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# 2-Linked Graphs

CARSTEN THOMASSEN

## 1. INTRODUCTION

Let $Z = (x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k)$ be an ordered set of distinct vertices of a graph $G$. A *Z-linkage* of $G$ is a set of $k$ pairwise disjoint paths $P_1, P_2, \ldots, P_k$ such that $P_i$ connects $x_i$ with $y_i$ for $i = 1, 2, \ldots, k$. We say that $G$ is *k-linked* if $G$ has at least $2k$ vertices and, for any ordered set $Z$ of $2k$ vertices, $G$ has a $Z$-linkage.

A necessary condition for $G$ to be $k$-linked is that $G$ is $(2k-1)$-connected. This condition is not sufficient unless $k = 1$. Larman and Mani [8] and Jung [7] proved independently that there exists a (smallest) integer $f(k)$ such that every $f(k)$-connected graph is $k$-linked. The proof is based on a result of Mader [11] on subdivisions of large complete graphs.

A complete characterization of $k$-linked graphs is not known. A partial result for $k = 2$ was obtained by Watkins [22] and improved by Jung [7] who demonstrated that a 4-connected graph is 2-linked unless it is a non-maximal planar graph. In such a graph we can select four vertices $x_1, x_2, y_1, y_2$ along a facial cycle of length at least four and it follows easily that the graph has no $(x_1, x_2, y_1, y_2)$-linkage. In particular, $f(2) = 6$ while $f(3)$ is unknown.

In this paper we describe completely when a graph does not contain an $(x_1, x_2, y_1, y_2)$-linkage. This result was obtained independently by Seymour [16] but stated without proof in [16].

As applications of this characterization we obtain the result of Jung [7] on 2-linked graphs and, as pointed out in [16, 20], the characterization also yields a polynomially bounded algorithm for deciding whether or not a graph has an $(x_1, x_2, y_1, y_2)$-linkage and for producing such a linkage if it exists. Such an algorithm has also been obtained by Shiloach [17]. In contrast to this, Fortune, Hopcroft and Wyllie [4] have shown that the analogous 2-linkage problem for directed graphs is NP-complete.

If $Z = (x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k)$ is an ordered set of $2k$ (not necessarily distinct) vertices of a multigraph $G$, then a *weak Z-linkage* in $G$ is a set of $k$ pairwise edge-disjoint paths $P_1, P_2, \ldots, P_k$ in $G$ such that $P_i$ connects $x_i$ and $y_i$ for $i = 1, 2, \ldots, k$. We say that $G$ is *weakly k-linked* if $G$ has at least two vertices and, for each ordered set $Z$ of $2k$ vertices, $G$ has a weak $Z$-linkage. The problem of finding weak $Z$-linkages may be regarded as the integer $k$-commodity flow problem for undirected graphs. The non-integer 2-commodity flow problem was treated by Hu [6], and a variant of the integer $k$-commodity flow problem for undirected graphs was shown to be NP-complete by Even, Itai and Shamir [3]. From the characterization of non-2-linked graphs we get the characterization of non-weakly-2-linked multigraphs found independently by Seymour [16] and the author [20].

The concept of $k$-linkage and weak $k$-linkage can be extended to directed graphs in the obvious way. The problem of finding an $(x_1, x_2, y_1, y_2)$-linkage or a weak $(x_1, x_2, y_1, y_2)$-linkage in a directed multigraph is NP-complete by the above-mentioned result of Fortune *et al.* [4] and the problem of characterizing $k$-linked directed graphs is no easier than that of characterizing $k$-linked undirected graphs. However, weakly $k$-linked directed multigraphs have a simpler characterization than weakly $k$-linked undirected multigraphs at least when $k = 2$. An obvious necessary condition for a directed multigraph to be weakly



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$k$-linked is that it is $k$-edge-connected. This condition is also sufficient. For if $Z = (x_1, x_2, \ldots, x_k, y_1, \ldots, y_k)$ is an ordered set of $2k$ vertices in a $k$-edge-connected directed multigraph $D$, then we add a new vertex $x_0$ and $k$ edges going from $x_0$ to $x_1, x_2, \ldots, x_k$, respectively. By a result of Edmonds [2] (for a short proof, see [10]), the resulting directed multigraph has $k$ edge-disjoint branchings from $x_0$ and clearly, the union of these branchings contains a weak $Z$-linkage.

