Decompositions of regular bipartite graphs

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Abstract

In this paper we discuss isomorphic decompositions of regular bipartite graphs into trees and forests. We prove that: (1) there is a wide class of r-regular bipartite graphs that are decomposable into any tree of size r, (2) every r-regular bipartite graph decomposes into any double star of size \( r \), and (3) every 4-regular bipartite graph decomposes into paths \( P_4 \).

1. Introduction

In this paper we discuss decompositions of regular bipartite graphs into trees. Let \( G \) and \( H \) be graphs. A collection of edge disjoint subgraphs of \( G \) covering all edges of \( G \), and such that each graph of the collection is isomorphic to \( H \), is called a decomposition of \( G \) into \( H \). A thorough discussion of the subject is given in [5]. The main problem is to give necessary and sufficient conditions for \( G \) to be decomposable into \( H \). In the case when \( G \) is \( r \)-regular (this is the case that was most extensively studied, and that we study in the paper) there are two simple necessary conditions, often referred to as divisibility conditions:

(i) \( |E(H)| \) divides \( |E(G)| \), and
(ii) \( \gcd\{d(v): v \in V(H)\} \) divides \( r \).

Most of the work on the problem has been concerned with the case when \( G \) is a complete multigraph \( K_n^{(A)} \). A beautiful, general result of Wilson (see [9–10])
states that for fixed \( \lambda \) and \( H \), \( K_n^{(\lambda)} \) decomposes into \( H \) if \( n \) is sufficiently large and the divisibility conditions (i) and (ii) hold. Another general result, this time concerning an arbitrary graph \( G \), states that if only \( G \) has sufficiently many edges, and this number is divisible by \( k \), then \( G \) decomposes into subgraphs, each isomorphic to \( K_{1,k} \), \( kK_2 \) or \( K_{1,k-1} \cup K_2 \) (see [7]; it is also known that the number of edges needed in \( G \) is \((1 + o(1))k^3\), see [4]).

In this paper we study a special case of the decomposition problem namely, decompositions of an \( r \)-regular bipartite graph \( G \) (\( G \) is \((2r)\)-regular, in one case) into a tree (forest) \( T \) of size \( r \). Note that in this case the necessary divisibility conditions for the existence of a decomposition are trivially satisfied (\(|E(G)|\) is a multiple of \( r \), and \( \gcd(d(v) : v \in V(T)) = 1 \)). We exhibit a class of \( r \)-regular bipartite graphs with the property that they are decomposable into any forest of size \( r \). By a different method, we show that the \( r \)-dimensional cube has the same property. We also discuss decompositions of \( r \)-regular bipartite graphs into homomorphic images of trees of size \( r \). Our results imply, in particular, that any \( r \)-regular bipartite graph decomposes into any double star of size \( r \) (by a double star we mean a tree of diameter at most 3). Another result states that an arbitrary 4-regular bipartite graph can be decomposed into paths of length 4. Thus, if \( r \leq 4 \), any \( r \)-regular bipartite graph decomposes into any tree of size \( r \). In view of our results, we propose the following conjecture.

**Conjecture 1.1.** Let \( T \) be a tree of size \( r \). Then every \( r \)-regular bipartite graph \( G \) is decomposable into \( T \).

Finally, some notation. A bipartite graph \( G \) with color classes \( X \) and \( Y \), and with the edge set \( E \) is denoted by \((X, Y; E)\). The pair \(|X|, |Y|\) is called the **color pattern** and \( G \) is referred to as an \((|X| + |Y|)\)-graph.

