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The Distribution of Zeros of Solutions of Neutral Advanced Differential Equations

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Abstract—The distribution of zeros of solutions of the neutral advanced differential equations

$$[x(t)+P(t)x(t+\tau)]' - Q(t)x(t+\sigma) = 0, \qquad t \geq t_0,$$

is investigated, where $P(t), Q(t) \in C([t_0,\infty), R^+)$, $\tau, \sigma \in R^+$. The estimate for the distance between adjacent zeros of the oscillatory solution of the above equation is obtained.

Keywords—Distribution of zeros, Neutral advanced equation.

1. INTRODUCTION

Recently, there are a lot of activities concerning the distribution of zeros of solutions of delay or neutral delay differential equations; for example, see [1–7]. But, for the distribution of zeros of solutions of advanced differential equations, compared with those of delay differential equations, less is known up to now. This paper is devoted to the study of the distribution of zeros of solutions of the following neutral advanced differential equations:

$$[x(t)+P(t)x(t+\tau)]' - Q(t)x(t+\sigma) = 0, \qquad t \geq t_0, \tag{1}$$

where

$$P(t), Q(t) \in C\left([t_0,\infty), R^+\right), \qquad \tau, \sigma \in R^+. \tag{2}$$

In this paper, we first give several lemmas which will enable us to prove our main results. Next, we study the distribution of zeros of solutions of equation (1). The estimate for the distance between adjacent zeros of the oscillatory solution of equation (1) is obtained. Finally, two examples are given to illustrate our results.

2. LEMMAS

First, we define a sequence $\{f_n(\rho)\}$, $0 < \rho < 1$, by [5]

$$f_0(\rho) = 1, \quad f_{n+1}(\rho) = e^{\rho f_n(\rho)}, \qquad n = 0,1,2,\dots. \tag{3}$$

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It is easily seen, that for $\rho > 0$,

$$f_{n+1}(\rho) > f_n(\rho), \qquad n = 1, 2, \dots.$$

Observe by [5], that when $0 < \rho \le 1/e$, then there exists a function $f(\rho)$ such that

$$\lim_{n\to\infty} f_n(\rho) = f(\rho), \qquad 1 \le f(\rho) \le e,$$

and

$$f(\rho) = e^{\rho f(\rho)}. \tag{4}$$

However, when $\rho > 1/e$, then

$$\lim_{n\to\infty} f_n(\rho) = +\infty.$$

Next, we also define a sequence $\{g_m(\rho)\}$, $0 < \rho < 1$, by [5]

$$g_1(\rho) = \frac{2(1-\rho)}{\rho^2}, \quad g_{m+1}(\rho) = \frac{2(1-\rho)}{\rho^2 + 2/g_m^2(\rho)}, \qquad m = 1, 2, \dots. \tag{5}$$

It is easily seen that for $0 < \rho < 1$,

$$g_{m+1}(\rho) < g_m(\rho), \qquad m = 1, 2, \dots.$$

Observe by [5], that when $0 < \rho \le 1/e$, then there exists a function $g(\rho)$ such that

$$\lim_{m\to\infty} g_m(\rho) = g(\rho)$$

and

$$g(\rho) = \frac{2}{1 - \rho - \sqrt{1 - 2\rho - \rho^2}}, \qquad \text{for } 0 < \rho \le \frac{1}{e}. \tag{6}$$

To prove our main results, we need the following lemmas.

Consider the advanced differential inequality

$$x'(t) - Q(t)x(t+\sigma) \ge 0. \tag{7}$$

LEMMA 1. *Suppose that $Q(t) \in C([t_0, \infty), R^+)$, $\sigma \in R^+$ and let $x(t)$ be a solution of inequality (7) on $[t_0, \infty)$. Further, assume that there exist $t_1 \ge t_0$ and $0 < \rho < 1$ such that*

$$\int_t^{t+\sigma} Q(s)\, ds \ge \rho, \qquad \text{for } t \ge t_1, \tag{8}$$

and that there exist $T_0 \ge t_1$ and $T \ge T_0 + 3\sigma$ such that $x(t)$ is positive on $[T_0, T]$. Then, for any $n \ge 1$ such that $T - (2+n)\sigma \ge T_0$,

$$\frac{x(t+\sigma)}{x(t)} \ge f_n(\rho), \qquad \text{for } t \in [T_0, T - (2+n)\sigma], \tag{9}$$

where $f_n(\rho)$ is defined by (3).

