



The modular hierarchy of the Toda lattice

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Abstract

The modular vector field plays an important role in the theory of Poisson manifolds and is intimately connected with the Poisson cohomology of the space. In this paper we investigate its significance in the theory of integrable systems. We illustrate in detail the case of the Toda lattice both in Flaschka and natural coordinates.

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1. Introduction

The infinitesimal generator of the modular automorphism group is important in the cohomology theory of Poisson manifolds. On the other hand, one of the main characteristics of integrable Hamiltonian systems is the formation of hierarchies consisting of Poisson tensors. Each one of these tensors carries naturally a modular vector field. One may, therefore, speak of a hierarchy of modular vector fields. The purpose of this paper is to begin an investigation of the role of this sequence of vector fields in the theory of integrable systems. As an illustrative example, we choose the famous Toda lattice, a system that is one of the most basic in the theory of finite dimensional integrable models and closely related to the theory of simple Lie groups. There is no doubt, however, that the results of this paper hold for other similar systems as well.

The modular vector field appears first in a paper of Koszul [17], as a special case of an operator on contravariant differential forms of degree -1 . The same vector field was later used by Dufour and Haraki in [8] to classify quadratic Poisson brackets in \mathbb{R}^3 . It was called “curl”, which in \mathbb{R}^3 is dual to “divergence”. In fact, the operator of Koszul applied to a vector field, gives the usual divergence of differential calculus with respect to the standard volume form. One can therefore speak of the divergence of a Poisson bracket; this is what the modular vector field is. The divergence of a Poisson bracket is useful in the classification of Poisson structures in low dimensions, e.g., [13,19].

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One class of operators considered by Koszul is the following. Let A_p be the space of covariant antisymmetric tensors on a smooth orientable manifold M of dimension n and let A^p denote the space of covariant differential forms. Choose a volume form ω and define an operator

$$D: A_p \rightarrow A_{p-1}$$

which satisfies

$$D^2 = 0.$$

The volume form ω induces an isomorphism Φ from A_p to A^{n-p} . Define the modular operator as:

$$D = \Phi^{-1} \circ d \circ \Phi,$$

where d is the exterior derivative.

Weinstein [29] gives a different interpretation. Suppose one has a Poisson tensor π and a smooth positive volume form ω . Consider the operator X_ω , which sends a smooth function f to $\text{div}_\omega \mathcal{X}_f$, where \mathcal{X}_f is the Hamiltonian vector field generated by f with respect to π . It turns out that this operator is a derivation and hence a vector field. It coincides with the Koszul operator, D , at the level of contravariant 2-tensors. It satisfies:

$$L_{X_\omega} \omega = 0, \quad L_{X_\omega} \pi = 0. \quad (1)$$

Even though we deal with spaces that are orientable, we remark that in the non-orientable case one can either replace volumes by densities or use the recent approach of [14].

In local coordinates (x_1, \dots, x_n) , the modular vector field of the Poisson tensor π is given by the formula:

$$D(\pi) = \sum_{j=1}^n \left(\sum_{i=1}^n \frac{\partial \pi_{ji}}{\partial x_i} \right) \frac{\partial}{\partial x_j}. \quad (2)$$

Lichnerowicz [18] considers the following cohomology defined on the space of contravariant tensors of a Poisson manifold. Let (M, π) be a Poisson manifold. Define a coboundary operator ∂_π which assigns to each p -tensor A , a $(p+1)$ -tensor $\partial_\pi A$ given by

$$\partial_\pi A = -[\pi, A],$$

where $[\cdot, \cdot]$ denotes the Schouten bracket. We have that $\partial_\pi^2 A = [\pi, [\pi, A]] = 0$ and consequently ∂_π defines a cohomology. More details on the Schouten bracket can be found in [18,22,27]. An element A is a p -cocycle if $[\pi, A] = 0$. An element B is a p -coboundary if $B = [\pi, C]$, for some $(p-1)$ -tensor C . Let

$$Z^n(M, \pi) = \{A: [\pi, A] = 0\}$$

and

$$B^n(M, \pi) = \{B: B = [\pi, C]\}.$$

The quotient

$$H^n(M, \pi) = \frac{Z^n(M, \pi)}{B^n(M, \pi)}$$

is the n th cohomology group. The elements of the first cohomology group are the infinitesimal automorphisms of the Poisson structure modulo the Hamiltonian vector fields. It follows from (1) that the modular vector field is an element of the first cohomology group. If one replaces ω by $a\omega$, where a is a positive smooth function, then

$$X_{a\omega} = X_\omega + \mathcal{X}_{\ln a}. \quad (3)$$

Therefore, the modular vector field is a well-defined element of the first cohomology group in the Lichnerowicz cohomology. Since it forms an element of the cohomology group, Weinstein in [29] uses the term “modular class”. It makes sense to consider two such vector fields as equal if in fact they differ by a Hamiltonian vector field. The reader can also refer to [18,27,28] for more details on Poisson manifolds and cohomology.

