the proofs of these lemma's we recall that $Z_n(G)$ is the set of all periods of $K_{n+1}(G)$. Moreover, if ϕ is a homomorphism of G, then clearly

$$\phi([g_1, \ldots, g_n]) = [\phi(g_1), \ldots, \phi(g_n)],\ g_j \in G.$$

1.2. LEMMA. Let (α, β) be an n-isoclinism of G_1 and G_2. Then the following holds:

a) If $Z_n(G) \leqslant H_1 < G_1$ and $\alpha(H_1/Z_n(G_1)) = H_2/Z_n(G_2)$, then $H_1 \underset{n}{\sim} H_2$.
b) β is an operator-isomorphism in the following sense: if $g_1 \in G_1$, $g_2 \in \alpha(g_1 Z_n(G_1))$ and $k_1 \in K_{n+1}(G_1)$, $k_2 = \beta(k_1)$, then $\beta(g_1^{-1} k_1 g_1) = g_2^{-1} k_2 g_2$.
c) If $N_1 \lhd G_1$, $N_1 \leqslant K_{n+1}(G_1)$, then $G_1/N_1 \underset{n}{\sim} G_2/\beta(N_1)$.

PROOF. a) If $Z_n(G_1) \leqslant H_1$, then $Z_n(G_1) \leqslant Z_n(H_1)$.
Similarly $Z_n(G_2) \leqslant Z_n(H_2)$. We define two isomorphisms

$$\bar{\alpha}\colon\ H_1/Z_n(H_1) \to H_2/Z_n(H_2)$$

$$\bar{\beta}\colon\ K_{n+1}(H_1) \to K_{n+1}(H_2),$$

as follows:

$$\bar{\alpha}(h_1 Z_n(H_1)) = h_2 Z_n(H_2),\ \text{if } h_1 \in H_1 \text{ and } h_2 \in \alpha(h_1 Z_n(G_1)),$$

$$\bar{\beta}(k_1) = k_2,\ \text{if } k_1 \in K_{n+1}(H_1) \text{ and } \beta(k_1) = k_2.$$

It can be easily checked that the pair $(\bar{\alpha}, \bar{\beta})$ is an isoclinism between H_1 and H_2. We omit the verification.

b) Without loss of generality we may assume:

$$k_1 = [a_1, a_2, \ldots, a_{n+1}] \text{ and } k_2 = [b_1, b_2, \ldots, b_{n+1}],$$

where $a_j \in G_1$ and $b_j \in \alpha(a_j Z_n(G_1))$.
Then

$$\beta(g_1^{-1} k_1 g_1) = \beta([g_1^{-1} a_1 g_1, \ldots, g_1^{-1} a_{n+1} g_1]) = [g_2^{-1} b_1 g_2, \ldots, g_2^{-1} b_{n+1} g_2] = g_2^{-1} k_2 g_2.$$

c) Denote $\bar{G}_1 = G_1/N_1$ and $\bar{G}_2 = G_2/\beta(N_1)$. We define two isomorphisms

$$\bar{\alpha}:\ \bar{G}_1/Z_n(\bar{G}_1) \to \bar{G}_2/Z_n(\bar{G}_2)$$

$$\bar{\beta}:\ K_{n+1}(\bar{G}_1) \to K_{n+1}(\bar{G}_2),$$

as follows:

$$\bar{\alpha}(\bar{g}_1 Z_n(\bar{G}_1)) = \bar{g}_2 Z_n(\bar{G}_2),\ \text{ if } g_2 \in \alpha(g_1/Z_n(G_1)),$$

$$\bar{\beta}([\bar{a}_1, \ldots, \bar{a}_{n+1}]) = [\bar{b}_1, \ldots, \bar{b}_{n+1}],\ \text{ if } b_i \in \alpha(a_i Z_n(G_1)).$$

Now $(\bar{\alpha}, \bar{\beta})$ is an n-isoclinism between $\bar{G}_1$ and $\bar{G}_2$, since (α, β) is an isoclinism between G_1 and G_2.