PROOF. From (7), we obtain

$$x'(t) \ge Q(t)x(t+\sigma) \ge 0, \qquad \text{for } t \in [T_0, T - \sigma], \tag{10}$$

which implies that $x(t)$ is nondecreasing on $[T_0, T - \sigma]$. It follows that

$$\frac{x(t+\sigma)}{x(t)} \ge 1 = f_0(\rho), \qquad \text{for } t \in [T_0, T - 2\sigma]. \tag{11}$$

When $T_0 \le t \le T - 3\sigma$, by dividing (7) by $x(t)$ and integrating from t to $t+\sigma$, we get

$$\ln \frac{x(t+\sigma)}{x(t)} - \int_t^{t+\sigma} Q(s) \frac{x(s+\sigma)}{x(s)} \, ds \ge 0. \tag{12}$$

By using (8), (11), and (12), we have

$$\ln \frac{x(t+\sigma)}{x(t)} \ge \int_t^{t+\sigma} Q(s) \frac{x(s+\sigma)}{x(s)} \, ds \ge \rho f_0(\rho).$$

It follows that

$$\frac{x(t+\sigma)}{x(t)} \ge e^{\rho f_0(\rho)} = f_1(\rho), \qquad \text{for } t \in [T_0, T - 3\sigma].$$

Repeating the above procedure, we get

$$\frac{x(t+\sigma)}{x(t)} \ge e^{\rho f_{n-1}(\rho)} = f_n(\rho), \qquad \text{for } t \in [T_0, T - (2+n)\sigma].$$

The proof of Lemma 1 is complete.

LEMMA 2. *Suppose that $Q(t) \in C([t_0, \infty), R^+)$, $\sigma \in R^+$ and let $x(t)$ be a solution of inequality (7) on $[t_0, \infty)$. Assume that there exist $t_1 \ge t_0$ and a positive constant $\rho < 1$ such that*

$$\int_t^{t+\sigma} Q(s) \, ds \ge \rho, \qquad \text{for } t \ge t_1, \tag{13}$$

and

$$\int_t^{s+\sigma} Q(u) \, du \ge \int_{t-\sigma}^{s} Q(u) \, du, \qquad \text{for } t_1 \le t \le s+\sigma \le t+\sigma. \tag{14}$$

Further, assume that there exist $T_0 \ge t_1 + \sigma$ and a positive integer $N \ge 4$ such that $x(t)$ is positive on $[T_0, T_0 + N\sigma]$. Then, for any $m \le N-3$,

$$\frac{x(t+\sigma)}{x(t)} < g_m(\rho), \qquad \text{for } t \in [T_0 + m\sigma, T_0 + (N-3)\sigma], \tag{15}$$

where $g_m(\rho)$ is defined by (5).

PROOF. From (13), we know that

$$\int_{t-\sigma}^{t} Q(s) \, ds \ge \rho, \qquad \text{for } t \ge t_1 + \sigma.$$

Note that $F(\lambda) = \int_{t-\sigma}^{\lambda} Q(s) \, ds$ is a continuous function. $F(t-\sigma) = 0$ and $F(t) \ge \rho$. Thus, there exists a λ_t such that $\int_{t-\sigma}^{\lambda_t} Q(s) \, ds = \rho$, where $t - \sigma < \lambda_t \le t$. When $T_0 + \sigma \le t \le T_0 + (N-3)\sigma$, integrating both sides of (7) for $t - \sigma$ to λ_t, we obtain

$$x(\lambda_t) - x(t-\sigma) \ge \int_{t-\sigma}^{\lambda_t} Q(s) x(s+\sigma) \, ds. \tag{16}$$