### 2. Cyclic decompositions

First, we describe a class of \( r \)-regular bipartite graphs for which our conjecture is true, i.e., that are decomposable into any tree of size \( r \). Let \( A \subseteq \{0, 1, \ldots\} \) be finite, and let \( n \) be an integer such that \( n > \max\{x \in A\} \). By \( B(A; n) \) we denote the bipartite graph with color classes \( \{x_0, \ldots, x_{n-1}\} \) and \( \{y_0, \ldots, y_{n-1}\} \), where \( x_i \) and \( y_j \) are connected with an edge if \( i - j(\text{mod } n) \in A \). Similarly, if \( A, A' \subseteq \{0, 1, \ldots\} \) are finite, \( |A| = |A'| \), and \( n > \max\{x \in A \cup A'\} \), by \( B(A, A'; n) \) we denote the bipartite graph with classes of bipartition as above, where \( x_i \) and \( y_j \) are connected if \( i - j(\text{mod } n) \in A \cup A' \). If, in addition \( i - j(\text{mod } n) \in A \cap A' \), \( x_i \) and \( y_j \) are joined by two parallel edges. Hence, in general, \( B(A, A'; n) \) is a multigraph. Finally, define \( B(A; n) \) as \( B(A, A'; n) \), where \( A' = \{n - x(\text{mod } n) : x \in A\} \). Clearly, \( B(A; n) \) is \(|A|\)-regular, and \( B(A, A'; n) \) and \( B(A, n) \) are \( 2|A| \)-regular.
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Notice also, that if $0 \notin A$ and $n > 2 \max \{x \in A\}$, then $\bar{B}(A; n)$ has no multiple edges. Graphs $B(A; n)$ and $\bar{B}(A; n)$ form a wide class of regular bipartite graphs. In particular,

$$B(\{0, 1, \ldots, n - 1\}; n) = K_{n,n}, B(\{0, 1, \ldots, n - 2\}, n) = K_{n,n} - nK_2, \bar{B}(\{0, 1, \ldots, n - 1\}; n) = K_{n,n}^{(2)},$$

and

$$\bar{B}(\{1, 2, \ldots, n\}; 2n + 1) = K_{2n+1,2n+1} - (2n + 1)K_2.$$

Our first results concern decompositions of graphs $B(A; n)$ and $\bar{B}(A; n)$.

**Lemma 2.1.** Let $G = (X, Y; E)$ be a subgraph of $B(A; n)$, where $X \subseteq \{x_0, \ldots, x_{n - 1}\}$ and $Y \subseteq \{y_0, \ldots, y_{n - 1}\}$, and let $|E| = |A|$. If

for every $a \in A$ there is an edge $x_iy_j \in E$ such that $i - j \equiv a \pmod{n}$, (*).

then $B(A; n)$ decomposes into $G$.

**Proof.** For $k = 0, 1, \ldots, n - 1$, define graphs $G_k = (X_k, Y_k; E_k)$ by:

$$X_k = \{x_i + k \pmod{n}: x_i \in X\}, \quad Y_k = \{y_i + k \pmod{n}: y_i \in Y\},$$

$$E_k = \{x_i + k \pmod{n}y_j + k \pmod{n}: x_i, y_j \in E\}.$$

Clearly, all graphs $G_k$ are isomorphic to $G$. Moreover, for each $a \in A$, there are exactly $n$ edges $x_iy_j$ in $B(A; n)$ such that $j - i \equiv a \pmod{n}$. Each such edge is covered by a different graph $G_k$. Hence, each edge of $B(A; n)$ is an edge of a unique $G_k$. □

Similarly, we prove the analog of Lemma 2.1 for graphs $B(A, A'; n)$.

**Lemma 2.2.** Let $G = (X, Y; E)$ be a subgraph of $B(A, A'; n)$, where $X \subseteq \{x_0, \ldots, x_{n - 1}\}$ and $Y \subseteq \{y_0, \ldots, y_{n - 1}\}$, and let $|E| = 2|A|$. If

for every $a \in A \cup A'$ there is an edge (two edges, if $a \in A \cap A'$) $x_iy_j \in E$ such that $i - j \equiv a \pmod{n}$, (**')

then $B(A, A'; n)$ decomposes into $G$. □

It follows from Lemmas 2.1 and 2.2 that in order to show decomposability of $B(A; n)$ (or $\bar{B}(A; n)$) into $G$, it suffices to find an embedding $\phi$ of $G$ in $B(A; n)$ (or $\bar{B}(A; n)$), such that $\phi(G)$ satisfies (**) (or (**')). To construct such embeddings we proceed in a standard way namely, we use certain vertex labelings (see e.g., Rosa [8]). Let $G = (X, Y; E)$ be any bipartite graph with $|E| = |A|$. A vertex labeling $f : X \cup Y \to \{0, 1, \ldots\}$ such that:

(1) $f | X$ is one-to-one,

(2) $f | Y$ is one-to-one, and

(3) $|f(x) - f(y)| : xy \in E = A,$
is called an *A-labeling* of $G$. If (1) and (2) hold, and instead of (3), a stronger condition

$$(3') \{f(x) - f(y): x \in X, y \in Y, xy \in E\} = A,$$

holds, then $f$ is called a *strong A-labeling* of $G$. (In particular, if (3') holds, $f(x) \not\equiv f(y)$, for every $x \in X$ and $y \in Y$ such that $xy \in E$.)

