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Integrals of products of Bernoulli polynomials ☆

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ABSTRACT

Extending known results on integrals of products of two or three Bernoulli polynomials with limits of integration 0 and 1, we obtain identities for such integrals with limits of integration 0 and x, for a variable x. As applications we obtain certain quadratic and cubic identities for Bernoulli polynomials.

1. Introduction

The Bernoulli polynomials $B_n(x)$, which are usually defined by the exponential generating function

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad\bigl(|t|<2\pi\bigr),\tag{1.1}$$

play an important role in different areas of mathematics, including number theory and the theory of finite differences. Since they satisfy the well-known relation

$$\frac{d}{dx}B_n(x)=nB_{n-1}(x)\tag{1.2}$$

(for all $n\geqslant 1$), which follows easily from (1.1), it is to be expected that integrals figure prominently in the study of these polynomials. The most immediate integral formula is obtained by integrating (1.2):

$$B_{n+1}(x)=(n+1)\int_0^x B_n(t)\,dt+B_{n+1},\tag{1.3}$$

with the Bernoulli numbers B_n, $n=0,1,2,\dots$, defined by $B_n=B_n(0)$ or, equivalently, by the exponential generating function

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}\quad\bigl(|t|<2\pi\bigr).\tag{1.4}$$

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Numerous identities in the literature are concerned with writing Bernoulli numbers or polynomials as integrals of other functions. However, relatively fewer publications deal with integrals having Bernoulli polynomials as integrands. Some such results can be found in papers of Mikolás [7] and Mordell [8], and the following interesting integral for a product of two Bernoulli polynomials appears in the book by Nörlund [10, p. 31]: For all $k+m \geqslant 2$,

$$\int_0^1 B_k(t)B_m(t)\,dt = (-1)^{k-1}\frac{k!m!}{(k+m)!}B_{k+m}. \tag{1.5}$$

This was extended by Carlitz [3] to integrals of products of three and four Bernoulli polynomials, and later Wilson [11] evaluated the integral

$$\int_0^1 \overline{B}_k(ax)\overline{B}_m(bx)\overline{B}_n(cx)\,dx,$$

where $\overline{B}_n(x)$ is the periodic extension of $B_n(x)$ on $[0,1)$, and a, b, c are pairwise coprime integers. For $a=b=c=1$ this reduces to Carlitz's result. More recently similar integral evaluations were used by Espinosa and Moll [5] in their study of Tornheim's double sum.

While all these integrals have *fixed* limits of integration, our approach in this paper will be to consider integrals of products of Bernoulli polynomials, with upper limit of integration a variable x. As consequences we obtain certain convolution identities for Bernoulli polynomials. In Section 2 we will do this for products of two Bernoulli polynomials, and in Section 3 for products of three. Finally, in Section 4 we derive some further consequences.

2. Products of two Bernoulli polynomials

As mentioned in the Introduction, the idea is to take the integral on the left of (1.5) from 0 to some x (instead of 1). In particular, we will prove

Proposition 1. *For all $k, m \geqslant 0$ we have*

$$\int_0^x B_k(t)B_m(t)\,dt = \frac{k!m!}{(k+m+1)!}\sum_{j=0}^{k}(-1)^j\binom{k+m+1}{k-j}\big(B_{k-j}(x)B_{m+j+1}(x) - B_{k-j}B_{m+j+1}\big). \tag{2.1}$$

Before proving this, we consider the special case $x=1$. For this we need the following special value of the Bernoulli polynomials:

$$B_n(1) = \begin{cases} B_n & \text{for } n \neq 1, \\ -B_1 = \frac{1}{2} & \text{for } n = 1. \end{cases} \tag{2.2}$$

This follows from the facts that $B_n(0) = B_n$ and $B_{2j+1} = 0$ for $j \geqslant 1$, and from the reflection identity

$$B_n(1-x) = (-1)^n B_n(x), \tag{2.3}$$

which itself can be obtained by easy manipulations of (1.1).

