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Huard type second-order converse duality for nonlinear programming ☆

X.M. Yang[a,*], X.Q. Yang[b], K.L. Teo[b]

[a] Department of Mathematics, Chongqing Normal University, Chongqing 400047, PR China
[b] Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, China

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Abstract

In this paper, we establish a Huard type converse duality for a second-order dual model in nonlinear programming using Fritz John necessary optimality conditions.

Keywords: Fritz John second order dual model; Huard type converse duality; Nonlinear programming

1. Introduction

Consider the nonlinear programming problem **NP**

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } g(x) \leq 0, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, f and g are twice differentiable functions from $\mathbb{R}^n$ into $\mathbb{R}$ and $\mathbb{R}^m$, respectively.

A second-order dual for such a nonlinear programming problem was introduced by Mangasarian [1]. Later, Mond [2] proved duality theorems under a condition which is called “second-order convexity”.

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* Corresponding author.
E-mail address: xmyang@cqnu.edu.cn (X.M. Yang).

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This condition is much simpler than that used by Mangasarian. Furthermore, Mond and Weir [3] reformulated the second-order dual.

Recently, Husain et al. formulated another second-order dual: **ND**:

$$\text{maximize } f(x) - \frac{1}{2}p^T\nabla^2 f(x)p,$$

$$\text{subject to } r(\nabla f(x) + \nabla^2 f(x)p) + \nabla(y^T g(x)) + \nabla^2(y^T g(x))p = 0, \tag{2}$$

$$y^T g(x) - \frac{1}{2}p^T\nabla^2(y^T g(x))p \geq 0, \tag{3}$$

$$(r, y) \geq 0, \tag{4}$$

$$(r, y) \neq 0, \tag{5}$$

where $p \in \mathbb{R}^n$, $r \in \mathbb{R}$ and for any function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$, the symbol $\nabla^2\phi(x)$ designates the $n \times n$ symmetric matrix of second-order partial derivatives. It is based on the Fritz John necessary optimality condition, while the Mond and Weir dual model uses the Karush–Kuhn–Tucker necessary optimality condition. Husain et al. [4] give a weak duality, a strong duality, a Mangasarian type strict converse duality and a Huard type converse duality under the conditions that f is pseudobonvex and $y^T g$ is semi-strictly pseudobonvex, where "pseudobonvexity" was defined by Mond and Weir as an extension of the second-order convexity. Thus, the duality relation does not require a constraint qualification. In particular, they prove the following Huard type converse duality theorem.

Theorem 1 (*Converse Duality (see Theorem 2.4 in [4])*). *Let* (r^*, x^*, y^*, p^*) *be an optimal solution of (ND) at which*

(A1) *the* $n \times n$ *Hessian matrix* $\nabla[r^*\nabla^2 f(x^*) + \nabla^2(y^{*T} g(x^*))]p^*$ *is positive or negative definite,*

(A2) $\nabla(y^{*T} g(x^*)) + \nabla^2(y^{*T} g(x^*))p^* \neq 0$, *and*

(A3) *the vector* $\{[\nabla^2 f(x^*)]_j, [\nabla^2(y^{*T} g(x^*))]_j, j = 1, 2, \dots, n\}$ *are linearly independent, where* $[\nabla^2 f(x^*)]_j$ *is the* j*th row of* $[\nabla^2 f(x^*)]$ *and* $[\nabla^2(y^{*T} g(x^*))]_j$ *is the* j*th row of* $[\nabla^2(y^{*T} g(x^*))]$.

If, for all feasible (r^*, x^*, y^*, p^*), $f(\cdot)$ is pseudobonvex and $y^{*T} g(\cdot)$ is semi-strictly pseudobonvex, then x^* is an optimal solution of (NP).

We note that the matrix $\nabla[r^*\nabla^2 f(x^*) + \nabla^2(y^{*T} g(x^*))]p^*$ is positive or negative definite in the assumption (A1) of Theorem 1, and the result of Theorem 1 implies $p^* = 0$; see the proof of Theorem 2.4 in [4]. It is obvious that the assumption and the result are inconsistent. In this note, we will give an appropriate modification for this deficiency contained in Theorem 1.

2. Huard type second-order converse duality

In the section, we will present a new Huard type second-order converse duality theorem which is a correction of Theorem 1.

Theorem 2 (*Converse Duality*). *Let* (r^*, x^*, y^*, p^*) *be an optimal solution of (ND) at which*

(B1) *either* (a) *the* $n \times n$ *Hessian matrix* $\nabla^2(y^{*T} g(x^*))$ *is positive definite and* $p^{*T}\nabla g(x^*) \geq 0$ *or* (b) *the* $n \times n$ *Hessian matrix* $\nabla^2(y^{*T} g(x^*))$ *is negative definite and* $p^{*T}\nabla g(x^*) \leq 0$,

(A2) $\nabla(y^{*T} g(x^*)) + \nabla^2(y^{*T} g(x^*))p^* \neq 0$, *and*

(A3) *the vector* $\{[\nabla^2 f(x^*)]_j, [\nabla^2(y^{*T} g(x^*))]_j, j = 1, 2, \dots, n\}$ *are linearly independent, where* $[\nabla^2 f(x^*)]_j$ *is the* j*th row of* $[\nabla^2 f(x^*)]$ *and* $[\nabla^2(y^{*T} g(x^*))]_j$ *is the* j*th row of* $[\nabla^2(y^{*T} g(x^*))]$.

If, for all feasible (r^*, x^*, y^*, p^*), $f(\cdot)$ is pseudobonvex and $y^{*T}g(\cdot)$ is semi-strictly pseudobonvex, then x^* is an optimal solution of (NP).