The modular operator D is a graded derivation. The following relations hold for a general 2-tensor π and vector fields X, Y :

$$D[\pi, X] = [D(\pi), X] - [\pi, D(X)], \tag{4}$$

$$D[X, Y] = XD(Y) - YD(X). \tag{5}$$

We note the minus sign in (4) that appears because we use the convention $\mathcal{X}_f^\pi(g) = \{f, g\}_\pi$, in the definition of the Hamiltonian vector field \mathcal{X}_f^π .

The purpose of this paper is to present a complete study of the modular hierarchy in the case of the Toda lattice. In Section 2 we review the finite, classical Toda lattice and its multi-Hamiltonian nature. In Section 3 we examine the modular hierarchy in Flaschka coordinates $(a, b) \in \mathbb{R}^{2N-1}$. In particular, we establish the Hamiltonian character of the modular vector fields associated to the well-known hierarchy of Poisson tensors for the Toda lattice, see Theorem 3, and present some of their basic properties in Corollary 1. A new bi-Hamiltonian formulation of the Toda lattice is provided in Theorem 2. In Section 4 we study the infinite modular sequence in natural coordinates $(q, p) \in \mathbb{R}^{2N}$, and present a formula that iteratively produces all members of this sequence. In Section 5 we comment on the results of this paper and present them in compact form.

2. The Toda lattice

The Hamiltonian of the Toda lattice is given by

$$H(q_1, \dots, q_N, p_1, \dots, p_N) = \sum_{i=1}^N \frac{1}{2} p_i^2 + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}}. \tag{6}$$

This type of Hamiltonian was first considered by Morikazu Toda [26]. Eq. (6) is known as the classical, finite, non-periodic Toda lattice to distinguish the system from the many and various other versions, e.g., the relativistic, quantum, infinite, periodic, etc. The integrability of the system was established in 1974 independently by Flaschka [11], Henon [15] and Manakov [21]. The original Toda lattice can be viewed as a discrete version of the Korteweg–de Vries equation. It is called a lattice, as in atomic lattice, since interatomic interaction was studied. This system also appears in cosmology and the work of Seiberg and Witten on supersymmetric Yang–Mills theories. It has applications in analog computing and numerical computation of eigenvalues. However, the Toda lattice is mainly a theoretical mathematical model which is important due to the rich mathematical structure encoded in it.

Hamilton’s equations take the form

$$\begin{aligned} \dot{q}_j &= p_j \\ \dot{p}_j &= e^{q_{j-1} - q_j} - e^{q_j - q_{j+1}}, \quad j = 1, \dots, N. \end{aligned}$$

The system is integrable. One can find a set of independent functions $\{H_1, \dots, H_N\}$ which are constants of motion for Hamilton’s equations. To determine the constants of motion, one uses Flaschka’s transformation:

$$\begin{aligned} a_i &= \frac{1}{2} e^{\frac{1}{2}(q_i - q_{i+1})}, \quad i = 1, \dots, N - 1, \\ b_i &= -\frac{1}{2} p_i, \quad i = 1, \dots, N. \end{aligned} \tag{7}$$

The equations of motion become

$$\begin{aligned} \dot{a}_i &= a_i(b_{i+1} - b_i), \\ \dot{b}_i &= 2(a_i^2 - a_{i-1}^2). \end{aligned} \tag{8}$$

These equations can be written as a Lax pair $\dot{L} = [B, L]$, where L is the Jacobi matrix

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \dots & \dots & 0 \\ a_1 & b_2 & a_2 & \dots & \dots & \vdots \\ 0 & a_2 & b_3 & \ddots & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & a_{N-1} \\ 0 & \dots & \dots & a_{N-1} & b_N & \end{pmatrix},$$

and

$$B = \begin{pmatrix} 0 & a_1 & 0 & \dots & \dots & 0 \\ -a_1 & 0 & a_2 & \dots & \dots & \vdots \\ 0 & -a_2 & 0 & \ddots & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & a_{N-1} \\ 0 & \dots & \dots & -a_{N-1} & 0 & \end{pmatrix}.$$

This is an example of an isospectral deformation; the entries of L vary over time but the eigenvalues remain constant, i.e. $\dot{\lambda}_i = 0$. It follows that the functions $H_j = \frac{1}{j} \text{tr } L^j$ are constants of motion. We note that

$$H_1 = \lambda_1 + \lambda_2 + \dots + \lambda_N = \sum_{i=1}^N b_i$$

corresponds to the total momentum and

$$H_2 = \frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \dots + \lambda_N^2) = \frac{1}{2} \sum_{i=1}^N b_i^2 + \sum_{i=1}^{N-1} a_i^2$$

is the Hamiltonian.