Since $t - \sigma \le s \le t$, we easily see that $t \le s + \sigma \le t + \sigma \le T_0 + (N-2)\sigma$. Integrating both sides of (7) from t to $s + \sigma$, we get

$$x(s+\sigma) - x(t) \ge \int_t^{s+\sigma} Q(u) x(u+\sigma) \, du.$$

From (10), it is clear that $x(u+\sigma)$ is nondecreasing on $T_0 \le u \le T_0 + (N-2)\sigma$. By (14), we get

$$x(s+\sigma) \ge x(t) + x(t+\sigma) \int_t^{s+\sigma} Q(u)\,du \ge x(t) + x(t+\sigma) \int_{t-\sigma}^{s} Q(u)\,du. \tag{17}$$

From (16) and (17), we have

$$\begin{aligned} x(\lambda_t) &\ge x(t-\sigma) + \int_{t-\sigma}^{\lambda_t} Q(s)x(s+\sigma)\,ds \\ &\ge x(t-\sigma) + \int_{t-\sigma}^{\lambda_t} Q(s)\left[x(t) + x(t+\sigma)\int_{t-\sigma}^{s} Q(u)\,du\right] ds \\ &= x(t-\sigma) + \rho x(t) + x(t+\sigma)\int_{t-\sigma}^{\lambda_t} Q(s)\,ds \int_{t-\sigma}^{s} Q(u)\,du \\ &= x(t-\sigma) + \rho x(t) + \frac{\rho^2}{2} x(t+\sigma). \end{aligned}$$

Noting that $x(\lambda_t) \le x(t)$, we get

$$x(t) \ge x(t-\sigma) + \rho x(t) + \frac{\rho^2}{2} x(t+\sigma). \tag{18}$$

Again since $x(t-\sigma) > 0$ for $t \in [T_0 + \sigma, T_0 + (N-3)\sigma]$, by (18), we obtain

$$\frac{x(t+\sigma)}{x(t)} < \frac{2(1-\rho)}{\rho^2} = g_1(\rho), \qquad \text{for } t \in [T_0+\sigma, T_0+(N-3)\sigma]. \tag{19}$$

When $T_0 + 2\sigma \le t \le T_0 + (N-3)\sigma$, we easily see that $T_0 + \sigma \le t - \sigma \le T_0 + (N-4)\sigma$. Thus, by (19), we have

$$x(t-\sigma) > \frac{x(t)}{g_1(\rho)} > \frac{x(t+\sigma)}{g_1^2(\rho)}.$$

Substituting this into (18), we have

$$x(t) > \frac{x(t+\sigma)}{g_1^2(\rho)} + \rho x(t) + \frac{\rho^2}{2} x(t+\sigma), \qquad \text{for } t \in [T_0+2\sigma, T_0+(N-3)\sigma].$$

Therefore,

$$\frac{x(t+\sigma)}{x(t)} < \frac{2(1-\rho)}{\rho^2 + 2/g_1^2(\rho)} = g_2(\rho), \qquad \text{for } t \in [T_0+2\sigma, T_0+(N-3)\sigma].$$

Repeating the above procedure, we obtain

$$\frac{x(t+\sigma)}{x(t)} < \frac{2(1-\rho)}{\rho^2 + 2/g_{m-1}^2(\rho)} = g_m(\rho), \qquad \text{for } t \in [T_0+m\sigma, T_0+(N-3)\sigma]. \tag{20}$$

The proof of Lemma 2 is complete.

3. MAIN RESULTS

THEOREM 1. *Suppose that (2) holds and that $R(t) \in C^1([t_0,\infty), R^+)$, where*

$$R(t) = \frac{Q(t)P(t+\sigma)}{Q(t+\tau)}. \tag{21}$$

Further, assume that

$$R'(t) \geq 0, \qquad \sigma > \tau > 0, \tag{22}$$

and that there exists $t_1 \geq t_0$ such that

$$\int_t^{t+\sigma-\tau} \frac{Q(s)}{1+R(s+\sigma-\tau)}\, ds \geq 1, \qquad \text{for } t \geq t_1. \tag{23}$$

Then, for any $T \geq t_1$, every solution of equation (1) has at least a zero on $[T, T+3\sigma-\tau]$.