2-Linkages in undirected graphs can be applied to electrical networks. Consider a 2-connected electrical network such that an edge $x_1x_2$ represents a voltage generator and all other edges represent resistances. Let one of these edges be $y_1y_2$ and we now consider the problem of adjusting the resistances such that the current through $y_1y_2$ becomes zero. Frank Nielsen (private communication) has pointed out that this is possible, by Kirchhoff's rule (see e.g. [15]), if and only if the network contains an $(x_1, x_2, y_1, y_2)$-linkage and an $(x_1, x_2, y_2, y_1)$-linkage. Note that Menger's theorem guarantees the existence of at least one of these linkages.

We also consider $k$-linked infinite graphs. We characterize completely the infinite maximal non-2-linked graphs. In particular, every 4-connected non-planar graph is 2-linked. As pointed out by Mader [12], there are infinite non-2-linked planar graphs of arbitrarily high (finite) connectivity. However, we apply a result of Halin [5] to prove that every uncountable $2k$-connected graph is $k$-linked. This result is best possible. Furthermore, we conjecture that, for each $k \geq 2$, every finite $(2k+2)$-connected graph is $k$-linked. If true this is best possible. We also conjecture that, for each odd integer $k$, $k \geq 3$, a finite or infinite multigraph is weakly $k$-linked if and only if it is $k$-edge-connected, and that, for each integer $k \geq 2$, an infinite directed multigraph is weakly $k$-linked if and only if it is $k$-edge-connected.

## 2. Terminology

We use standard terminology. The *vertex* set of a graph $G$ is denoted $V(G)$ and the *edge* joining two vertices $x$ and $y$ is denoted $xy$. If $A \subseteq V(G)$, then $G - A$ is the subgraph obtained from $G$ by *deleting* $A$ and $G(A)$ is the subgraph of $G$ *induced* by $A$, i.e. $G(A) = G - [V(G) \backslash A]$.

If $G$ is a connected graph and $R$, $S$, $T$ are vertex sets, we say that $R$ *separates* $S$ from $T$ if each of $S \backslash R$ and $T \backslash R$ are non-empty, and every $S - T$ path of $G$ (i.e. a path with ends in $S$ and $T$, respectively) contains a vertex of $R$.

A *plane* graph (finite or infinite) is a graph drawn in the plane such that any two edges have at most an end in common.

A *planar* graph is an abstract graph isomorphic to a plane graph.

A *facial cycle* of a plane graph is a cycle whose interior or exterior does not intersect the graph, and a facial cycle in a planar graph is a cycle which is facial in some plane representation of the graph. If the graph is 3-connected, a facial cycle is facial in any plane representation of the graph (see [19]).

Only in Sections 4 and 5 the graphs under consideration may be infinite.

## 3. Maximal Non-2-linked Graphs

In order to characterize the graphs which contain no $(x_1, x_2, y_1, y_2)$-linkage and which are edge-maximal under this restriction we consider a plane graph $G_0$ such that the unbounded face is bounded by a 4-cycle $S_0: x_1x_2y_1y_2x_1$ and such that every other face is bounded by a 3-cycle. Suppose in addition that $G_0$ has no separating 3-cycle (i.e. a 3-cycle which is not a facial cycle). For each 3-cycle $S$ of $G_0$ we add $K^S$, a possible empty complete graph disjoint from $G_0$, and we join all vertices of $K^S$ to all vertices of $S$. The resulting graph $G$ is called an $(x_1, x_2, y_1, y_2)$-*web* with *frame* $S_0$ and *rib* $G_0$. If $G_0$ has more than four vertices, $S_0$ and the rib $G_0$ are uniquely determined, and it follows from well-known results on planar graphs that $G_0$ (and hence also $G$) is 3-connected and that any separating set of three vertices of $G_0$ is of the form $\{x_1, y_1, z\}$ or $\{x_2, y_2, z\}$. A simple argument shows that $G$ has no $(x_1, x_2, y_1, y_2)$-linkage.

THEOREM 1. *Let $x_1, x_2, y_1, y_2$ be vertices of a graph $G$. If $G$ has no $(x_1, x_2, y_1, y_2)$-linkage and the addition of any edge to $G$ results in a graph containing an $(x_1, x_2, y_1, y_2)$-linkage, then $G$ is an $(x_1, x_2, y_1, y_2)$-web. Conversely, any $(x_1, x_2, y_1, y_2)$-web is maximal with respect to the property of not containing an $(x_1, x_2, y_1, y_2)$-linkage.*

PROOF. We prove the first part of the theorem by induction on the number of vertices of $G$. If $G$ has only four vertices, the statement is trivial so we proceed to the induction step. It is easy to see that $G$ contains the cycle $S_0: x_1x_2y_1y_2x_1$. Also, $G$ is 2-connected, for if $x$ is a cutvertex of $G$ and $y, z$ are neighbours of $x$ belonging to distinct components of $G-x$, then clearly the addition of $yz$ to $G$ does not create an $(x_1, x_2, y_1, y_2)$-linkage.