Now, by (2.2) we have

$$B_1(1)B_{m+k}(1) - B_1B_{m+k} = B_{m+k},$$
$$B_{k-j}(1)B_{m+j+1}(1) = B_{k-j}B_{m+j+1} \quad \text{if } j \neq k-1.$$

Therefore almost all the terms in the sum on the right of (2.1) disappear, with only $(-1)^{k+1}(k+m+1)B_{k+m}$ remaining. This immediately gives (1.5).

Proof of Proposition 1. We use integration by parts which, along with (1.2), gives

$$\int_0^x B_k(t)B_m(t)\,dt = \frac{1}{m+1}\big(B_k(x)B_{m+1}(x) - B_kB_{m+1}\big) - \frac{k}{m+1}\int_0^x B_{k-1}(t)B_{m+1}(t)\,dt. \tag{2.4}$$

If we set

$$I_{a,b} := \int_0^x B_a(t)B_b(t)\,dt, \qquad C_{a,b} := B_a(x)B_b(x) - B_aB_b, \tag{2.5}$$

then (2.4) becomes

$$I_{k,m}=\frac{1}{m+1}C_{k,m+1}-\frac{k}{m+1}I_{k-1,m+1},$$

and similarly we have for all $j=0,1,\dots,k-1$,

$$I_{k-j,m+j}=\frac{1}{m+1+j}C_{k-j,m+1+j}-\frac{k-j}{m+1+j}I_{k-1-j,m+1+j} \tag{2.6}$$

and also, by (1.3),

$$I_{0,m+k}=\frac{1}{m+k+1}C_{0,m+k+1}. \tag{2.7}$$

We now substitute (2.7) into (2.6) for $j=k-1$, then this into (2.6) for $j=k-2$, etc.; then we get

$$\begin{aligned}I_{n,m}&=\sum_{j=0}^{k}(-1)^j\frac{k(k-1)\cdots(k-j+1)}{(m+j+1)(m+j)\cdots(m+1)}C_{k-j,m+j+1}\\&=\sum_{j=0}^{k}(-1)^j\frac{k!m!}{(m+j+1)!(k-j)!}C_{k-j,m+j+1},\end{aligned}$$

and with (2.5) this is exactly (2.1). □

If we now interchange the parameters k and m in (2.1), we immediately obtain, for all $k,m\geqslant 0$,

$$\begin{aligned}&\sum_{j=0}^{k}(-1)^j\binom{k+m+1}{k-j}\big(B_{k-j}(x)B_{m+j+1}(x)-B_{k-j}B_{m+j+1}\big)\\&\quad=\sum_{j=0}^{m}(-1)^j\binom{m+k+1}{m-j}\big(B_{m-j}(x)B_{k+j+1}(x)-B_{m-j}B_{k+j+1}\big).\end{aligned} \tag{2.8}$$

Next we evaluate the sums of products of Bernoulli numbers that occur in the two sums. We have the following summation formula.

Lemma 1. *For all $k,m\geqslant 0$ we have*

$$\sum_{j=0}^{k}(-1)^j\binom{k+m+1}{k-j}B_{k-j}B_{m+j+1}-\sum_{j=0}^{m}(-1)^j\binom{m+k+1}{m-j}B_{m-j}B_{k+j+1}=(-1)^m(k+m)B_{k+m+1}. \tag{2.9}$$

Proof. We rewrite (2.9) as

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k+m+1}{j}B_jB_{m+k+1-j}-\sum_{j=0}^{m}(-1)^{m-j}\binom{m+k+1}{j}B_jB_{m+k+1-j}=(-1)^m(k+m)B_{k+m+1}. \tag{2.10}$$

First let $k+m$ be even. In this case always one of $j,m+k+1-j$ is odd, so that $B_jB_{m+k+1-j}=0$, with the exception of B_1B_{m+k}, but this gets canceled. On the right-hand side we have $B_{k+m+1}=0$ for $k+m>0$, and when $k+m=0$, the right-hand side vanished trivially.