PROOF. Otherwise, without loss of generality, we may assume that $x(t)$ is a solution of equation (1) satisfying $x(t) > 0$ for $t \in [T, T+3\sigma-\tau]$. For the convenience, in the sequel, we denote

$$z(t) = x(t) + P(t)x(t+\tau). \tag{24}$$

Clearly, we have

$$z(t) > 0, \qquad \text{for } t \in [T, T+3\sigma-2\tau], \tag{25}$$

and

$$z'(t) = Q(t)x(t+\sigma) \geq 0, \qquad \text{for } t \in [T, T+2\sigma-\tau], \tag{26}$$

which implies that $z(t)$ is nondecreasing on $[T, T+2\sigma-\tau]$. By (26) and (24), we obtain

$$\begin{aligned} z'(t) &= Q(t)x(t+\sigma) \\ &= Q(t)[z(t+\sigma) - P(t+\sigma)x(t+\sigma+\tau)] \\ &= Q(t)z(t+\sigma) - \frac{Q(t)P(t+\sigma)}{Q(t+\tau)} z'(t+\tau). \end{aligned}$$

That is,

$$z'(t) + R(t)z'(t+\tau) - Q(t)z(t+\sigma) = 0, \qquad \text{for } t \geq T, \tag{27}$$

where $R(t) = Q(t)P(t+\sigma)/Q(t+\tau) \geq 0$. We set

$$w(t) = z(t) + R(t)z(t+\tau), \qquad \text{for } t \geq T. \tag{28}$$

Thus, we have by (25) and (28)

$$w(t) > 0, \qquad \text{for } t \in [T, T+3(\sigma-\tau)], \tag{29}$$

and by (22), (25), and (27),

$$w'(t) = R'(t)z(t+\tau) + Q(t)z(t+\sigma) \geq 0, \qquad \text{for } t \in [T, T+2(\sigma-\tau)]. \tag{30}$$

Since $z(t)$ is nondecreasing on $[T, T+2\sigma-\tau]$, we have by (28)

$$w(t) \leq [1+R(t)]z(t+\tau), \qquad \text{for } t \in [T, T+2(\sigma-\tau)].$$

Thus,

$$z(t+\tau) \geq \frac{w(t)}{1+R(t)}, \qquad \text{for } t \in [T, T+2(\sigma-\tau)].$$

It follows by (22) and (30), that

$$w'(t) - \frac{Q(t)}{1+R(t+\sigma-\tau)} w(t+\sigma-\tau) \geq w'(t) - Q(t)z(t+\sigma) \geq 0, \qquad \text{for } t \in [T, T+\sigma-\tau]. \tag{31}$$

Integrating both sides of (31) from T to $T+\sigma-\tau$, we obtain by (23),(30)

$$\begin{aligned} w(T+\sigma-\tau) &\geq w(T) + \int_T^{T+\sigma-\tau} \frac{Q(s)}{1+R(s+\sigma-\tau)} w(s+\sigma-\tau)\, ds \\ &\geq w(T) + w(T+\sigma-\tau). \end{aligned}$$

That is, $w(T) \leq 0$, which contradicts (29) and completes the proof.

COROLLARY 1. *Assume that the conditions in Theorem 1 hold. Then, the distances between adjacent zeros of every solution of equation (1) on* $[t_1, \infty)$ *are less than* $3\sigma - \tau$.