1.3. Lemma. Let G be a group with subgroups H, K and let N be a normal subgroup of G. Then

a) $H \underset{n}{\sim} HZ_n(G)$. In particular if $G = HZ_n(G)$, then $G \underset{n}{\sim} H$.
Conversely, if $|G/Z_n(G)| < \infty$ and $G \underset{n}{\sim} H$, then $G = HZ_n(G)$.

b) $G/N \underset{n}{\sim} G/(N \cap K_{n+1}(G))$. In particular, if $N \cap K_{n+1}(G) = 1$, then $G \underset{n}{\sim} G/N$.
Conversely, if $|K_{n+1}(G)| < \infty$ and $G \underset{n}{\sim} G/N$, then $N \cap K_{n+1}(G) = 1$.

Proof. a) We define $\alpha(hZ_n(H)) = hZ_n(HZ_n(G))$. Since $Z_n(HZ_n(G)) = Z_n(H)Z_n(G)$, α is an isomorphism of $H/Z_n(H)$ onto $HZ_n(G)/Z_n(HZ_n(G))$, and α induces the identity on $K_{n+1}(H) = K_{n+1}(HZ_n(G))$. Thus $H \underset{n}{\sim} HZ_n(G)$, and if $G = HZ_n(G)$, then $G \underset{n}{\sim} H$. Conversely, if H is a subgroup of G such that $G \underset{n}{\sim} H$, then we may assume by part a), that $H > Z_n(G)$, so that $Z_n(H) > Z_n(G)$. Since $H/Z_n(G) \simeq H_1/Z_n(H)$, $H_1 < H$, this implies $H_1 = H$ and $Z_n(H) = Z_n(G)$, so that $G/Z_n(G) \simeq H/Z_n(G)$. Thus, if $|G/Z_n(G)| < \infty$, then $G = HZ_n(G)$.

b) We denote $G = \bar{G}/N$ and $\tilde{G} = G/(N \cap K_{n+1}(G))$. If $k_1 \in K_{n+1}(G)$ and $k_2 \in K_{n+1}(G)$, then $\bar{k}_1 = \bar{k}_2 \Leftrightarrow \tilde{k}_1 = \tilde{k}_2$.

We have therefore,

$$[\bar{g}_1, \ldots, \bar{g}_j\bar{g}, \ldots, \bar{g}_{n+1}] = [\bar{g}_1, \ldots, \bar{g}_j, \ldots, \bar{g}_{n+1}]$$

if and only if

$$[\tilde{g}_1, \ldots, \tilde{g}_j\tilde{g}, \ldots, \tilde{g}_{n+1}] = [\tilde{g}_1, \ldots, \tilde{g}_j, \ldots, \tilde{g}_{n+1}].$$

This implies: $\bar{g} \in Z_n(\bar{G})$ if and only if $\tilde{g} \in Z_n(\tilde{G})$.

If $\alpha(\bar{g}Z_n(\bar{G})) = \tilde{g}Z_n(\tilde{G})$, then α is an isomorphism of $\bar{G}/Z_n(\bar{G})$ onto $\tilde{G}/Z_n(\tilde{G})$. Let $k \in K_{n+1}(G)$ and denote $\beta(\bar{k}) = \tilde{k}$.

Then β defines an isomorphism of $K_{n+1}(\bar{G})$ onto $K_{n+1}(\tilde{G})$ and β is induced by α in the sense of definition 1.1.

Conversely, if $N \lhd G$ and $G \underset{n}{\sim} G/N$, then

$$K_{n+1}(G) \simeq K_{n+1}(G/N) = K_{n+1}(G)N/N \simeq K_{n+1}(G)/(N \cap K_{n+1}(G)).$$

Thus, if $|K_{n+1}(G)| < \infty$, then $N \cap K_{n+1}(G) = 1$.

The relationship of n-isoclinic groups is made clear by the following theorem, which can be obtained by a direct generalization of a result of P. M. Weichsel [7].