Now let $k+m$ be odd. Then we have $(-1)^{m-j}=-(-1)^{k-j}$. Without loss of generality we may assume that $k>m$; in this case we take the sum from 0 to m out of the first summation on the left-hand side of (2.10) and rewrite it as

$$\sum_{j=0}^{m}(-1)^{k-j}\binom{k+m+1}{j}B_jB_{m+k+1-j}=\sum_{j=k+1}^{m+k+1}(-1)^{j-m-1}\binom{k+m+1}{j}B_{m+k+1-j}B_j,$$

having switched the direction of the summation. Now we note that $(-1)^{j-m-1}=(-1)^{k-j}$, and thus the left-hand side of (2.10) becomes

$$\left(\sum_{j=m+1}^{k}+\sum_{j=k+1}^{m+k+1}+\sum_{j=0}^{m}\right)(-1)^{k-j}\binom{k+m+1}{j}B_jB_{m+k+1-j}. \tag{2.11}$$

But these three sums cover the whole range of summation, and this can be easily evaluated using the well-known convolution identity

$$\sum_{j=0}^{n}\binom{n}{j}B_j(y)B_{n-j}(x)=n(x+y-1)B_{n-1}(x+y)-(n-1)B_n(x+y); \tag{2.12}$$

see, e.g., [6, (50.11.2)]. Indeed, if we take $y=1$ and $x=0$, then (2.12) becomes, with (2.3) and the fact that $B_n(0)=B_n$,

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}B_jB_{n-j}=-(n-1)(-1)^nB_n.$$

This, applied to (2.11) with $n=k+m+1$, immediately gives (2.10). □

It is now clear that (2.9) and (2.8) combined give the following reciprocity relation for sums of products of Bernoulli polynomials:

Proposition 2. *For all $k, m\geqslant 0$ we have*

$$\sum_{j=0}^{k}(-1)^j\binom{k+m+1}{k-j}B_{k-j}(x)B_{m+j+1}(x)$$
$$-\sum_{j=0}^{m}(-1)^j\binom{m+k+1}{m-j}B_{m-j}(x)B_{k+j+1}(x)=(-1)^m(k+m)B_{k+m+1}. \tag{2.13}$$

Remarks: 1. Lemma 1 is of a similar nature to two reciprocity relations given in [2]. As its proof showed, the identity (2.9) is basically equivalent to a special case of (2.12), which can also be seen as an alternating version of the well-known Euler's formula

$$\sum_{j=0}^{n}\binom{n}{j}B_jB_{n-j}=-nB_{n-1}-(n-1)B_n \quad (n\geqslant 1);$$

(2.9) was obtained by a simple rearrangement of the summands. For a different family of extensions of Euler's formula, see [1].

2. Proposition 2 can be proved in the following alternative way, without the use of integrals. When $k+m$ is odd, proceed exactly as in the proof of Lemma 1 for that case. In the end we use (2.12) again, this time with $y=1-x$. Then the right-hand side evaluates to $-(n-1)(-1)^nB_n$ again.

Now let $k+m$ be even. After first rewriting (2.13) in a form analogous to (2.10), we assume, without loss of generality, that $k>m$. Then, since $(-1)^{m-j}=(-1)^{k-j}$ in this case, the left-hand side reduces to

$$I:=\sum_{j=m+1}^{k}(-1)^{k-j}\binom{k+m+1}{j}B_j(x)B_{m+k+1-j}(x),$$

while the right-hand side is always 0. Now, changing the direction of the summation in I, we get

$$I=\sum_{j=m+1}^{k}(-1)^{m+1-j}\binom{k+m+1}{k+m+1-j}B_{m+k+1-j}(x)B_j(x)=-I,$$

since m and k have the same parity. Thus $I=0$, and we are done.