If one of the sets $\{x_1, y_1\}$, $\{x_2, y_2\}$ is a separating set of $G$, then it is easy to see that $G$ is an $(x_1, x_2, y_1, y_2)$-web with rib $S_0 \cup \{x_1y_1\}$ or $S_0 \cup \{x_2y_2\}$ so assume none of these sets separate $G$. We can then prove that $G$ is 3-connected. For suppose $\{x, y\}$ separates $G$ and let $H$ be a component of $G-\{x, y\}$ not intersecting $S_0$. The maximality property of $G$ easily implies that the edge $xy$ is present and that $G-V(H)$ is also maximal with respect to the property of not containing an $(x_1, x_2, y_1, y_2)$-linkage. By the induction hypothesis, $G-V(H)$ is an $(x_1, x_2, y_1, y_2)$-web with rib, say, $G_0$. Then $G_0$ contains a 3-cycle $S$ such that every $V(H)-V(S_0)$ path intersects $S$. But then it is easy to see that the addition of an edge from $H$ to $S$ does not create an $(x_1, x_2, y_1, y_2)$-linkage, a contradiction to the maximality property of $G$.

So we can assume that $G$ is 3-connected.

We next consider the case where $G$ contains a set $A$ of three vertices such that $G-A$ contains a component $H$ not intersecting $S_0$. Then the maximality of $G$ easily implies that $G(A)$ is complete, that $H$ is complete and that all vertices of $H$ are joined to all vertices of $A$. Moreover, it is easy to see that $G-V(H)$ is maximal with respect to the property of not containing an $(x_1, x_2, y_1, y_2)$-linkage. So by the induction hypothesis, $G-V(H)$, is an $(x_1, x_2, y_1, y_2)$-web with rib, say, $G_0$. Let $S$ be the unique 3-cycle of $G_0$ such that every $V(H)-V(S_0)$ path intersects $S$. The maximality of $G$ implies that every vertex of $H$ is joined to every vertex of $S$. So $A = V(S)$ and it follows easily that $G$ is an $(x_1, x_2, y_1, y_2)$-web.

So we can assume that any separating set of three vertices of $G$ (if any) is of the form $\{x_1, y_1, z\}$ or $\{x_2, y_2, z\}$ where $z \notin V(S_0)$. We now consider the situation where $G$ contains a separating set $A$ with four vertices such that a component $H$ of $G-A$ does not intersect $S_0$, and $H$ has at least two vertices. By Menger's theorem, $G$ contains four disjoint $V(S_0)-A$ paths $P_1, P_2, P_3, P_4$. Suppose w.l.g. that these paths form an $(x_1, x_2, y_1, y_2, x_1', x_2', y_1', y_2')$-linkage. We shall prove that $G$ contains the cycle $S_0': x_1'x_2'y_1'y_2'x_1'$.

Let $H'$ be the subgraph of $G$ induced by $A \cup V(H)$. Since $H$ has at least two vertices, $H'$ has no vertex $z$ which separates $\{x_1', x_2'\}$ from $\{y_1', y_2'\}$ for if this were the case, then either $\{x_1', x_2', z\}$ or $\{y_1', y_2', z\}$ would separate $G$, contrary to the assumption of the previous paragraph. So by Menger's theorem, $H'$ contains paths $P_5$, $P_6$ forming an $(x_1', x_2', y_1', y_2')$-linkage or an $(x_1', x_2', y_2', y_1')$-linkage.