**Remark 2.3.** (a) An A-labeling $f$ of $G = (X, Y; E)$ such that $f$ is one-to-one and $f(X \cup Y) \subseteq \{0, 1, \ldots, |E|\}$ is a graceful labeling (see [1, 3]). It is conjectured that all trees are graceful (Ringel-Kotzig Conjecture, [3]) and the conjecture is known to be true in several special cases, see [1, 3, 6, 8].

(b) Rosa [8] introduced the notion of an *α-valuation* (see [6] or [8] for the definition). It is easy to see that if $G = (X, Y; E)$ has an α-valuation, then $G$ has a strong $\{1, 2, \ldots, |A|\}$-labeling $f$ such that $f(X \cup Y) \subseteq \{0, 1, \ldots, |A|\}$.

(c) There exists a strong $\{1, 2, \ldots, |A|\}$-labeling $f$ of $G$ such that $f(X \cup Y) \subseteq \{0, 1, \ldots, |A|\}$ if and only if there exists a strong $\{0, 1, \ldots, |A| - 1\}$-labeling $f'$ of $G$ such that $f'(X \cup Y) \subseteq \{0, 1, \ldots, |A| - 1\}$.

### 2.1. Strong A-labelings

**Proposition 2.1.1.** If $G = (X, Y; E)$ has a strong A-labeling $f$, then for every $n > \max\{f(v): v \in X \cup Y\}$, $B(A; n)$ decomposes into $G$.

**Proof.** Consider the embedding $\phi$ of $G$ into $B(A; n)$ given by $\phi(v) = x_{f(v)}$, if $v \in X$, and $\phi(v) = y_{f(v)}$, if $v \in Y$. Its easy to see that $\phi(G)$ satisfies (*). □

**Corollary 2.1.2.** If a tree $T$, $|V(T)| = n + 1$, has a strong $\{0, 1, \ldots, n - 1\}$-labeling $f$ such that $f(V(T)) \subseteq \{0, 1, \ldots, n - 1\}$, then for every $m \geq n$, $B((0, 1, \ldots, n - 1); m)$ decomposes into $T$. In particular, $K_{n,n}$ and $K_{n+1,n+1} - (n + 1)K_2$ decompose into $T$. □

Corollary 2.1.2. applies to all trees that have an α-valuation, in particular to all caterpillars, see [8]. In fact, the set of trees with α-valuation is very rich, see [6] for some general results.

**Corollary 2.1.3.** Let $A \subseteq \{0, 1, \ldots\}$, $|A| = n$, and let $T$ be a caterpillar, $|V(T)| = n + 1$. There exists $m_0$ such that for every $m > m_0$, $B(A; m)$ decomposes into $T$.

**Proof.** We will construct a strong A-labeling $f$ for $T$. Then, it will suffice to take $m_0 = \max\{f(v): v \in V(T)\}$. Consider the following geometric representation of the caterpillar $T$. Let vertices of the color classes of $T$ lie on two different, horizontal, parallel lines (one line for the vertices of one class) in such a way that edges do not intersect in their interior points. Assign elements of $A$ to the edges of $T$, so that they increase as we move from left to right. Let $v$ be the rightmost
vertex on the top line. Assign \( f(v) = 0 \) and observe that this can be uniquely extended to a strong \( A \)-labeling \( f \) of \( T \) such that for every \( e = xy \in E(T) \), where \( x \) is on the bottom line, \( f(x) - f(y) \) is the element of \( A \) assigned to \( e \).

The following question arises now.

**Problem 2.1.4.** Which trees of size \( r \) have a strong \( A \)-labeling for every \( A \) with \( |A| = r \)? In particular, do all trees of size \( r \) have a strong \( A \)-labeling for every \( r \)-element set \( A \)?

Note that unlike in the case of graceful labelings (or \( \alpha \)-valuations) we allow any nonnegative integers to be assigned to vertices. Moreover, the same label can be assigned to two different vertices, as long as they are in different color classes. Thus, these two questions should be more manageable than the Ringel–Kotzig conjecture.