Consider \mathbb{R}^{2N} with coordinates $(q_1, \dots, q_N, p_1, \dots, p_N)$, the standard symplectic bracket

$$\{f, g\}_s = \sum_{i=1}^N \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right),$$

and the mapping $F: \mathbb{R}^{2N} \rightarrow \mathbb{R}^{2N-1}$ defined by

$$F: (q_1, \dots, q_N, p_1, \dots, p_N) \rightarrow (a_1, \dots, a_{N-1}, b_1, \dots, b_N).$$

The Flaschka transformation F is a symplectic realization of a degenerate Lie Poisson bracket on \mathbb{R}^{2N-1} , i.e. there exists a Poisson bracket on \mathbb{R}^{2N-1} which satisfies

$$\{f, g\} \circ F = \{f \circ F, g \circ F\}_s.$$

This bracket (up to a constant multiple) is given by

$$\begin{aligned} \{a_i, b_i\} &= -a_i, \\ \{a_i, b_{i+1}\} &= a_i; \end{aligned} \tag{9}$$

all other brackets are zero. We denote this bracket by π_1 . Its Lie algebraic interpretation can be found in [16]. In π_1 , the only Casimir is $H_1 = b_1 + b_2 + \dots + b_N$, and the Hamiltonian is $H_2 = \frac{1}{2} \text{tr } L^2$. The invariants H_i are in involution with respect to π_1 . For a proof of these facts see [4].

The quadratic Toda bracket appears in conjunction with isospectral deformations of Jacobi matrices. Let λ be an eigenvalue of L with normalized eigenvector v . Using standard perturbation theory one obtains,

$$\nabla\lambda = (2v_1v_2, \dots, 2v_{N-1}v_N, v_1^2, \dots, v_N^2)^t := U^\lambda,$$

where $\nabla\lambda$ denotes $(\frac{\partial\lambda}{\partial a_1}, \dots, \frac{\partial\lambda}{\partial a_{N-1}}, \frac{\partial\lambda}{\partial b_1}, \dots, \frac{\partial\lambda}{\partial b_N})$. Some manipulations show that U^λ satisfies

$$\pi_2 U^\lambda = \lambda \pi_1 U^\lambda,$$

where π_1 and π_2 are skew-symmetric matrices. The defining relations for the Poisson tensor π_2 are the following quadratic functions of the a_i and b_i :

$$\begin{aligned} \{a_i, a_{i+1}\} &= \frac{1}{2}a_i a_{i+1}, \\ \{a_i, b_i\} &= -a_i b_i, \\ \{a_i, b_{i+1}\} &= a_i b_{i+1}, \\ \{b_i, b_{i+1}\} &= 2a_i^2; \end{aligned}$$

all other brackets are zero. The quadratic Toda bracket appeared in a paper of Adler [1] in 1979. It is a Poisson bracket in which the Hamiltonian vector field generated by H_1 is the same as the Hamiltonian vector field generated by H_2 with respect to the π_1 bracket. The tensor π_2 has $\det L$ as Casimir and $H_1 = \text{tr } L$ as the Hamiltonian. The eigenvalues of L (and therefore the H_i as well) are still in involution. Furthermore, π_2 is compatible with π_1 . We have

$$\pi_2 \nabla H_j = \pi_1 \nabla H_{j+1}, \quad j = 1, 2, \dots \tag{10}$$

These relations are similar to the Lenard relations for the KdV equation; they are generally called the Lenard relations. Taking $j = 1$ in (10), we conclude that the Toda lattice is bi-Hamiltonian. Bi-Hamiltonian structures were introduced by Magri in [20]. Using results from [5], one proves that the Toda lattice is multi-Hamiltonian:

$$\pi_2 \nabla H_1 = \pi_1 \nabla H_2 = \pi_0 \nabla H_3 = \pi_{-1} \nabla H_4 = \dots \tag{11}$$

The Hamiltonian hierarchies of the Toda lattice are well-known. The results are usually presented either in the natural (q, p) coordinates or in the more convenient Flaschka coordinates (a, b) . In the former case the hierarchy of higher invariants are generated by the use of a recursion operator [7,10]. We remark that recursion operators were first introduced by Olver [25]. The system is bi-Hamiltonian and one of the brackets is symplectic. Thus, one can find a recursion operator by inverting the symplectic tensor. The recursion operator is then applied to the initial symplectic bracket to produce an infinite sequence of Poisson tensors. However, in the case of the Toda lattice in Flaschka variables (a, b) , the first two Poisson brackets π_1 and π_2 are non-invertible and therefore this method fails. The absence of a recursion operator for the finite Toda lattice is also mentioned in Morosi and Tondo [23], where a Nijenhuis tensor for the infinite Toda lattice is calculated. The family of Poisson tensors in this case is constructed using master symmetries. Invariant functions and Hamiltonian vector fields are preserved by master symmetries. New Poisson brackets are generated using Lie derivatives in the direction of these vector fields, and they satisfy interesting deformation relations. We quote the results from Refs. [3,4].