3. An important tool in Carlitz's paper [3] and in other papers in this area is the following identity which goes back to at least Nielsen [9, p. 75]:

$$B_k(x)B_m(x)=\sum_{j=0}^{\lfloor\frac{k+m}{2}\rfloor}\left[\binom{k}{2j}m+\binom{m}{2j}k\right]\frac{B_{2j}B_{k+m-2j}(x)}{k+m-2j}+(-1)^{k+1}\frac{k!m!}{(k+m)!}B_{k+m} \tag{2.14}$$

(valid for $k+m\geqslant 2$). Using this, one can easily obtain a different sum for the integral in (2.1); this was done by Carlitz [3] for the corresponding integral from 0 to 1. However, the point here has been to obtain the identity in Proposition 2.

3. Products of three Bernoulli polynomials

To deal with integrals of products of three Bernoulli polynomials, we adopt notation similar to that in Section 2. Once again suppressing the variable x for simplicity, we set

$$I_{n,m,k} := \int_0^x B_n(t)B_m(t)B_k(t)\,dt, \tag{3.1}$$

$$C_{a,b,c} := B_a(x)B_b(x)B_c(x) - B_aB_bB_c. \tag{3.2}$$

Using integration by parts, along with (1.2), we get

$$\begin{aligned} I_{n,m,k} &= \frac{1}{k+1}\int_0^x B_n(t)B_m(t)\frac{d}{dt}B_{k+1}(t)\,dt \\ &= \frac{1}{k+1}\big[B_n(t)B_m(t)B_{k+1}(t)\big]_0^x - \frac{1}{k+1}\int_0^x \frac{d}{dt}\big(B_n(t)B_m(t)\big)B_{k+1}(t)\,dt \\ &= \frac{1}{k+1}C_{n,m,k+1} - \frac{n}{k+1}\int_0^x B_{n-1}(t)B_m(t)B_{k+1}(t)\,dt - \frac{m}{k+1}\int_0^x B_n(t)B_{m-1}(t)B_{k+1}(t)\,dt, \end{aligned}$$

and thus

$$I_{n,m,k} = \frac{1}{k+1}C_{n,m,k+1} - \frac{1}{k+1}\{nI_{n-1,m,k+1} + mI_{n,m-1,k+1}\}. \tag{3.3}$$

We also see that it is a reasonable convention to set

$$I_{n,m,k} = 0 \quad \text{when } \min\{n,m,k\} < 0.$$

It will be convenient to deal with

$$\widetilde{I}_{n,m,k} := \frac{1}{n!m!k!}I_{n,m,k}, \qquad \widetilde{C}_{n,m,k} := \frac{1}{n!m!k!}C_{n,m,k}. \tag{3.4}$$

Then it is easily seen that (3.3) becomes

$$\widetilde{I}_{n,m,k} = \widetilde{C}_{n,m,k+1} - \widetilde{I}_{n-1,m,k+1} - \widetilde{I}_{n,m-1,k+1}. \tag{3.5}$$

Now we claim that for any $\mu \geqslant 1$ we have

$$\widetilde{I}_{n,m,k} = \sum_{a=0}^{\mu-1}(-1)^a\sum_{i=0}^{a}\binom{a}{i}\widetilde{C}_{n-a+i,m-i,k+a+1} + (-1)^\mu\sum_{i=0}^{\mu}\binom{\mu}{i}\widetilde{I}_{n-\mu+i,m-i,k+\mu}. \tag{3.6}$$

This is best proved by induction. For $\mu = 1$ it just reduces to (3.5). Now assume that (3.6) holds for some $\mu \geqslant 1$, and let S be the second sum in (3.6). Then by (3.5) we have

$$\begin{aligned} S &= \sum_{i=0}^{\mu}\binom{\mu}{i}\widetilde{C}_{n-\mu+i,m-i,k+\mu+1} - \sum_{i=0}^{\mu}\binom{\mu}{i}\widetilde{I}_{n-\mu+i-1,m-i,k+\mu+1} - \sum_{i=0}^{\mu}\binom{\mu}{i}\widetilde{I}_{n-\mu+i,m-i-1,k+\mu+1} \\ &= \sum_{i=0}^{\mu}\binom{\mu}{i}\widetilde{C}_{n-\mu+i,m-i,k+\mu+1} - \sum_{i=0}^{\mu}\binom{\mu}{i}\widetilde{I}_{n-(\mu+1)+i,m-i,k+(\mu+1)} - \sum_{i=1}^{\mu+1}\binom{\mu}{i-1}\widetilde{I}_{n-(\mu+1)+i,m-i,k+(\mu+1)}. \end{aligned}$$