We finish this subsection with the result that shows that for some sets \( A \), every tree of size \( |A| \) has a strong \( A \)-labeling. First, note that in a bipartite graph \( G = (X, Y; E) \), the color classes \( X \) and \( Y \) are distinguished by their order. Trivially, this is immaterial when \( A \)-labelings are considered, and the following simple statement shows that this is the case with strong \( A \)-labelings, too.

**Proposition 2.1.5.** Let \( A \) be a given set of \( r \) nonnegative integers. A bipartite graph \( G = (X, Y; E) \) of \( r \) edges has a strong \( A \)-labeling if and only if \( G' = (Y, X; E) \) has one.

**Proof.** Let \( f : X \cup Y \rightarrow \{0, 1, \ldots, m\} \) be a strong labeling of \( G \), where \( m = \max\{f(v) : v \in X \cup Y\} \). Define \( f'(v) = m - f(v) \), for \( v \in X \cup Y \). Then, \( f' \) is a strong \( A \)-labeling of \( G' \). \( \square \)

Our next result shows that for some sets \( A \), \( |A| = r \), all trees of size \( r \) have a strong \( A \)-labeling. Let \( A = \{a_1, \ldots, a_r\} \), \( 0 \leq a_1 < \cdots < a_r \). Then, \( A \) is called sparse, if for all \( 2 \leq i \leq r \),

\[
a_i > a_{i-1} + \sum_{j=1}^{[(i-2)/2]} (a_{i-1-j} - a_j). \tag{**}\]

(For instance, if \( k \geq 2 \), and \( a_i = k^i \), then \( A \) is sparse.)

**Theorem 2.1.6.** If \( A \) is a sparse set of cardinality \( r \), then every tree of size \( r \) has a strong \( A \)-labeling.

**Proof.** Let \( G = (X, Y; E) \) be a tree of size \( r \). By Proposition 2.1.5 we may assume that \( X \) contains a vertex \( x \) of degree 1. Let \( y \in Y \) be the neighbor of \( x \), and denote by \( G' \) the graph obtained from \( G \) by the deletion of \( x \).
Trivially, the assertion is true for \( r = 1 \). So, assume that \( r > 1 \). Let \( A' = A - \{a_r\} \). Clearly, \( A' \) is sparse hence, by the induction hypothesis, there exists a strong \( A' \)-labeling \( f' : (X - \{x\}) \cup Y \rightarrow \{0, 1, \ldots\} \) of \( G' \). Define \( f(v) = f'(v) \), for \( v \in (X - \{x\}) \cup Y \), and \( f(x) = f'(y) + a_r \).

We claim that \( f \) is a strong \( A \)-labeling of \( G \). By our assumptions on \( f' \), the only fact to be shown is that \( f(x) \neq f(z) \), for \( z \in X - \{x\} \). Since \( G - x \) is connected, there is a path \( y_1, x_1, y_2, \ldots, y_t, x_t \) (\( t \geq 1 \)) such that \( y_1 = y \) and \( x_t = z \). For an edge \( xy \) of \( G \), define \( f(xy) = f(x) - f(y) \). Using this notation we have

\[
f(z) = f(y) + (f(x_1, y_1) - f(x_1, y_2)) + \cdots + (f(x_{r-1}, y_{r-1}) - f(x_r, y_r)) + f(x_t, y_t).
\]

Clearly,

\[
f(x_1, y_1) + \cdots + f(x_r, y_r) \leq a_{r-1} + \cdots + a_{r-t},
\]

and

\[
f(x_1, y_2) + \cdots + f(x_{r-1}, y_{r-1}) \geq a_1 + \cdots + a_{t-1}.
\]

Thus,

\[
f(z) \leq f(y) + a_{r-1} + \cdots + a_{r-t} - a_1 + \cdots + a_{t-1} < f(y) + a_r = f(x),
\]

by (**). \( \square \)

This result and Proposition 2.1.1 imply, in particular, that there is an infinite class of \( r \)-regular bipartite graphs that decompose into any tree of size \( r \). Now, we pass on to the case of \( A \)-labelings.

2.2. \textit{A-labelings}

\textbf{Proposition 2.2.1.} If \( G = (X, Y; E) \) has an \( A \)-labeling \( f \) then for every \( n > \max\{f(v) : v \in X \cup Y\} \), \( \tilde{B}(A; n) \) decomposes into \( G \).