Since $\bigcup_{i=1}^{6} P_i$ does not form an $(x_1, x_2, y_1, y_2)$-linkage, $P_5, P_6$ must be an $(x_1', x_2', y_2', y_1')$-linkage. Now it is not difficult to see that the edge $e = x_2'y_1'$ is present in $G$. For otherwise we add this edge to $G$ and obtain an $(x_1, x_2, y_1, y_2)$-linkage consisting of the two paths

$P_7$, $P_8$ where $P_7$, say, contains $e$. But then we obtain an $(x_1, x_2, y_1, y_2)$-linkage of $G$ by replacing $e$ by $P_6$ and, if necessary, a segment of $P_8$ by $P_5$. This contradiction shows that $x_2'y_1'$ and, by symmetry, $x_1'y_2'$ are present in $G$. By considering the sets $\{x_2', y_1'\}$ and $\{x_1', y_2'\}$ instead of $\{x_1', x_2'\}$ and $\{y_1', y_2'\}$, respectively, we conclude as above that also $x_1'x_2'$ and $y_1'y_2'$ are present in $G$. Since $H'$ has no $(x_1', x_2', y_1', y_2')$-linkage none of the edges $x_1'y_1'$ and $x_2'y_2'$ are present in $G$. Hence $G(A) = S_0'$.

Let $G'$ denote the graph obtained by contracting $H$ into a vertex $z_0$ (which is then adjacent to precisely the vertices of $S_0'$). It is easy to see that $G'$ has no $(x_1, x_2, y_1, y_2)$-linkage so by the induction hypothesis, $G'$ is contained in an $(x_1, x_2, y_1, y_2)$-web $G''$ with rib $G_0'$ say. For each vertex $u \in V(S_0') \cup \{z_0\}$, $G'$ contains four $u - V(S_0)$ paths having only $u$ in common pair by pair, so $A \cup \{z_0\} \subseteq V(G_0')$.Since two consecutive vertices of $S_0'$ do not separate $G$, each of the four 3-cycles of $G_0'$ containing $z_0$ are facial cycles of $G_0'$ and each complete graph of $G''$ attached to these 3-cycles is empty. Also, by the connectivity properties of $G$, each complete graph of $G'$ attached to any other 3-cycle of $G_0'$ is empty. So it follows that $G - V(H)$ has a plane representation such that $S_0$ and $S_0'$ are facial cycles. By the maximality of $G$ all other facial cycles are 3-cycles.

Now $H' = G(V(H) \cup A)$ has no $(x_1', x_2', y_1', y_2')$-linkage and is therefore contained in an $(x_1', x_2', y_1', y_2')$-web $H''$. By the connectivity property of $G$ it follows that $H''$ has no separating 3-cycle so $H''$ is planar. It now follows that $G$ is planar and that $S_0$ is a facial cycle. So we have proved that $G$ is an $(x_1, x_2, y_1, y_2)$-web in the case where $G$ has a four-vertex separating set $A$ such that a component $H$ of $G - A$ has at least two vertices and is disjoint from $S_0$.

We can therefore assume that whenever $A$ is a set of at most four vertices separating $G$ such that a component $H$ of $G - A$ does not intersect $S_0$, then $|A| = 4$ and $H$ consists of a vertex of degree 4 in $G$, and $G - A$ does not contain another component $H'$ disjoint from $S_0$. We consider any edge $e$ of $G$ not joining two vertices of $V(S_0)$ and we let $G'$ denote the graph obtained from $G$ by contracting $e$. Then $G'$ has no $(x_1, x_2, y_1, y_2)$-linkage, so, by the induction hypothesis, $G'$ is contained in an $(x_1, x_2, y_1, y_2)$-web $G''$. If $K^S$ has more than one vertex for some facial cycle $S$ of the rib of $G''$, then the three vertices of $S$ separate this $K^S$ from $V(S_0)$. But then $G$ has a set of three or four vertices which separates more than one vertex from $S_0$. However, this is a contradiction to the initial assumption of this paragraph. So each $K^S$ has at most one vertex and hence $G''$ is planar.

We have shown that the contraction of any edge $e$ not in $S_0$ results in a planar graph. We shall show that this implies $G$ to be planar. For suppose $G$ is non-planar. By Kuratowski's theorem, $G$ contains a subgraph $H$ which is a subdivision of $K_5$ or $K_{3,3}$. . Now $V(S_0) \subseteq V(H)$. For if $x_1$, say, is not in $H$, then the contraction of any edge incident with $x_1$ and not in $S_0$ results in a non-planar graph (and such an edge exists since $G$ is 3-connected). Also, if $x$ is a vertex of degree 2 in $H$, then $S_0$ contains the two edges of $H$ incident with $x$ because the contraction of any such edge results in a non-planar graph. This implies that $G$ is obtained from $K_5$ or $K_{3,3}$ by possibly inserting one or two vertices of degree 2 and then adding some edges such that the resulting graph is 3-connected. It is easy to see that such a graph is 2-linked.