The first of the three sums in this last expression adds the case $a = \mu$ to the double sum in (3.6), while the second and third sums above combine, and we have exactly (3.6) with $\mu + 1$ in place of μ. This completes the proof by induction.

Now we set $\mu = n + m + 1$ in (3.6). Then each of the terms $\widetilde{I}_{n-\mu+i,m-i,k+\mu}$ vanishes since one of $n - \mu + i$ and $m - i$ is always negative. Thus the second sum in (3.6) vanishes, and with (3.4) and (3.6) we obtain the following evaluation of the integral in (3.1).

Proposition 3. *For all $n, m, k \geqslant 0$ we have*

$$\frac{I_{n,m,k}}{n!m!k!} = \sum_{a=0}^{n+m}(-1)^a\sum_{i=0}^{a}\binom{a}{i}\frac{C_{n-a+i,m-i,k+a+1}}{(n-a+i)!(m-i)!(k+a+1)!}. \tag{3.7}$$

Once again we note that a different type of evaluation is easily possible by an iterated use of the identity (2.14). We now use (3.7) to obtain the known evaluation of the integral from 0 to 1 which we denote by $I_{n,m,k}(1)$. To simplify notation we set, in analogy to (3.4),

$$\widetilde{B}_n := \frac{1}{n!}B_n.$$

This notation was also used in [5].

Corollary 1. *(See Carlitz [3].) For all $n, m, k \geqslant 1$ we have*

$$I_{n,m,k}(1) = (-1)^{k+1}n!m!k!\sum_{a=0}^{n+m}\left[\binom{a}{m-1}+\binom{a}{n-1}\right]\widetilde{B}_{k+a+1}\widetilde{B}_{n+m-a-1}. \tag{3.8}$$

Proof. With (3.2) and (2.3) we have for $x = 1$,

$$C_{n-a+i,m-i,k+a+1} = \left((-1)^{n+m+k+1}-1\right)B_{n-a+i}B_{m-i}B_{k+a+1}. \tag{3.9}$$

So, clearly $I_{n,m,k}(1) = 0$ when $n+m+k$ is odd, and this is consistent with the right-hand side of (3.8). On the other hand, when $n+m+k$ is even, then we get with (3.7),

$$\frac{I_{n,m,k}(1)}{n!m!k!} = 2\sum_{a=0}^{n+m}(-1)^{a+1}\widetilde{B}_{k+a+1}\sum_{i=0}^{a}\binom{a}{i}\widetilde{B}_{n-a+i}\widetilde{B}_{m-i}. \tag{3.10}$$

Now we use the fact that $B_1 = -1/2$ and $B_{2j+1} = 0$ for $j \geqslant 1$. If k is even, then a must be odd, and since n and m must have the same parity, one of $n-a+i$ and $m-i$ is always odd. Similarly, if k is odd, then a must be even, and since n and m must have different parities, again $n-a+i$ or $m-i$ is odd. The only nonzero terms therefore occur when $i = m-1$ and when $i = a+1-n$, which corresponds to the terms

$$\frac{-1}{2}\binom{a}{m-1}\widetilde{B}_{n+m-a-1} \quad \text{and} \quad \frac{-1}{2}\binom{a}{a+1-n}\widetilde{B}_{n+m-a-1}$$

respectively, as only nonzero terms from the inner sum in (3.10). Hence the right-hand side of (3.10) has reduced to

$$\sum_{a=0}^{n+m}(-1)^a\left[\binom{a}{m-1}+\binom{a}{n-1}\right]\widetilde{B}_{k+a+1}\widetilde{B}_{n+m-a-1}.$$