Theorem 1. *There exists a sequence of vector fields X_i , for $i \geq -1$, and a sequence of contravariant 2-tensors π_j , $j \geq 1$, satisfying:*

- (i) π_j are all Poisson.
- (ii) The functions H_i , $i \geq 1$, are in involution with respect to all of the π_j .
- (iii) $X_i(H_j) = (i + j)H_{i+j}$, $i \geq -1$, $j \geq 1$.
- (iv) $L_{X_i}\pi_j = (j - i - 2)\pi_{i+j}$, $i \geq -1$, $j \geq 1$.
- (v) $[X_i, X_j] = (j - i)X_{i+j}$, $i \geq 0$, $j \geq 0$.
- (vi) $\pi_j \nabla H_i = \pi_{j-1} \nabla H_{i+1}$, where π_j denotes the Poisson matrix of the tensor π_j .

Remark 1. Theorem 3 was extended for all integer values of the index in [5].

3. The modular class in Flaschka coordinates

We consider the modular vector field of the Poisson tensor π_j , denoted by $Y_j = D(\pi_j)$, where D is the Koszul operator. We begin with some preliminary results needed to prove the main theorem that establishes the Hamiltonian character of the modular class. The vector fields of [Theorem 1](#), denoted by X_j , are master symmetries for the Toda lattice system. For example, X_1 is given as follows:

$$X_1 = \sum_{i=1}^{N-1} A_i \frac{\partial}{\partial a_i} + \sum_{i=1}^N B_i \frac{\partial}{\partial b_i},$$

where

$$\begin{aligned} A_i &= -ia_i b_i + (i+2)a_i b_{i+1}, \\ B_i &= (2i+3)a_i^2 + (1-2i)a_{i-1}^2 + b_i^2. \end{aligned}$$

If we define the function $f = \ln(a_1 \cdots a_{N-1})$, then we have the following proposition:

Proposition 1.

$$D(X_1) = X_1(f) + 2H_1.$$

Proof. By definition

$$\begin{aligned} D(X_1) &= \sum_{i=1}^{N-1} \frac{\partial A_i}{\partial a_i} + \sum_{i=1}^N \frac{\partial B_i}{\partial b_i} \\ &= \sum_{i=1}^{N-1} [-ib_i + (i+2)b_{i+1}] + 2 \sum_{i=1}^N b_i \\ &= X_1(f) + 2\text{Tr}(L) = X_1(f) + 2H_1. \end{aligned}$$

A similar relation holds for the second master symmetry X_2 that is defined as

$$X_2 = \sum_{i=1}^{N-1} C_i \frac{\partial}{\partial a_i} + \sum_{i=1}^N D_i \frac{\partial}{\partial b_i},$$

where

$$\begin{aligned} C_i &= (2-i)a_{i-1}^2 a_i + (1-i)a_i b_i^2 + a_i b_i b_{i+1} \\ &\quad + (i+1)a_i a_{i+1}^2 + (i+1)a_i b_{i+1}^2 + a_i^3 + \left(\sum_{j=1}^{i-1} b_j \right) a_i (b_{i+1} - b_i) \\ D_i &= 2 \left(\sum_{j=1}^{i-1} b_j \right) a_i^2 - 2 \left(\sum_{j=1}^{i-2} b_j \right) a_{i-1}^2 + (2i+2)a_i^2 b_i + (2i+1)a_i^2 b_{i+1} \\ &\quad + (3-2i)a_{i-1}^2 b_{i-1} + (4-2i)a_{i-1}^2 b_i + b_i^3. \quad \square \end{aligned}$$

Proposition 2.

$$D(X_2) = X_2(f) + 6H_2.$$

Proof. Similar to the proof of [Proposition 1](#). \square

We can generalize the results of [Propositions 1 and 2](#) as follows:

Proposition 3.

$$D(X_n) = X_n(f) + n(n + 1)H_n.$$

Proof. The proof is inductive. The result holds for $n = 1, 2$. We assume that the result holds for $n \geq 2$ and prove it for $n + 1$.

$$\begin{aligned} D(X_{n+1}) &= \frac{1}{n-1} D[X_1, X_n] \\ &= \frac{1}{n-1} (X_1 D(X_n) - X_n D(X_1)) \\ &= \frac{1}{n-1} (X_1 (X_n(f) + n(n+1)H_n) - X_n (X_1(f) + 2H_1)) \\ &= \frac{1}{n-1} ([X_1, X_n](f) + n(n+1)X_1(H_n) - 2X_1(H_n)) \\ &= \frac{1}{n-1} ((n-1)X_{n+1}(f) + (n(n+1)-2)X_1(H_n)) \\ &= X_{n+1}(f) + (n+2)X_1(H_n) = X_{n+1}(f) + (n+1)(n+2)H_{n+1}. \quad \square \end{aligned}$$

Below we prove a statement that we will use in [Theorem 3](#), but it is also important in its own right. It is a new bi-Hamiltonian formulation of the Toda lattice.