THEOREM 2. *Assume that (2), (21), and (22) hold and that there exist* $t_1 \geq t_0$ *and a positive constant* ρ, $1/e < \rho < 1$, *such that*

$$\int_t^{t+\sigma-\tau} \frac{Q(s)}{1+R(s+\sigma-\tau)}\, ds \geq \rho. \tag{32}$$

Further, assume that

$$\int_t^{s+\sigma-\tau} \frac{Q(u)}{1+R(u+\sigma-\tau)}\, du \geq \int_{t-\sigma+\tau}^{s} \frac{Q(u)}{1+R(u+\sigma-\tau)}\, du, \qquad t_1 \leq t \leq s+\sigma-\tau \leq t+\sigma-\tau. \tag{33}$$

Then, for any $T \geq t_1 + \sigma - \tau$, *every solution of equation (1) has at least a zero on* $[T, T+2\sigma + k(\sigma-\tau)]$, *where*

$$k = \min_{n\geq 1, m\geq 1} \{n+m \mid f_n(\rho) \geq g_m(\rho)\}. \tag{34}$$

PROOF. Otherwise, without loss of generality, we assume that $x(t)$ is a solution of equation (1) satisfying $x(t) > 0$ for $t \in [T, T+2\sigma+k(\sigma-\tau)]$. By the proof of Theorem 1, we obtain

$$w'(t) - \frac{Q(t)}{1+R(t+\sigma-\tau)} w(t+\sigma-\tau) \geq 0, \qquad \text{for } t \in [T, T+k(\sigma-\tau)],$$

and

$$w(t) > 0, \qquad \text{for } t \in [T, T+(k+2)(\sigma-\tau)].$$

Let $k = n^* + m^*$ satisfy

$$f_{n^*}(\rho) \geq g_{m^*}(\rho). \tag{35}$$

By Lemma 1, we have

$$\frac{w(t+\sigma-\tau)}{w(t)} \geq f_{n^*}(\rho), \qquad \text{for } t \in [T, T+(k-n^*)(\sigma-\tau)]. \tag{36}$$

On the other hand, by Lemma 2, we obtain

$$\frac{w(t+\sigma-\tau)}{w(t)} < g_{m^*}(\rho), \qquad \text{for } t \in [T+m^*(\sigma-\tau), T+(k-1)(\sigma-\tau)]. \tag{37}$$

Setting $t^* = T + (k-n^*)(\sigma-\tau) = T + m^*(\sigma-\tau)$ in (36) and (37), we have

$$f_{n^*}(\rho) \leq \frac{w(t^*+\sigma-\tau)}{w(t^*)} < g_{m^*}(\rho),$$

which contradicts (35) and completes the proof.

COROLLARY 2. *Assume that the conditions in Theorem 2 hold. Then, the distances between the adjacent zeros of every solution of equation (1) on* $[t_1+\sigma-\tau, \infty)$ *are less than* $2\sigma + k(\sigma-\tau)$, *where* k *is defined by (34).*

THEOREM 3. *Assume that (2), (21), (22), and (33) hold and that there exist $t_1 \ge t_0$ and a constant ρ, $0 < \rho \le 1/e$, such that (32) holds. Further, assume that there exists a sequence $\{T_i\} : T_i \to \infty$ as $i \to \infty$ such that*

$$\int_{T_i}^{T_i+\sigma-\tau} \frac{Q(s)}{1+R(s+\sigma-\tau)}\, ds \ge L > \frac{1+\ln f(\rho)}{f(\rho)} - \frac{1-\rho-\sqrt{1-2\rho-\rho^2}}{2}, \tag{38}$$

where $f(\rho)$ satisfies equation (4) on $[1, e]$. Then, every solution of equation (1) has at least a zero on $[T_i - m^*(\sigma-\tau), T_i + 2\sigma + n^*(\sigma-\tau)]$, where m^* and n^* satisfy $T_i \ge t_1 + (m^*+1)(\sigma-\tau)$ and

$$n^* + m^* = \min_{n\ge1, m\ge1} \left\{ n+m \mid L > \frac{1+\ln f_{n-1}(\rho)}{f_{n-1}(\rho)} - \frac{1}{g_m(\rho)} \right\}. \tag{39}$$