Finally, since $n, m, k \geqslant 1$, we need $k+a+1$ even to get nonzero contributions to the sum, so we may replace $(-1)^a$ by $(-1)^{k+1}$. This completes the proof. □

Remark. The right-hand side of (3.10) differs from Carlitz's result, but the two are equivalent. One could also sum over every second value of a; however, we left the sum in the form (3.10) for greater simplicity and symmetry.

4. Some further consequences

We will now derive a few further easy consequences of Proposition 3. The following is obvious from the definition of the integral $I_{n,m,k}$.

Corollary 2. *Let $T_{n,m,k}(x)$ be the right-hand side of* (3.7)*, and let $\sigma \in S_3$, where S_3 is the symmetric group of degree* 3*. Then for all $n, m, k \geqslant 0$,*

$$T_{n,m,k}(x) = T_{\sigma(n),\sigma(m),\sigma(k)}(x).$$

In some cases we obtain the following simpler statement.

Proposition 4. *For $n, m, k \geqslant 1$ with $n+m+k$ even, let*

$$U_{n,m,k}(x) := \sum_{a=0}^{n+m}(-1)^a\frac{B_{k+a+1}(x)}{(k+a+1)!}\sum_{i=0}^{a}\binom{a}{i}\frac{B_{n-a+i}(x)}{(n+a+1)!}\frac{B_{m-i}(x)}{(m-i)!}. \tag{4.1}$$

Then for any $\sigma \in S_3$ we have

$$U_{n,m,k}(x) = U_{\sigma(n),\sigma(m),\sigma(k)}(x).$$

Proof. Using the definition (3.2), we separate the Bernoulli numbers from the Bernoulli polynomials in (3.7). By (3.9) and (3.10) the Bernoulli numbers add up to $-\frac{1}{2}I_{n,m,k}(1)/(n!m!k!)$, and this sum is therefore independent of any permutation of $\{n,m,k\}$. We can therefore delete the Bernoulli *numbers* component and rewrite the double sum with the remaining Bernoulli *polynomials* in the same form as the right-hand side of (3.10). After dividing by -2 we get the expression in (4.1). The result then follows from Corollary 2. □

Remark. As can be seen by (3.9), when $n+m+k$ is odd, a connection between the double sum of Bernoulli numbers and the integral $I_{n,m,k}(1)$ cannot be established. In fact, examples show that the corresponding sum of Bernoulli numbers is not invariant under all permutations. (However, the inner sum on the right of (3.10) shows that n and m can be interchanged.) Accordingly, Proposition 4 is in general not true when $n+m+k$ is odd.

Example.

$$\begin{aligned} 7!U_{1,2,3}(x) &= 105B_4(x)B_2(x)B_1(x) - 42B_5(x)B_1(x)^2 - 21B_5(x)B_2(x) + 21B_6(x)B_1(x) - 3B_7(x), \\ 7!U_{3,1,2}(x) &= 140B_3(x)^2B_1(x) - 105B_4(x)B_2(x)B_1(x) - 35B_4(x)B_3(x) \\ &\quad + 42B_5(x)B_1(x)^2 + 42B_5(x)B_2(x) - 28B_6(x)B_1(x) + 4B_7(x). \end{aligned}$$

By equating the two, simplifying, and dividing by 7, we obtain the cubic recurrence relation

$$B_7(x) = -20B_3(x)^2B_1(x) + 30B_4(x)B_2(x)B_1(x) + 5B_4(x)B_3(x) - 12B_5(x)B_1(x)^2 - 9B_5(x)B_2(x) + 7B_6(x)B_1(x).$$

As this example shows, Proposition 4 is of a quite different nature from identities for sums of products of a fixed number of Bernoulli polynomials that were obtained in [4].

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