Theorem 2.

$$\mathcal{X}_{H_1}^{\pi_j} = \mathcal{X}_g^{\pi_{j+1}}, \quad \text{where } g = \ln(\det(L)), \text{ and } j \geq 1.$$

Proof. We use the Lenard relations for the eigenvalues [\[4,6\]](#), $\pi_j \nabla \lambda_i = \lambda_i \pi_{j-1} \nabla \lambda_i$.

$$\mathcal{X}_{H_1}^{\pi_j} = \mathcal{X}_{(\lambda_1 + \dots + \lambda_N)}^{\pi_j} = \pi_j \nabla (\lambda_1 + \dots + \lambda_N) = \sum_{i=1}^N \pi_j \nabla \lambda_i.$$

On the other hand

$$\mathcal{X}_g^{\pi_{j+1}} = \mathcal{X}_{\ln \lambda_1 \dots \lambda_N}^{\pi_{j+1}} = \pi_{j+1} \nabla \ln \lambda_1 \dots \lambda_N = \sum_{i=1}^N \pi_{j+1} \frac{1}{\lambda_i} \nabla \lambda_i = \sum_{i=1}^N \pi_j \nabla \lambda_i = \mathcal{X}_{H_1}^{\pi_j}. \quad \square$$

The following theorem investigates the Hamiltonian character of the divergence of the infinite sequence of Poisson tensors π_j . In particular, it states that the divergence of the Poisson tensor π_j is the Hamiltonian vector field given by the function $h := \ln(a_1 \cdots a_{N-1}) + (j - 1) \ln(\det(L))$.

Theorem 3.

$$Y_j = \mathcal{X}_{f+(j-1)g}^{\pi_j}, \quad \text{where } f = \ln(a_1 \cdots a_{N-1}), \text{ and } g = \ln(\det(L)).$$

Proof. We will prove the theorem inductively in two steps. First we will show that it holds for $j = 1$ and then for $2 \leq j \leq 3$. Consequently, we will show that it holds for $j = 4$ and then for $j \geq 5$.

We have that $Y_1 = D(\pi_1)$, where π_1 is given by

$$\begin{aligned} \{a_i, b_i\} &= -a_i, \\ \{a_i, b_{i+1}\} &= -a_i. \end{aligned}$$

Thus, using [\(2\)](#) we obtain

$$Y_j = \frac{\partial}{\partial b_1} - \frac{\partial}{\partial b_n} = (0, \dots, 0, 1, 0, \dots, 0, -1)^t.$$

On the other hand, it is not hard to show that the Hamiltonian vector field induced by $f = \ln(a_1 \cdots a_{N-1})$ with respect to the bracket π_1 has the form:

$$\begin{aligned}\dot{a}_i &= 0, & i &= 1, \dots, N-1, \\ \dot{b}_1 &= 1, & \dot{b}_N &= -1, \\ \dot{b}_i &= 0, & i &= 2, \dots, N-1.\end{aligned}$$

Hence $Y_1 = \mathcal{X}_f^{\pi_1}$.

Now, let us assume that the theorem holds for $j \geq 4$. We will then show that it holds for $j+1$. By definition $L_{X_1}\pi_j = [X_1, \pi_j] = [\pi_j, X_1]$ and using (iv) of [Theorem 1](#), we have that $\pi_{j+1} = \frac{1}{(j-3)}[\pi_j, X_1]$. Thus,

$$\begin{aligned}Y_{j+1} &= D(\pi_{j+1}) = \frac{1}{(j-3)}D([\pi_j, X_1]) \\ &= \frac{1}{(j-3)}([D(\pi_j), X_1] - [\pi_j, D(X_1)]) \\ &= \frac{1}{(j-3)}([Y_j, X_1] - [\pi_j, D(X_1)]) \\ &= \frac{1}{(j-3)}([\mathcal{X}_f^{\pi_j}, X_1] + (j-1)[\mathcal{X}_g^{\pi_j}, X_1] - [\pi_j, D(X_1)]) \\ &= \frac{1}{(j-3)}([f, \pi_j], X_1] + (j-1)[\mathcal{X}_{H_1}^{\pi_{j-1}}, X_1] - [\pi_j, D(X_1)]) \\ &= \frac{1}{(j-3)}([\pi_j, [X_1, f]] + [f, [\pi_j, X_1]] + (j-1)(j-2)\mathcal{X}_{H_1}^{\pi_j} - [\pi_j, D(X_1)]) \\ &= \frac{1}{(j-3)}([\pi_j, X_1(f)] + (j-3)[f, \pi_{j+1}] + (j-1)(j-2)\mathcal{X}_{H_1}^{\pi_j} - [\pi_j, D(X_1)]) \\ &= \frac{1}{(j-3)}([\pi_j, D(X_1)] - 2[\pi_j, H_1] + (j-3)\mathcal{X}_f^{\pi_{j+1}} + (j-1)(j-2)\mathcal{X}_{H_1}^{\pi_j} - [\pi_j, D(X_1)]) \\ &= \mathcal{X}_f^{\pi_{j+1}} + \frac{(j-1)(j-2)-2}{(j-3)}\mathcal{X}_{H_1}^{\pi_j} = \mathcal{X}_f^{\pi_{j+1}} + j\mathcal{X}_g^{\pi_{j+1}}.\end{aligned}$$