PROOF. Otherwise, without loss of generality, we assume that $x(t)$ is a solution of equation (1) satisfying $x(t) > 0$ for $t \in [T_i - m^*(\sigma-\tau), T_i + 2\sigma + n^*(\sigma-\tau)]$. By the proof of Theorem 1, we have

$$w'(t) - \frac{Q(t)}{1+R(t+\sigma-\tau)} w(t+\sigma-\tau) \ge 0, \qquad \text{for } t \in [T_i - m^*(\sigma-\tau), T_i + n^*(\sigma-\tau)], \tag{40}$$

and $w(t) > 0$, for $t \in [T_i - m^*(\sigma-\tau), T_i + (n^*+2)(\sigma-\tau)]$. By Lemmas 1 and 2, we have

$$\frac{w(t+\sigma-\tau)}{w(t)} \ge f_{n^*}(\rho), \qquad \text{for } t \in [T_i - m^*(\sigma-\tau), T_i], \tag{41}$$

$$\frac{w(t+\sigma-\tau)}{w(t)} \ge f_{n^*-1}(\rho), \qquad \text{for } t \in [T_i - m^*(\sigma-\tau), T_i + \sigma - \tau], \tag{42}$$

and

$$\frac{w(t)}{w(t+\sigma-\tau)} > \frac{1}{g_{m^*}(\rho)}, \qquad \text{for } t \in [T_i, T_i + (n^*-1)(\sigma-\tau)]. \tag{43}$$

Clearly, from (30) we have

$$w'(t) \ge 0, \qquad \text{for } t \in [T_i - m^*(\sigma-\tau), T_i + (n^*+1)(\sigma-\tau)].$$

That is, $w(t)$ is nondecreasing on $t \in [T_i - m^*(\sigma-\tau), T_i + (n^*+1)(\sigma-\tau)]$. From (39), n^* and m^* satisfy

$$L > \frac{1+\ln f_{n^*-1}(\rho)}{f_{n^*-1}(\rho)} - \frac{1}{g_{m^*}(\rho)}. \tag{44}$$

Since $\{f_n(\rho)\}$ is increasing, by (41) we have

$$\frac{w(T_i+\sigma-\tau)}{w(T_i)} \ge f_{n^*-1}(\rho). \tag{45}$$

Since $w(t)$ is nondecreasing, there exists a $t_i^* \in (T_i, T_i+\sigma-\tau)$ such that

$$\frac{w(T_i+\sigma-\tau)}{w(t_i^*)} = f_{n^*-1}(\rho). \tag{46}$$

Integrating (40) from T_i to t_i^* and noting that $w(t)$ is nondecreasing, we obtain

$$\begin{aligned} w(t_i^*) - w(T_i) &\ge \int_{T_i}^{t_i^*} \frac{Q(s)}{1+R(s+\sigma-\tau)} w(s+\sigma-\tau)\, ds \\ &\ge w(T_i+\sigma-\tau) \int_{T_i}^{t_i^*} \frac{Q(s)}{1+R(s+\sigma-\tau)}\, ds, \end{aligned}$$

which implies

$$\int_{T_i}^{t_i^*} \frac{Q(s)}{1+R(s+\sigma-\tau)}\,ds \le \frac{w(t_i^*)}{w(T_i+\sigma-\tau)} - \frac{w(T_i)}{w(T_i+\sigma-\tau)}. \tag{47}$$

From (43), (46), and (47), we obtain

$$\int_{T_i}^{t_i^*} \frac{Q(s)}{1+R(s+\sigma-\tau)}\,ds \le \frac{1}{f_{n^*-1}(\rho)} - \frac{1}{g_{m^*}(\rho)}. \tag{48}$$