\textbf{Proof.} Consider the subgraph \( H \) of \( \tilde{B}(A; n) \) defined as the edge-disjoint union of embeddings \( \phi_i(G) \) of \( G \), \( i = 0, 1 \), into \( \tilde{B}(A; n) \), where

\[
\begin{align*}
\phi_0(v) &= x_{f(v)} \quad \text{for} \ v \in X, \\
\phi_1(v) &= y_{f(v)} \quad \text{for} \ v \in Y, \\
\phi_0(v) &= x_{f(v)} \quad \text{for} \ v \in Y, \\
\phi_1(v) &= y_{f(v)} \quad \text{for} \ v \in X.
\end{align*}
\]

It is easy to see that \( H \) satisfies \( (\ast \ast) \). \( \square \)

\textbf{Corollary 2.2.2.} If a tree \( T \), \( |V(T)| = n + 1 \), has a graceful labeling, then for every \( m > n \), \( \tilde{B}(\{1, 2, \ldots, n\}; m) \) decomposes into \( T \). In particular, \( K_{2n+1, 2n+1} - (2n + 1)K_2 \) decomposes into \( T \).

\textbf{Corollary 2.2.3.} Let \( A \subseteq \{0, 1, \ldots\}, |A| = n \), and let \( T \) be a tree, \( |V(T)| = n + 1 \). There exists \( m_0 \) such that for every \( m > m_0 \), \( \tilde{B}(A; m) \) decomposes into \( T \).
Proof. As in the proof of Corollary 2.1.3, it suffices to show that $T$ has an $A$-labeling. To this end, let us choose a vertex $v \in V(T)$ and label the vertices of $T$ according to the breadth-first search starting in $v$. Suppose $v_0, v_1, \ldots, v_n$ is the labeling obtained. This labeling induces a unique labeling of the edges of $T$, $e_1, \ldots, e_n$ such that $e_i$ is incident with $v_i$, $i = 1, \ldots, n$. Assign elements of $A$ to the edges of $T$ so that they increase (with respect to the labeling defined). Then set $f(v) = 0$. Clearly, this can be uniquely extended to a labeling $f$ of $T$ such that for every $e = xy \in E(T)$, $|f(x) - f(y)|$ is the element of $A$ assigned to $e$. \qed

Remark 2.2.4. The proof of Corollary 2.2.3 solves a variant of Problem 2.1.4 for $A$-labelings, i.e., it shows that for every tree $T$ of size $n$ and for every set $A \subseteq \{0, 1, \ldots\}$ of cardinality $n$ there is an $A$-labeling of $T$.

3. Homomorphic decompositions

Our next results are concerned with decompositions of regular bipartite graphs into homomorphic images of trees.

Proposition 3.1. Let $G$ be an $r$-regular bipartite $(n + n)$-graph, and let $T'$ be a given tree of size $r$. Then, there exists a decomposition of $G$ into $n$ subgraphs $G_1, \ldots, G_n$, each of size $r$, such that every $G_i$ is a homomorphic image of $T'$.

Proof. We show that there exists a decomposition of $G$ such that:

(i) each graph of the decomposition is a homomorphic image of $T'$,

(ii) for any $x \in T'$, the images of $x$ (under respective homomorphisms) are all distinct, and form a color class of $G$.

If $r = 1$, $G$ is a matching and $T' = K_2$. In this case the statement is obvious. Let $x \in T'$ be an endpoint and denote $T'^{-1} = T' - \{x\}$. By the König–Hall Theorem, $G$ has a perfect matching $M$. The graph $G' = G - M$ is an $(r - 1)$-regular bipartite graph, so that $G'$ has a decomposition into subgraphs, each homomorphic to $T'^{-1}$, and such that for any $y \in V(T'^{-1})$, the images of $y$ form one color class of $G'$. In particular, this property holds for the neighbor of $x$ in $T'$, and using the edges of $M$, the decomposition $G'$ can be extended to a decomposition of $G$ satisfying (i) and (ii). \qed

This fact can be generalized to the case when the graphs in the decomposition are homomorphic images of (possibly) different trees, as long as these trees have the same color pattern.
Proposition 3.2. Let $T_1, \ldots, T_n$ be $(p + q)$-trees of size $r = p + q - 1$, and let $G$ be an $r$-regular $(n + n)$-bipartite graph. Then, there exist homomorphisms $\phi_1, \ldots, \phi_n$, $(\phi_i : T_i \to G)$ such that $\phi_1(T_1), \ldots, \phi_n(T_n)$ is a decomposition of $G$.