In an identical manner one proves that the theorem holds for $2 \leq j \leq 3$. To show that the theorem holds for $j=4$ we use a similar argument as the one used above, however, we employ the relation $\pi_4 = -\frac{1}{2}L_{X_2}\pi_2 = -\frac{1}{2}[X_2, \pi_2]$, to avoid division by zero. \square

Corollary 1.

- (a) $Y_i(H_j) = Y_j(H_i)$,
- (b) $L_{Y_i}\pi_j = -L_{Y_j}\pi_i$.

Proof. For part (a) we have:

$$\begin{aligned}Y_i(H_j) &= (X_f^{\pi_i} + (i-1)\mathcal{X}_{H_1}^{\pi_{i-1}})(H_j) = \mathcal{X}_f^{\pi_i}(H_j) = \{f, H_j\}_{\pi_i} \\ &= \nabla f \pi_i \nabla H_j = \nabla f \pi_j \nabla H_i = \{f, H_i\}_{\pi_j} = Y_j(H_i).\end{aligned}$$

For part (b),

$$[[h, \pi_j], \pi_i] + [[\pi_j, \pi_i], h] + [[\pi_i, h], \pi_j] = 0.$$

Since $[\pi_i, \pi_j] = 0$ the result follows. \square

4. The modular hierarchy in (q, p) -coordinates

The bi-Hamiltonian structure of the Toda lattice appears in a paper of Das and Okubo in 1989 [7]. The generation of master symmetries and Poisson tensors using Oevel’s theorem and the connection with the results in Flaschka coordinates are due to Fernandes [10]. In principle, the method is general and works for other finite dimensional systems as well. For example, this approach was used by Nunes da Costa and Marle [2] in the case of the relativistic Toda lattice. The procedure goes as follows. One defines a second Poisson bracket in the space of canonical variables $(q_1, \dots, q_N, p_1, \dots, p_N)$. This gives rise to a recursion operator, call it \mathcal{R} . The presence of a conformal symmetry, as defined by Oevel, allows one to use the recursion operator and generate an infinite sequence of master symmetries. These, in turn, project to the space of the new variables (a, b) to produce a sequence of master symmetries in the reduced space. We quote the relevant theorem of Oevel that appears in [24].

Theorem 4. *Suppose that Z_0 is a conformal symmetry for both Poisson tensors J_1, J_2 and function h_1 , i.e. for some scalars λ, μ , and ν we have*

$$\mathcal{L}_{Z_0}J_1 = \lambda J_1, \quad \mathcal{L}_{Z_0}J_2 = \mu J_2, \quad \mathcal{L}_{Z_0}h_1 = \nu h_1.$$

Then the vector fields $Z_i = \mathcal{R}^i Z_0$ are master symmetries and we have,

- (a) $\mathcal{L}_{Z_i}h_k = (\nu + (k - 1 + i)(\mu - \lambda))h_{i+k}$,
- (b) $\mathcal{L}_{Z_i}J_k = (\mu + (k - i - 2)(\mu - \lambda))J_{i+k}$,
- (c) $[Z_i, Z_k] = (\mu - \lambda)(k - i)Z_{i+k}$.

We note that master symmetries were introduced in [12].

We proceed with the computation of the modular vector fields that are associated to the family of Poisson tensors given in (q, p) -coordinates. The first Poisson tensor in the hierarchy is the standard canonical symplectic tensor:

$$\hat{J}_1 = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix},$$

where I_N denotes the $N \times N$ identity matrix. The second Poisson tensor has the form,

$$\hat{J}_2 = \begin{pmatrix} A_N & B_N \\ -B_N & C_N \end{pmatrix},$$

where A_N is the $N \times N$ skew-symmetric matrix defined by $a_{ij} = 1 = -a_{ji}$, for $i < j$, $B_N = \text{diag}(-p_1, -p_2, \dots, -p_N)$, and C_N is the $N \times N$ skew-symmetric matrix whose non-zero terms are given by $c_{i,i+1} = -c_{i+1,i} = e^{q_i - q_{i+1}}$, for $i = 1, 2, \dots, N - 1$. If we let $J_1 = 4\hat{J}_1$ and $J_2 = 2\hat{J}_2$ then J_1 and J_2 are mapped precisely onto the brackets π_1 and π_2 under the Flaschka transformation.