This contradiction proves that $G$ is planar. In order to prove that $S_0$ is a facial cycle it is sufficient to prove that $G - V(S_0)$ is connected. But if this is not the case, we select vertices $z_1$, $z_2$ in distinct components and consider for $i = 1, 2$ four $z_i - V(S_0)$ paths having only $z_i$ in common pair by pair. This easily gives us an $(x_1, x_2, y_1, y_2)$-linkage, a contradiction.

So $G$ is planar and $S_0$ is a facial cycle. By the maximality of $G$, $G$ is an $(x_1, x_2, y_1, y_2)$-web and the proof of the first part of the theorem is complete.

In order to prove the second part we consider an $(x_1, x_2, y_1, y_2)$-web $G$ with rib $G_0$. We shall prove that $G \cup e$ has an $(x_1, x_2, y_1, y_2)$-linkage for any edge $e$ not in $G$. Since $G_0$ has no separating 3-cycle it contains no $K_4$ and hence $G \cup \{e\}$ contains a path of length one, two or three having only its ends $x$ and $y$ in common with $G_0$ such that $x$ and $y$ are non-adjacent in $G$. Now $G_0 \cup \{xy\}$ is not a spanning subgraph of an $(x_1, x_2, y_1, y_2)$-web and has therefore, by the first part of Theorem 1, an $(x_1, x_2, y_1, y_2)$-linkage. Then $G$ also has such a linkage and the proof is complete.

COROLLARY 1 (JUNG [7]). *Let $G$ be a 4-connected graph containing vertices $x_1, x_2, y_1, y_2$. Then $G$ has an $(x_1, x_2, y_1, y_2)$-linkage unless $G$ is planar and contains a facial cycle containing $x_1, x_2, y_1, y_2$ in that cyclic order.*

It is possible to modify the proof of Theorem 1 such that Kuratowski's theorem is not used. It seems however, that this is not worth the effort since a very short proof of Kuratowski's theorem is presented in [19]. Also, Theorem 1 has a short proof using Jung's result [7].

As pointed out in [16, 20] we can derive the following result on edge-disjoint paths from Theorem 1 using the line graph operation.

THEOREM 2 (SEYMOUR [16], THOMASSEN [20]). *Let $G$ be a 2-connected multigraph with vertices $x_1, x_2, y_1, y_2$. Then $G$ contains two edge-disjoint paths connecting $x_1$ with $y_1$ and $x_2$ with $y_2$, respectively, unless $G$ is contractible to a 4-cycle or to a graph $G'$ (such that $x_1, x_2, y_1, y_2$ are contracted into $x'_1, x'_2, y'_1, y'_2$, respectively) which is obtained from a 2-connected planar cubic graph by selecting a facial cycle and inserting the vertices $x'_1, x'_2, y'_1, y'_2$ in that cyclic order on edges of that cycle.*

*Conversely, if $G$ has this property, then it does not contain two edge-disjoint paths which connect $x_1$ with $y_1$ and $x_2$ with $y_2$, respectively.*

Alternative proofs of Theorem 2 are given in [16, 20].

Let $g(k)$ denote the smallest function such that every $g(k)$-edge-connected multigraph is weakly $k$-linked. It follows easily from Menger's theorem that $g(k) \leq 2k$. Results containing this as a corollary can also be derived from the fact that every $k$-edge-connected directed multigraph is $k$-linked combined with Nash-Williams' result [13] that every $2k$-edge-connected multigraph has a $k$-edge-connected orientation, and from the result of Edmonds [2], Nash-Williams [14] and Tutte [22] which implies that every $2k$-edge-connected multigraph has $k$ edge-disjoint spanning trees. From Theorem 2 it follows that $g(2) = 3$.

We offer the following conjecture.

CONJECTURE 1. For each odd integer $k \geq 3, g(k) = k$ and, for each even integer $k \geq 2, g(k) = k + 1$.

We have already observed that $g(k) \geq k$. To see that $g(k) > k$ when $k$ is even, we consider the multigraph $G$ obtained from a cycle $x_1 x_2 \cdots x_k y_1 y_2 \cdots y_k x_1$ of length $2k$ by replacing each edge by a multiple edge consisting of $\frac{1}{2}k$ edges. Then $G$ has no weak $(x_1, x_2, \ldots, x_k, y_1, \ldots, y_k)$-linkage. For such a linkage would have $k^2$ edges and hence the linkage would be a decomposition of $G$ into paths. But a simple parity argument shows that such a decomposition is impossible.