Proof. Since (r^*, x^*, y^*, p^*) is an optimal solution of (ND), by the generalized Fritz John necessary condition, there exist $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^n$, $\theta \in \mathbb{R}$, $\xi \in \mathbb{R}$, and $\eta \in \mathbb{R}^m$ such that

$$\begin{aligned}&-\alpha\left\{\nabla f(x^*) - \frac{1}{2}p^{*T}\nabla(\nabla^2 f(x^*)p^*)\right\}\\&+\beta^T\{r^*(\nabla^2 f(x^*) + \nabla(\nabla^2 f(x^*)p^*)) + \nabla^2(y^{*T}g(x^*)) + \nabla(\nabla^2(y^{*T}g(x^*))p^*)\}\\&-\theta\left\{\nabla(y^{*T}g(x^*)) - \frac{1}{2}p^{*T}\nabla(\nabla^2(y^{*T}g(x^*))p^*)\right\} = 0,\end{aligned} \tag{6}$$

$$\beta^T\left[\nabla(g(x^*)) + \nabla^2(g(x^*))p^*\right] - \theta\left[g(x^*) - \frac{1}{2}p^{*T}\nabla^2 g(x^*)p^*\right] - \eta = 0, \tag{7}$$

$$\beta^T[\nabla(f(x^*)) + \nabla^2(f(x^*))p^*] - \xi = 0, \tag{8}$$

$$(\alpha p^* + \beta r^*)^T[\nabla^2 f(x^*)] + (\theta p^* + \beta)^T[\nabla^2 y^{*T}g(x^*)] = 0, \tag{9}$$

$$\theta^T[y^{*T}g(x^*) - \frac{1}{2}p^{*T}\nabla^2(y^{*T}g(x^*))p^*] = 0, \tag{10}$$

$$\eta^T y^* = 0, \tag{11}$$

$$\xi^T r^* = 0, \tag{12}$$

$$(\alpha, \beta, \theta, \xi, \eta) \geq 0, \tag{13}$$

$$(\alpha, \beta, \theta, \xi, \eta) \neq 0. \tag{14}$$

Because of assumption (A3), (9) gives

$$\alpha p^* + r^*\beta = 0 \qquad \text{and} \qquad \theta p^* + \beta = 0. \tag{15}$$

Multiplying (7) by y^{*T} and then using (10) and (11), we have

$$\beta^T[\nabla(y^{*T}g(x^*)) + \nabla^2(y^{*T}g(x^*))p^*] = 0. \tag{16}$$

Using (2) in (6), we have

$$\begin{aligned}&(\alpha p^* + r^*\beta)^T[r^*(\nabla^2 f(x^*)) + \nabla(\nabla^2 f(x^*)p^*)]\\&\quad + r^*(\theta p^* + \beta)^T[\nabla^2 y^{*T}g(x^*) + \nabla(\nabla^2 y^{*T}g(x^*)p^*)] + (\alpha - r^*\theta)[\nabla y^{*T}g(x^*)\\&\quad + \nabla^2 y^{*T}g(x^*)p^*] - \frac{1}{2}r^*(\alpha p^*)^T\nabla(\nabla^2(f(x^*))p^*) - \frac{1}{2}r^*(\theta p^*)^T\nabla(\nabla^2(y^{*T}g(x^*))p^*) = 0.\end{aligned} \tag{17}$$

Using (15) and (17) gives

$$\begin{aligned}&(\alpha - r^*\theta)[\nabla y^{*T}g(x^*) + \nabla^2(y^{*T}g(x^*))p^*]\\&\quad + \frac{1}{2}(\beta r^*)^T\{\nabla(\nabla^2(f(x^*))) + \nabla^2(y^{*T}g(x^*))p^*\} = 0.\end{aligned} \tag{18}$$

We claim that $\alpha \neq 0$. Indeed, if $\alpha = 0$, then (15) gives

$$r^*\beta = 0.$$

In view of (A2), the equality constraint of (ND) implies $r^* \neq 0$ and so $\beta = 0$. Using $\beta = 0$ in (18), we have

$$(\alpha - r^*\theta)(\nabla y^{*T}g(x^*) + \nabla^2(y^{*T}g(x^*))p^*) = 0.$$

In view of (A2) again, this gives

$$\theta = \frac{\alpha}{r^*}. \tag{19}$$

So we have $\theta = 0$. Now from (7) and (8) and $\beta = 0$, it follows that $\eta = \xi = 0$. Hence, $(\alpha, \beta, \theta, \xi, \eta) = 0$, which contradicts (14). Thus, $\alpha > 0$, and from (19), $\theta > 0$. Using $\theta > 0$ and (15) and (16) yields

$$p^{*T}[\nabla(y^{*T} g(x^*)) + \nabla^2(y^{*T} g(x^*))p^*] = 0. \tag{20}$$

We now prove that $p^* = 0$. Otherwise, assumption (B1) implies that $p^{*T}[\nabla(y^{*T} g(x^*)) + \nabla^2(y^{*T} g(x^*))p^*] \neq 0$, contradicting (20). Hence, $p^* = 0$. This gives

$$f(x^*) = f(x^*) - \frac{1}{2} p^{*T} \nabla^2 f(x^*) p^*.$$

From (15) and $p^* = 0$, we know that $\beta = 0$. Using $\theta > 0$, $\beta = 0$ and $p^* = 0$, (7) gives

$$g(x^*) \leq 0.$$

Thus, x^* is feasible for (NP), and the objective functions of (NP) and (ND) are equal.

If, for all feasible (r, x, y, p), $f(\cdot)$ is pseudobonvex and $y^{*T} g(\cdot)$ is semi-strictly pseudobonvex, by Theorem 2.1 in [4], x^* is an optimal solution of (NP). □

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