The proof is similar to that of Proposition 3.1. We use the following lemma.

Lemma 3.3. For $V(T) = \{x_1, \ldots, x_{p+q}\}$, denote by $T^{(k)}$ the subgraph induced by $\{x_1, \ldots, x_k\}$. If $T_1, \ldots, T_n$ are arbitrary $(p + q)$-trees, then there exists an ordering of their vertex sets $V(T_i) = \{x_1^i, \ldots, x_{p+q}^i\}$ such that for every $k, 1 \leq k \leq p + q$, and every $i, 1 \leq i \leq n$,

(i) $T^{(k)}_i$ is a connected tree (of size $k - 1$),
(ii) $x_k^i \in V(T_i)$ is an endpoint of $T^{(k)}_i$,
(iii) for a fixed $k$, all $T^{(k)}_i$s have the same color pattern, and the $x_k^i$s are in the same color class.

Proof. For each tree $T_i$, choose the color class with at least as many elements as the other one. Each such color class contains an endpoint of the tree. Choose this endpoint for the last point in the ordering. Now, the statement follows by induction.

Proof of Proposition 3.2. Let $V(T_i) = \{x_i^j : 1 \leq j \leq p + q\}$, where the labelings of the vertex sets satisfy (i)–(iii) of Lemma 3.3. Similarly as in the proof of Proposition 3.1, one can prove by induction on $p + q$, that there exist homomorphisms $\phi_i : T_i \to G$ such that for any fixed $j$, the $\phi_i(x_i^j), (1 \leq i \leq n)$, are all distinct and form a color class of $G$.

Corollary 3.4. Every $r$-regular bipartite graph can be decomposed into any double star of size $r$.

Proof. Any homomorphic image of a double star is a double star, and they are isomorphic.

Note that every tree of at most three edges is a double star, so that our conjecture is true for $r \leq 3$. The next result implies its validity for $r = 4$, as well, since $P_4$ (the path of length 4) is the unique tree of four edges that is not a double star.

Theorem 3.5. Every 4-regular bipartite graph can be decomposed into 4-paths $P_4$.

In the proof we use a special case of the following fact.

Lemma 3.6. Let $B$ and $R$ be arbitrary graphs on the same vertex set (multiple edges allowed) and suppose that $d_B(x) = d_R(x)$ for every vertex $x$. Then the edge set of $B \cup R$ can be decomposed into alternating $B$-$R$ closed walks.
Proof. By the degree condition, every maximal alternating walk is closed. Deleting one of them from \( B \cup R \) yields a graph for which the degree condition still holds, and the result follows by induction. \( \square \)

Proof of Theorem 3.5. Let \( G \) be a 4-regular bipartite graph. As a consequence of the König–Hall Theorem, \( G \) can be partitioned into two 2-regular graphs \( G_B \) and \( G_R \). Let \( V(G) = V_1 \cup V_2 \) (where \( V_i \) is the \( i^{th} \) color class of \( G \) under a fixed 2-coloring). We define two graphs \( B \) and \( R \) with the same vertex set \( V \), as follows: \( uv \in E(B) \) (resp. \( E(R) \)) if in \( G_B \) (resp. \( G_R \)) \( u \) and \( v \) have the same neighbor in \( V_1 \). Obviously, \( B \) and \( R \) are 2-regular graphs on \( V \). Applying Lemma 3.6, we obtain a decomposition of \( B \cup R \) into alternating closed walks, in which any consecutive pair of alternating edges from \( B \) and \( R \) corresponds to a 4-cycle or a 4-path of \( G \), in the latter case, with both endpoints in \( V_2 \).