1.4. THEOREM. Let G and H be finite groups. Then G and H are n-isoclinic if and only if there exist finite groups C, Z_G, Z_H, C_G and C_H such that $G \simeq C/Z_H$ and $H \simeq C/Z_G$ and the following two (equivalent) properties hold:

a) $G \simeq C/Z_H \underset{n}{\sim} C \underset{n}{\sim} C/Z_G \simeq H$

b) $C/Z_H \times C/K_{n+1}(C) \underset{n}{\sim} C_H \simeq C \simeq C_G \underset{n}{\sim} C/Z_G \times C/K_{n+1}(C)$, where C_H and C_G are subgroups of $C/Z_H \times C/K_{n+1}(C)$ and $C/Z_G \times C/K_{n+1}(C)$ respectively.

PROOF. One part of the theorem is trivial. Assume now $G \underset{n}{\sim} H$, and let β be the isomorphism between $K_{n+1}(G)$ and $K_{n+1}(H)$ given in definition 1.1. Finally, let C be the direct product of G and H with identified factor groups $G/Z_n(G)$ and $H/Z_n(H)$:

$$C := G \curlywedge H.$$

If

$$Z_H := \{(1, z) | z \in Z_n(H)\} \text{ and } Z_G := \{(z, 1) | z \in Z_n(G)\},$$

then we have

$$C/Z_H \simeq G \text{ and } C/Z_G \simeq H, \text{ where } Z_H \simeq Z_n(H) \text{ and } Z_G \simeq Z_n(G).$$

a) It follows from definition 1.1 that $K_{n+1}(C)$ is generated by elements of the form

$$([g_1, \ldots, g_{n+1}],\ \beta([g_1, \ldots, g_{n+1}])).$$

We claim that

$$K_{n+1}(C) \cap Z_H = K_{n+1}(C) \cap Z_G = 1.$$

For, if $(1, z) = (g, h) \in K_{n+1}(C)$, then $g = 1$ and since β is an isomorphism, also $h = 1$. Similarly for $K_{n+1}(C) \cap Z_G$. By lemma 1.3b we therefore have

$$C/Z_H \underset{n}{\sim} C \underset{n}{\sim} C/Z_G.$$

b) Let

$$C_G := \{(cZ_G, cK_{n+1}(C)) | c \in C\}.$$

C_G is a group which is isomorphic to C, since $K_{n+1}(C) \cap Z_G = 1$. Moreover, it follows from lemma 1.3a that $C_G \underset{n}{\sim} C/Z_G \times C/K_{n+1}(C)$, for we have, as we will show,

$$C_G Z_n(C/Z_G \times C/K_{n+1}(C)) = C/Z_G \times C/K_{n+1}(C).$$

Therefore, let $x = (c_1 Z_G, c_2 K_{n+1}(C))$ be an element of the direct product of the groups C/Z_G and $C/K_{n+1}(C)$.

Then $x = yz$, where $y = (c_1 Z_G, c_1 K_{n+1}(C)) \in C_G$, and $z = (Z_G, c_1^{-1} c_2 K_{n+1}(C))$. Since $Z_n(C/K_{n+1}(C)) = C/K_{n+1}(C)$ and Z_G is the identity of C/Z_G it follows that $z \in Z_n(C/Z_G \times C/K_{n+1}(C))$.

Similarly: $C \simeq C_H \underset{n}{\sim} C/Z_H \times C/K_{n+1}(C)$.

2. THE IRREDUCIBLE CHARACTERS OF ISOCLINIC GROUPS

In this section we consider only finite groups. If G is a group, then Irr (G) denotes the set of all irreducible complex characters of G. The number of the irreducible characters of G of degree d is denoted by $r_d(G)$. Suppose G and H are isoclinic groups. If H is a factor group of G, then the irreducible complex characters of G can be computed from the set Irr (H), see [1] lemma II.2.3. We state this result in a more explicit form in lemma 2.1. As a corollary of this lemma and theorem 1.4b we obtain results of P. Hall [2] and J. Tappe [6] on the irreducible characters of isoclinic groups.

2.1. LEMMA. Let G and G/N be isoclinic groups. If $\{\lambda_1, \ldots, \lambda_m\}$ is the set of irreducible characters of the (necessarily abelian) group N, then each λ_i can be extended to a linear character $\hat{\lambda}_i$ of G. If $\{\chi_1, \ldots, \chi_n\}$ is the set of irreducible characters of G with $N < \ker \chi_i$, then $\chi_j \hat{\lambda}_i = \chi_k \hat{\lambda}_t$ if and only if $j = k$ and $i = t$, and

$$\text{Irr}\,(G) = \{\chi_j \hat{\lambda}_i \mid j = 1, 2, \ldots, n, \ i = 1, 2, \ldots, m\}.$$

Hence we have $r_d(G) = |N| r_d(G/N)$.