If a graph $G$ is not $k$-linked and we add two new vertices and join them to all vertices of $G$, then the resulting graph is not $(k+1)$-linked and its connectivity exceeds the connectivity of $G$ by two. By Theorem 1, $f(2) = 6$ and, thus, $f(k) \geq 2k + 2$ for each $k \geq 2$.

CONJECTURE 2. For each $k \geq 2, f(k) = 2k + 2$.

This conjecture is also of interest in connection with the problem of finding cycles through specified edges [9, 18, 23].

## 4. 2-Linkages in Infinite Graphs

In the following two sections the graphs are allowed to be infinite. If $G$ is a graph containing vertices $x_1, x_2, y_1, y_2$ such that $G$ has no $(x_1, x_2, y_1, y_2)$-linkage, then it is an easy consequence of Zorn's Lemma that $G$ is a spanning subgraph of an edge-maximal graph containing no $(x_1, x_2, y_1, y_2)$-linkage. The purpose of this section is to characterize all such edge-maximal graphs.

We first extend Jung's result [7] to infinite graphs.

Lemma 1. *Let $G$ be a (possibly infinite) graph $G$ containing a 4-cycle $S_0 : x_1x_2y_1y_2x_1$. Suppose further that $G$ contains a subgraph $H$, which is a subdivision of $K_5$ or $K_{3,3}$, such that for every vertex $z$ of $H$ which has degree greater than 2 in $H$ there are in $G$ four $z - V(S_0)$ paths having only $z$ in common pair by pair. Then $G$ contains an $(x_1, x_2, y_1, y_2)$-linkage.*

Proof. It is sufficient to prove the lemma for the finite subgraph $G'$ of $G$ consisting of $S_0$, $H$ and the 20 or 24 $z - V(S_0)$ paths described in the lemma. If $G'$ has no $(x_1, x_2, y_1, y_2)$-linkage, then $G'$ is contained in an $(x_1, x_2, y_1, y_2)$-web $G''$ with rib $G_0$, say. Every vertex $z$ of $H$ which has degree greater than 2 in $H$ must be contained in $G_0$ because of the four $z - V(S_0)$ paths. But then it is easy to see that the planar graph $G_0$ contains a subdivision of $K_5$ or $K_{3,3}$, a contradiction.

We shall need a more general definition of a web. Let $G_0$ be a finite or countably infinite graph containing a 4-cycle $S_0 : x_1x_2y_1y_2x_1$ such that the following hold:

(i) $G_0$ is planar and $S_0$ is a facial cycle,
(ii) the addition to $G_0$ of any edge distinct from $x_1y_1$ and $x_2y_2$ results in a non-planar graph, and
(iii) either $G_0$ has four vertices and five edges or $G_0$ is 3-connected and has no separating 3-cycle.

Now we add, for each facial 3-cycle $S$ of $G_0$ a finite or infinite (possibly empty) complete graph $K^S$ and join it completely to $V(S)$, and for every edge $e$ of $G_0$ which is not contained in a 3-cycle we add a finite or infinite (possibly empty) complete graph $K^e$ and join it completely to the ends of $e$. The resulting graph is called an $(x_1, x_2, y_1, y_2)$-web with rib $G_0$. It is easy to give examples showing that $G_0$ need not contain any facial 3-cycle.

We can now extend Theorem 1 to infinite graphs.

Theorem 3. *The graphs which have no $(x_1, x_2, y_1, y_2)$-linkage and which are edge-maximal under this condition are precisely the $(x_1, x_2, y_1, y_2)$-webs.*

Proof. Suppose $G$ is a graph with no $(x_1, x_2, y_1, y_2)$-linkage and suppose $G$ is maximal under this condition. Clearly $G$ contains the cycle $S_0 : x_1x_2y_1y_2x_1$. As in the proof of Theorem 1 we show that $G$ is 2-connected and if $A$ is a minimal separating set of two or three vertices such that a component $H$ of $G-A$ does not intersect $S_0$, then $A$ induces a complete subgraph of $G$, $H$ is complete and is completely joined to $A$, and $G-A$ has only one such component $H$. Moreover, in this case there is no minimal separating set $A' \neq A$ with two or three vertices such that $G-A'$ has a component $H'$ which intersects $A$ but not $S_0$. For any such minimal set $A'$ would not intersect $H$ and hence $H'$ would contain $H$. But $H'$ (and hence also $H$) is joined completely to $A'$, which is clearly a contradiction. In other words, any vertex $z$ of $A$ either belongs to $V(S_0)$ or is connected to $V(S_0)$ by four paths having only $z$ in common pair by pair.