Using the path-cycle decomposition constructed above, we define now a decomposition that uses only paths. Let us consider a 4-cycle \( C \) of the decomposition. It corresponds to an alternating cycle \( xyx \) of length 2 in \( B \cup R \). Let \( z(z') \) be the common neighbor of \( x \) and \( y \) in \( G_B(G_R) \). Then, the two neighbors of \( z(z') \) in \( G_B(G_R) \) belong to another cycle or path \( P_1(P_z) \) of the decomposition of \( G \), starting in a point \( w \in V_2(w' \in V_2) \). Clearly, \( x \neq w \neq y \), and the path or cycle \( P_z \) can have at most one common point with \( \{x, y\} \) namely, its other endpoint (because \( w \) and the other neighbor of \( z \) are disjoint from \( \{x, y\} \)). Say, \( x \) is disjoint from \( P_z \). Interchange \( zx \) and \( zw \) in \( C \) and \( P_z \). Then, \( C \) becomes a path, as well as \( P_z \). Similarly, we could use the path or cycle \( P_z \) to convert \( C \), and if necessary \( P_z \), into paths. To complete the proof, let us observe that there is a one-to-one assignment of paths or cycles \( P_z \) or \( P_{z'} \) to all cycles \( C \) of the decomposition. It follows from the fact that each cycle can be assigned to two paths or cycles \( (P_z \) and \( P_{z'} \)), and each such path or cycle is assigned to at most two cycles. \( \square \)

4. Decompositions of the \( r \)-cube

Our last result concerns decompositions of \( r \)-dimensional cubes.

Theorem 4.1. Let \( T \) be a tree of size \( r \). Then the \( r \)-dimensional cube \( Q_r \) decomposes into \( T \).

We prove Theorem 4.1 in the following more general form.

Theorem 4.2. Let \( T_1, \ldots, T_k \) be arbitrary \( (p + q) \)-trees, \( k = 2^{-1}, p + q = r + 1 \). There exists a decomposition \( G_1 \cup \cdots \cup G_k = Q_r \) such that \( G_i \cong T_i \) for \( i = 1, \ldots, k \).
Proof. First, assume that the vertices of each $T_i$ are labeled $\{x_i^1, \ldots, x_i^{r+1}\}$, so that conditions (i)-(iii) of Lemma 3.3 holds. We prove (by induction) that $Q_r$ has a decomposition with the additional properties analogous to those of the proof of Proposition 3.1:

(i) each graph $G_i$, $i = 1, 2, \ldots, k$, of the decomposition is isomorphic to $T_r$,
(ii) for every $j = 1, 2, \ldots, r + 1$, images (under respective isomorphisms) of the vertices $x_i^j$, $i = 1, 2, \ldots, k$, form a color class of $Q_r$.

Let $T_i'$ be the tree obtained from $T_i$ by deleting $x_i^{r+1}$, and let $y_i \in V(T_i')$ be the neighbor of $x_i^{r+1}$ in $T_i$. Now, $Q_r = Q_{r-1} \times K_2$, and, by induction hypothesis, one $Q_{r-1}$ can be decomposed into $2^{r-2}$ subgraphs isomorphic to $T_i'$, such that the (images of) vertices $y_i$ form the 'even' vertex class of $Q_{r-1}$, and the other one can be decomposed so that these vertices form the 'odd' class. (More generally, vertices whose images form the 'even' (odd) class in one decomposition, form the 'odd' (even) class in the other.) In $Q_r$, the 'odd' points of the second $Q_{r-1}$ become 'even', and the matching between the two subgraphs $Q_{r-1}$ can be used to obtain a decomposition of $Q_r$, in which the (images of) vertices $x_i^{r+1}$ become the distinct 'odd' points of $Q_r$. □

Remark 4.3. (a) Our results in Theorem 4.1 are related to results of Bialostocki in [2]. He proved that $Q_r$ has an isomorphic decomposition into $t$ graphs (all isomorphic to some graph $H$) whenever $t \mid r2^{r-1}$. Moreover, his decomposition has an additional property (it is strong, i.e. for any two graphs of the decomposition there is an automorphism of $Q_r$ that maps one of them onto the other). Theorem 4.1 states that if $t = 2^{r-1}$, any tree of size $r$ can be chosen as a decomposition 'pattern’. In addition, our decomposition is strong, as well.

(b) In the decomposition of $Q_n$ obtained in the proof of Theorem 4.2, every $T_i$ is an induced subgraph ($Q_n$ is sufficiently 'sparse').

(c) Instead of $(p + q)$-trees, our results (Proposition 3.1, Proposition 3.2, Lemma 3.3, Theorem 4.1, and Theorem 4.2) can be stated for $(p + q)$-forests. A forest is called a $(p + q)$-forest if it does not contain isolated vertices (for the sake of Lemma 3.3, besides, isolated vertices are immaterial when discussing edge decompositions), and if it has (at least one) 2-coloring in which the color classes have $p$ and $q$ vertices, respectively.

References