PROOF. N is a central subgroup of G, for $[G, N] < N \cap G' = 1$. If $\lambda \in \text{Irr}\,(N)$, then λ has an extension $\bar{\lambda}$ to $G'N = G' \times N$, such that $\ker \bar{\lambda} > G'$. Thus λ can be viewed as an irreducible character of a subgroup of G/G', and thus λ has an extension to G. Let Irr $(N) = \{\lambda_1, \ldots, \lambda_m\}$, and denote each extension of λ_j to G by $\hat{\lambda}_j$. Then it follows (taking the restriction to N) by [4] theorem V.17.12b that the cardinality of the set $I(G) := \{\chi_i \lambda_j | \chi_i \in \text{Irr}\,(G/N)\}$ equals nm. To prove that $I(G) = \text{Irr}\,(G)$, assume that $\chi \in \text{Irr}\,(G)$. Then $\chi | N = \chi(1)\mu$, $\mu \in \text{Irr}\,(N)$. If $\hat{\mu}$ is the extension of μ to G, then we have by theorem V.17.12d of [4]:

$$\chi = \hat{\mu}\chi_j, \ \chi_j \in \text{Irr}\,(G/N).$$

Since $m = |N|$ and $\hat{\lambda}_j(1) = 1$, we therefore have $r_d(G) = |N| r_d(G/N)$. //

REMARK. The converse of the above lemma holds also. Thus, if N is a central subgroup of G, then the irreducible characters of N are simultaneously extendible to G if and only if $N \cap G' = 1$.

As a corollary we obtain the following results of P. Hall [2] and J. Tappe [6].

2.2. THEOREM. Let G and H be isoclinic groups. Then

a) (P. Hall) $|H| r_d(G) = |G| r_d(H)$.

b) (J. Tappe) The matrices of the irreducible complex representations of G and H only differ by scalar factors.

PROOF. a) By theorem 1.4a there exist groups C, Z_H and Z_G such that

$$G \simeq C/Z_H \sim C \sim C/Z_G \simeq H. \tag{1}$$

By lemma 2.1 and (1) we obtain the desired result:

$$\frac{r_d(G)}{r_d(H)} = \frac{r_d(C)|Z_H|}{r_d(C)|Z_G|} = \frac{|Z(G)|}{|Z(H)|} = \frac{|G|}{|H|}.$$

b) If χ_1 and χ_2 are irreducible characters of G and H respectively, then there exist linear characters λ_1 and λ_2 of C such that

$$\chi_1 \lambda_1 = \chi_2 \lambda_2.$$

The matrices of the irreducible representations of G and H differ therefore only by scalar factors. //

REMARK. Theorem 2.2 can also be proved via theorem 1.4b and lemma II.2.2 of [1].

3. INVARIANTS OF THE FAMILIES OF FINITE ISOCLINIC GROUPS

In [1] we have proved that the following hierarchy of classes of finite groups is invariant under isoclinisms: abelian, nilpotent, supersolvable, strongly-monomial, monomial, solvable.

The only non-trivial result here is that monomiality is an invariant of the families of isoclinic groups. However, since a finite group is monomial if and only if its irreducible complex matrix representations can be transformed into monomial form, this is now a direct consequence of theorem 2.2b, see also [6].

In general, nilpotency, supersolvability and solvability are invariants of the families of n-isoclinic groups. It is not known, whether monomiality is such an invariant if $n > 2$. We have however, the following result.

3.1. THEOREM. If G and H are finite n-isoclinic groups, then G is strongly-monomial if and only if H is strongly-monomial.

PROOF. A group G is called strongly-monomial (an $\tilde{M}$-group), if G and all its subgroups are monomial. Let S be the set of all ordered pairs (G_1, G_2), where G_1 and G_2 are finite solvable groups.