It is easy to see that we have a bi-Hamiltonian pair. We define

$$h_1 = -2(p_1 + p_2 + \dots + p_N),$$

and h_2 to be the Hamiltonian

$$h_2 = \sum_{i=1}^N \frac{1}{2} p_i^2 + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}}.$$

Under Flaschka’s transformation (7), h_1 is mapped onto $4(b_1 + b_2 + \dots + b_N) = 4 \text{tr } L = 4H_1$ and h_2 is mapped onto $2 \text{tr } L^2 = 4H_2$. Using the relationship

$$\pi_2 \nabla H_1 = \pi_1 \nabla H_2,$$

which follows from part (iv) of Theorem 1, we obtain, after multiplication by 4, the following pair:

$$J_1 \nabla h_2 = J_2 \nabla h_1.$$

The Lenard relations for the eigenvalues translate into

$$\pi_j \nabla \lambda_i = \lambda_i \pi_{j-1} \nabla \lambda_i. \quad (12)$$

We define the recursion operator as follows:

$$\mathcal{R} = J_2 J_1^{-1}.$$

This operator raises degrees and we therefore call it the positive Toda operator. In (q, p) coordinates, we use the symbol \mathcal{X}_i as a shorthand for $\mathcal{X}_{h_i}^{J_1}$, the Hamiltonian vector field of h_i with respect to the symplectic bracket J_1 . It is generated, as usual, by

$$\mathcal{X}_i = \mathcal{R}^{i-1} \mathcal{X}_1.$$

In a similar fashion we obtain the higher order Poisson tensors

$$J_i = \mathcal{R}^{i-1} J_1.$$

We then define the conformal symmetry

$$Z_0 = \sum_{i=1}^N (N - 2i + 1) \frac{\partial}{\partial q_i} + \sum_{i=1}^N p_i \frac{\partial}{\partial p_i}.$$

It is straightforward to verify that

$$\mathcal{L}_{Z_0} J_1 = -J_1,$$

$$\mathcal{L}_{Z_0} J_2 = 0.$$

In addition,

$$Z_0(h_1) = h_1,$$

$$Z_0(h_2) = 2h_2.$$

Consequently, Z_0 is a conformal symmetry for J_1 , J_2 and h_1 . The constants appearing in [Theorem 4](#) are $\lambda = -1$, $\mu = 0$ and $\nu = 1$. Therefore, we end-up with the following deformation relations:

$$[Z_i, h_k] = (i + k)h_{i+k}$$

$$L_{Z_i} J_k = (k - i - 2)J_{i+k}$$

$$[Z_i, Z_k] = (k - i)Z_{i+k}.$$

We compute the divergence of the master symmetry Z_1 , as we will use it in the proof of [Theorem 5](#). It was proved in [\[3\]](#) that $Z_1 = \sum_{i=1}^N \lambda_i^2 \frac{\partial}{\partial \lambda_i}$, where λ_i are the eigenvalues of the Jacobi matrix L . Using the definition of divergence we have that $D(Z_1) = 2 \sum_{i=1}^N \lambda_i = 2 \sum_{i=1}^N b_i = - \sum_{i=1}^N p_i = \frac{1}{2}h_1$. Therefore,

$$D(Z_1) = \frac{1}{2}h_1.$$

The following proposition will also come to use in the proof of [Theorem 5](#).

Proposition 4.

$$\mathcal{X}_{h_1}^{J_j} = 4\mathcal{X}_f^{J_{j+1}}, \quad \text{where } f = \ln(\sqrt{\det \mathcal{R}}) \text{ for } j \geq 1. \quad (13)$$

Proof. Similar to the proof of [Theorem 2](#). We note that one uses the fact that the eigenvalues of \mathcal{R} are the squares of the eigenvalues of L , see [\[6,9\]](#). \square

Theorem 5. For $j \geq 1$, Y_j is a Hamiltonian vector field given as

$$Y_j = (j - 1)\mathcal{X}_f^{J_j}, \quad \text{where } f = \ln(\sqrt{\det \mathcal{R}}). \quad (14)$$

We will prove the theorem for $j = 1, \dots, 4$, and then we will use induction for $j \geq 5$. First we observe that $Y_1 = \vec{0}$, since J_1 is standard symplectic. Using the general form of the tensor J_2 we obtain the following: $Y_2 = -2 \sum_{i=1}^N \frac{\partial}{\partial q_i}$. A simple calculation gives that $\mathcal{X}_{h_1}^{J_1} = -8 \sum_{i=1}^N \frac{\partial}{\partial q_i}$. Thus $Y_2 = \frac{1}{4} \mathcal{X}_{h_1}^{J_1} = \mathcal{X}_f^{J_2}$, using Proposition 4.