Now we define $G_0$ as the subgraph of $G$ induced by $V(S_0)$ and all vertices $z$ which are connected to $V(S_0)$ by four paths having only $z$ in common pair by pair. From the reasoning above it follows that $G_0$ contains every vertex of any minimal separating set $A$ with two or three vertices such that a component of $G-A$ does not intersect $S_0$. We prove that for any vertex $z \in V(G_0)\backslash V(S_0)$ there are in $G_0$ four $z - V(S_0)$ paths having only $z$ in common pair by pair. By definition of $G_0$, there are such four paths $P_1, P_2, P_3, P_1$ in $G$. We select these paths such that the total number of edges in these paths is the least possible. Then these paths are contained in $G_0$. For if a vertex $z'$ of $P_1$ is not in $G_0$, then $G$ contains a (minimal) set $A$ of two or three vertices separating $z'$ from $V(S_0)$. One of these vertices is on the segment of $P_1$ from $z'$ to $V(S_0)$ and another is on the segment of $P_1$ from $z'$ to $z$. Since $G(A)$ is complete we can replace $P_1$ by a shorter path, a contradiction to the choice of $P_1, P_2, P_3, P_4$.

From the connectivity property of $G_0$ established above it follows easily that $G_0$ either has four vertices only or else $G_0$ is 3-connected and has no separating 3-cycle. So $G_0$ satisfies condition (iii). Also, by Lemma 1, $G_0$ contains no subdivision of $K_5$ or $K_{3,3}$. By a result of Halin [2, Theorems 9.1, 9.4] any uncountable 3-connected graph contains a subdivision of $K_{3,3}$ so $G_0$ is countable. Now $G_0$ is planar by Kuratowski's theorem. Moreover, $S_0$ is a facial cycle of $G_0$ for otherwise $G_0$ would contain an $(x_1, x_2, y_1, y_2)$-linkage consisting of two paths one of which is in the interior and the other in the exterior of $S_0$ in some planar representation of $G_0$. So $G_0$ satisfies condition (i).

Now consider any vertex $x$ not in $G_0$. Then there exists a (smallest) set $A$ of two or three vertices of $G$ such that the component of $G-A$ containing $x$ does not intersect $S_0$. Then $A \subseteq V(G_0)$, $G(A)$ is complete, and $x$ is joined to all vertices of $A$. Since $G_0$ is 3-connected and has no separating 3-cycle, $x$ is not joined to any other vertex of $G_0$. So $A$ consists precisely of the vertices of $G_0$ adjacent to $x$. If another vertex $x'$ not in $G_0$ is joined to $A' \subseteq V(G_0)$ and $A' \subseteq A$, then the maximality property of $G$ implies that $A' = A$ and $x'$ and $x$ are adjacent. Also, if $A$ consists of two vertices $u$, $v$ only, then the maximality property of $G$ also implies that the edge $uv$ is not contained in a 3-cycle of $G_0$.

In order to prove that $G$ is an $(x_1, x_2, y_1, y_2)$-web with rib $G_0$ it only remains to prove that $G_0$ satisfies condition (ii). If we add an edge $e$ to $G_0(e \neq x_1y_1, e \neq x_2y_2)$ and $G_0 \cup \{e\}$ is still planar, then $S_0$ is facial cycle of this graph and hence $G_0 \cup \{e\}$ has no $(x_1, x_2, y_1, y_2)$-linkage. But then it is easy to see that also $G \cup \{e\}$ has no such linkage. This contradiction shows that (ii) is satisfied.

We have proved that every maximal graph with no $(x_1, x_2, y_1, y_2)$-linkage is an $(x_1, x_2, y_1, y_2)$-web. The converse is proved as in Theorem 1.

## 5. $k$-Linkages and Weak $k$-linkages in Infinite Graphs

Larman and Mani [8] and Jung [7] proved that a $2k$-connected graph which contains a subdivision of a complete graph with $3k$ vertices is $k$-linked. This also holds for infinite graphs but, as pointed out by Mader [12], there are infinite planar graphs of arbitrarily high finite connectivity which are not 2-linked. However, these graphs are countable, and by using a result of Halin [5], we can get a best possible sufficient condition, in terms of connectivity, for an uncountable graph to be $k$-linked.