We write $(G_1, G_2) \prec (H_1, H_2)$, if $|G_1| \leq |H_1|$ and $|G_2| \leq |H_2|$, while at least one of the inequalities is strict. Consider the following subset of S:

$$S_0 := \{(G_1, G_2) \in S \mid G_1 \underset{n}{\sim} G_2,\ G_1 \text{ an } \tilde{M}\text{-group},\ G_2 \text{ not an } \tilde{M}\text{-group}\}.$$

Let (G, H) be a minimal counterexample. Then $(G, H) \in S_0$ and there is no element $(G_1, H_1) \in S_0$ such that $(G_1, H_1) \prec (G, H)$.

STEP 1. H is a minimal (solvable) non-M-group, that is: H is non-monomial, but each proper subgroup and each proper factor group of H is monomial.

PROOF OF STEP 1. If H is monomial, then H has a non-monomial proper subgroup H_2. By lemma 1.2a there exists $H_1 \leqslant G$ such that $H_1 \underset{n}{\simeq} H_2$. Therefore $(H_1, H_2) \in S_0$ and $(H_1, H_2) \prec (G, H)$. Contradiction. Similarly, each proper subgroup of H is monomial. Let H/N be a proper non-monomial factor group of H. Then by lemma 1.2b there exists a factor group G/N_1 such that $G/N_1 \underset{n}{\simeq} H/N_2$. Since $(G/N_1, H/N_2) \in S_0$ this yields again a contradiction.

Our proof is now based on the structure of the solvable minimal non-M-group H. By theorem 1.4 of D. T. Price [5] the group H has a normal p-subgroup F such that:

a1) F is extra-special of exponent p, p prime, $p=2$.
a2) F is an extra-special 2-group, but not dihedral.
b) $H=FA$, where A acts trivially on $Z(F)$ and irreducibly on $F/Z(F)$.
c) Either A is a p'-group or $p=2$ and $A/O_{2'}(A)$ is a cyclic 2-group.
d) $O_{p'}(H)=1$.
e) If A is of odd order, then A is of prime order.

STEP 2. $F < K_{\infty}(H)$.

PROOF OF STEP 2. Since $[F, A]$ is an A-invariant subgroup of F, we have either $F'[F, A]=F$ or $[F, A] < F' = Z(F) \leqslant Z(H)$. If $[F, A] < Z(H)$, then $[F, O_{p'}(A)] < Z(H)$, so that $O_{p'}(A)Z(H) \lhd H$. This yields $O_{p'}(A) \lhd H$, and thus $O_{p'}(A) < O_{p'}(H) = 1$. But if $O_{p'}(A)=1$, then H is a p-group. For, if A is a p'-group, then $A = O_{p'}(A) = 1$, and if $p=2$ and $A/O_{2'}(A)$ is a cyclic 2-group, then $O_{2'}(A)=1$ would imply that A, whence also H, is a cyclic 2-group. Therefore, we have $F = F'[F, A]$, $F' = Z(F) < Z(H)$, so that $F' = [F, A]'$.

Conclusion: $F=[F, A]$ and $F < H'$.

This implies $F=[F, A] < [H', H] = K_3(H)$. With induction it follows that $F < K_{\infty}(H)$.

STEP 3. G is not an $\tilde{M}$-group.

PROOF OF STEP 3. $F/Z(F)$ is a chief section of H. Since $F < K_{n+1}(H)$, there exists by lemma 1.2b a group $F_1 < G$, such that $F_1 \simeq F$ and such that $F_1/Z(F_1)$ is a chief section of G. Since $Z(F)$ is a central subgroup of H we have by the same lemma that $Z(F_1)$ is a central subgroup of G. It has been proved by Price [5] theorem 4, that $Z(F)$ has a so-called ramified character λ, that is: λ is an H-invariant character of $Z(F)$ such that the induced character λ^F equals $e\chi$, where $\chi \in \mathrm{Irr}\,(F)$ and $e^2 = [F : Z(F)]$.

It will be clear now that $F_1/Z(F_1)$ is a chief section of G with at least one ramified character. But this contradicts theorem 1.3 of [5] stating that an $\tilde{M}$-group has no chief sections with a ramified character. //

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