We have that $J_3 = -[Z_1, J_2]$ and $D(Z_1) = \frac{1}{2}h_1$. Therefore,

$$\begin{aligned} Y_3 &= D(J_3) = -D[J_2, Z_1] = -([D(J_2), Z_1] - [J_2, D(Z_1)]) \\ &= -[Y_2, Z_1] + \frac{1}{2}[J_2, h_1] = -[\mathcal{X}_f^{J_2}, Z_1] + \frac{1}{2}\mathcal{X}_{h_1}^{J_2}. \end{aligned}$$

Using the super-Jacobi identity for the Schouten bracket, the first term equals

$$-[J_2, [Z_1, f]] - [f, [J_2, Z_1]].$$

Therefore

$$Y_3 = -\frac{1}{4}\mathcal{X}_{h_1}^{J_2} + [f, J_3] + \frac{1}{2}\mathcal{X}_{h_1}^{J_2} = \frac{1}{4}\mathcal{X}_{h_1}^{J_2} + \mathcal{X}_f^{J_3} = 2\mathcal{X}_f^{J_3}.$$

In the last step of the argument we have used that $\mathcal{X}_{h_1}^{J_2} = 4\mathcal{X}_f^{J_3}$. The proof of the formula $Y_4 = 3\mathcal{X}_f^{J_4}$ is identical to the one for Y_3 except that we employ the relation $[Z_2, J_2] = -2J_4$. An inductive argument based on the same technique that we have used for Y_3 , can also be used to show that the result of the theorem holds for $j \geq 5$. We omit the details.

In the theorem that follows, we present an iterative formula that produces all members of the modular class in terms of the recursion operator \mathcal{R} and the modular vector field $Y_2 = -2 \sum_{i=1}^N \frac{\partial}{\partial q_i} = (-2, \dots, -2, 0, \dots, 0)^t$.

Theorem 6. For $j \geq 2$,

$$Y_{j+1} = j\mathcal{R}^{j-1}Y_2.$$

Proof.

$$\begin{aligned} Y_{j+1} &= j\mathcal{X}_f^{J_{j+1}} = \frac{j}{4}\mathcal{X}_{h_1}^{J_j} = \frac{j}{4}\mathcal{X}_{h_j}^{J_1} = \frac{j}{4}\mathcal{X}_j = \frac{j}{4}\mathcal{R}^{j-1}\mathcal{X}_1 \\ &= \frac{j}{4}\mathcal{R}^{j-1}\mathcal{X}_{h_1}^{J_1} = \frac{j}{4}\mathcal{R}^{j-1}4\mathcal{X}_f^{J_2} = j\mathcal{R}^{j-1}Y_2. \quad \square \end{aligned}$$

5. Summary

In this paper we study the hierarchy of modular vector fields associated to the infinite family of Poisson tensors for the classical Toda lattice equations. We present analytical expressions of the modular vector fields both in Flaschka coordinates $(a, b) \in \mathbb{R}^{2N-1}$, as well as in natural coordinates $(q, p) \in \mathbb{R}^{2N}$. In both cases, all the members of the infinite modular family, denoted by $Y_j, j \geq 1$, are Hamiltonian. In (a, b) -variables we have that

$$Y_j = \mathcal{X}_{\ln(a_1 \dots a_{N-1}) + (j-1)\ln(\det(L))}^{\pi_j}$$

where L is the Jacobi matrix of the Lax pair. In natural coordinates (q, p) , the modular vector field takes the form

$$Y_j = (j - 1)\mathcal{X}_{\ln(\sqrt{\det \mathcal{R}})}^{J_j}$$

where \mathcal{R} is the recursion operator. It is not difficult to show that the term $\mathcal{X}_{\ln(\det(L))}^{\pi_j}$ is the projection of the vector field $\mathcal{X}_{\ln(\sqrt{\det \mathcal{R}})}^{J_j}$ under the Flaschka map. The term $\mathcal{X}_{\ln(a_1 \dots a_{N-1})}^{\pi_j}$ makes its appearance due to the change of coordinates.

The following properties are proved regarding the behavior of modular vector fields when they are applied on the constants of motion H_j , and the Lie derivative of the Poisson tensor π_j in the direction of the modular vector field Y_j :

- (i) $Y_i(H_j) = Y_j(H_i)$
- (ii) $L_{Y_i}\pi_j = -L_{Y_j}\pi_i$.

In (q, p) -variables, we presented a formula that iteratively produces all members of the infinite family of modular vector fields, in terms of the recursion operator. Namely,

$$Y_1 = \vec{0},$$

$$Y_2 = -2 \sum_{i=1}^N \frac{\partial}{\partial q_i} = (-2, \dots, -2, 0, \dots, 0)^t,$$

$$Y_{j+1} = j\mathcal{R}^{j-1}Y_2 \quad \text{for } j \geq 2.$$

We conclude with an alternate bi-Hamiltonian formulation of the Toda lattice given by the relation $\mathcal{X}_{H_1}^{\pi_j} = \mathcal{X}_{\ln(\det(L))}^{\pi_{j+1}}$, in addition to the existing one $\mathcal{X}_{H_1}^{\pi_{j+1}} = \mathcal{X}_{H_2}^{\pi_j}$.

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