Dividing (40) by $w(t)$ and integrating from t_i^* to $T_i+\sigma-\tau$ and noting (42), we have

$$\begin{aligned}\int_{t_i^*}^{T_i+\sigma-\tau} \frac{w'(s)}{w(s)}\,ds &\ge \int_{t_i^*}^{T_i+\sigma-\tau} \frac{Q(s)}{1+R(s+\sigma-\tau)}\cdot\frac{w(s+\sigma-\tau)}{w(s)}\,ds \\ &\ge f_{n^*-1}(\rho)\int_{t_i^*}^{T_i+\sigma-\tau} \frac{Q(s)}{1+R(s+\sigma-\tau)}\,ds,\end{aligned}$$

which implies

$$\int_{t_i^*}^{T_i+\sigma-\tau} \frac{Q(s)}{1+R(s+\sigma-\tau)}\,ds \le \frac{1}{f_{n^*-1}(\rho)}\int_{t_i^*}^{T_i+\sigma-\tau} \frac{w'(s)}{w(s)}\,ds = \frac{\ln f_{n^*-1}(\rho)}{f_{n^*-1}(\rho)}. \tag{49}$$

From (44), (48), and (49), we have

$$\int_{T_i}^{T_i+\sigma-\tau} \frac{Q(s)}{1+R(s+\sigma-\tau)}\,ds \le \frac{1+\ln f_{n^*-1}(\rho)}{f_{n^*-1}(\rho)} - \frac{1}{g_{m^*}(\rho)} < L,$$

which contradicts (38) and completes the proof.

COROLLARY 3. *Assume that (2), (21), (22), and (33) hold and that there exist $t_1 \ge t_0$ and a constant ρ, $0<\rho\le 1/e$, such that (32) holds. Further, assume that*

$$\limsup_{t\to\infty}\int_t^{t+\sigma-\tau} \frac{Q(s)}{1+R(s+\sigma-\tau)}\,ds \ge L > \frac{1+\ln f(\rho)}{f(\rho)} - \frac{1-\rho-\sqrt{1-2\rho-\rho^2}}{2}.$$

Then, every solution of equation (1) oscillates.

Finally, we give two examples.

EXAMPLE 1. Consider the neutral advanced differential equation

$$[x(t)+x(t+1)]' - 4x(t+1.5) = 0, \qquad t\ge 0, \tag{50}$$

where $P(t)=1$, $Q(t)=4$, $\tau=1$, $\sigma=1.5$. We have $R(t)=1$ and

$$\int_t^{t+\sigma-\tau} \frac{Q(s)}{1+R(s+\sigma-\tau)}\,ds = 1, \qquad t\ge 0.$$

Applying Theorem 1, we obtain for any $T\ge 0$ every solution of equation (50) has at least a zero on $[T, T+3.5]$. Therefore, the distances between adjacent zeros of every solution of equation (50) on $[0,\infty)$ are less than 3.5.

EXAMPLE 2. Consider the advanced differential equation

$$x'(t) - \frac{2}{5}(1+\cos 2\pi t)x(t+1) = 0, \qquad t\ge 0, \tag{51}$$

where $P(t) = 0$, $Q(t) = (2/5)(1 + \cos 2\pi t)$, $\tau = 0$, $\sigma = 1$. We have $R(t) = 0$,

$$\int_t^{t+\sigma-\tau} \frac{Q(s)}{1 + R(s+\sigma-\tau)}\, ds = \frac{2}{5} = \rho > \frac{1}{e}, \qquad t \geq 0,$$

and

$$\int_t^{s+\sigma-\tau} \frac{Q(u)}{1 + R(u+\sigma-\tau)}\, du = \int_{t-\sigma+\tau}^{s} \frac{Q(u)}{1 + R(u+\sigma-\tau)}\, du, \qquad 0 \leq t \leq s + \sigma \leq t + \sigma.$$

It is easy to see that

$$\begin{aligned} &f_n(\rho) < 5 < g_m(\rho), && 1 \leq n \leq 10, \quad m \geq 1; \\ &f_{11}(\rho) > g_m(\rho), && m \geq 3; \\ &f_{12}(\rho) > g_m(\rho), && m \geq 1. \end{aligned}$$

Hence, we have $k = 12 + 1 = 13$ and $2\sigma + k(\sigma - \tau) = 15$. By Corollary 2, the distances between adjacent zeros of every solution of equation (51) on $[1, \infty)$ are less than 15.

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