The webs of Section 4 may be uncountable and 3-connected so we conclude, as in the remark preceding Conjecture 2 that, for each $k \geq 2$, there are uncountable $(2k-1)$-connected graphs which are not $k$-linked. So the following result is best possible.

THEOREM 4. *Every uncountable $2k$-connected graph $G$ is $k$-linked.*

PROOF. Let $Z=(x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k)$ be any ordered set of $2k$ distinct vertices of $G$. By a result of Halin [5, Theorems 9.1, 9.4], $G$ contains a subdivision $H$ of the complete bipartite graph with $2k$ vertices in one class (which we denote by $A$) and countably many vertices in the other class. Now we consider $2k$ disjoint $Z-A$ paths. These paths contain only finitely many vertices of $H$, and so we can extend the $Z-A$ paths into a $Z$-linkage using appropriate paths of $H$.

We believe that Conjecture 1 is also valid for infinite graphs and that an infinite directed multigraph is weakly $k$-linked if and only if it is $k$-edge-connected.

## REFERENCES


1. J. Edmonds, Minimum partition of a matroid into independent subsets, *J. Res. Nat. Bur. Standards Sect. B* **69** (1965), 67–72.
2. J. Edmonds, Edge-disjoint branchings, in *Combinatorial Algorithms*, Academic Press, New York, 1973, pp. 91–96.
3. S. Even, A. Itai, and A. Shamir, On the complexity of time-table and multi-commodity flow problems, *SIAM J. Comput.* **5** (1976), 691–703.
4. S. Fortune, J. Hopcroft, and J. Wyllie, The directed subgraph homeomorphism problem, to appear.
5. R. Halin, Simplicial decompositions of infinite graphs, *Ann. Discrete Math.* **3** (1978), 93–109.
6. T. C. Hu, Multicommodity network flows, *Oper. Res.* **11** (1963), 344–360.
7. H. A. Jung, Eine Verallgemeinerung des $n$-fachen zusammenhangs für Graphen, *Math. Ann.* **187** (1970), 95–103.
8. D. G. Larman and P. Mani, On the existence of certain configurations within graphs and the 1-sceletons of polytopes, *Proc. London Math. Soc.* **20** (1970), 144–160.
9. L. Lovász, Problem 5, *Period. Math. Hungar.* **4** (1973), 82.
10. L. Lovász, On two minimax theorems in graph, *J. Combin. Theory Ser. B* **21** (1976), 96–103.
11. W. Mader, Hinreichende Bedingungen für die Existenz von Teilgraphen, die zu einem vollständigen Graphen homöomorph sind, *Math. Nachr.* **53** (1972), 145–150.
12. W. Mader, Connectivity and edge-connectivity in finite graphs, in *Surveys in Combinatorics* (B. Bollobás, ed.), *London Math. Soc. Lecture Note Series* **38** (1979), 66–95.
13. C. St. J. A. Nash-Williams, On orientations, connectivity and odd-vertex-pairings in finite graphs, *Canad. J. Math.* **12** (1960), 555–567.
14. C. St. J. A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, *J. London Math. Soc.* **36** (1961), 445–450.
15. S. Seshu and M. B. Reed, *Linear Graphs and Electrical Networks*, Addison-Wesley, London, 1961.
16. P. D. Seymour, Disjoint paths in graphs, *Discrete Math.* **29** (1980), 293–309.
17. Y. Shiloach, A polynomial solution to the undirected two paths problem, *J. Assoc. Comput. Mach.* **3** (1980), 445–456.
18. C. Thomassen, Note on circuits containing specified edges, *J. Combin. Theory Ser. B* **22** (1977), 279–280.
19. C. Thomassen, Planarity and duality of finite and infinite graphs, *J. Combin. Theory Ser. B* **29** (1980), 244–271.
20. C. Thomassen, 2-linked graphs, Preprint Series 1979/80, No. 17, Matematisk Institut, Aarhus Universitet, 1979.
21. W. T. Tutte, On the problem of decomposing a graph into $n$ connected factors, *J. London Math. Soc.* **36** (1961), 221–230.
22. W. Watkins, On the existence of certain disjoint arcs in graphs, *Duke Math. J.* **35** (1968), 231–246.
23. D. R. Woodall, Circuits containing specified edges, *J. Combin. Theory Ser. B* **22** (1977), 274–278.


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C. THOMASSEN
*Matematisk Institüt, Aarhus Universität,*
*8000 Aarhus C. Denmark*