Master Polytopes for Cycles of Binary Matroids*

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

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ABSTRACT

Prior work on the cycle polytopes $P(M)$ of binary matroids M has almost exclusively concentrated on regular matroids. Yet almost all binary matroids are nonregular, and almost nothing is known about their cycle polytopes. In this paper we introduce a class of binary matroids L_k, $k \geqslant 1$, the *complete binary matroids of order* k. We show that the facets of the cycle polytopes $P(L_k)$ have a rather simple description which may be used to deduce easily some, and in principle all, facets of the cycle polytopes of general binary matroids M. For this reason we call the polytopes $P(L_k)$ *master polytopes*. Specifically, we describe two methods by which facets of $P(M)$ can be deduced from the facets of certain master polytopes. One method produces a complete description of $P(M)$ but is not computationally efficient. The other one produces a subset of the facets of $P(M)$ by an efficient lifting procedure.

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1. INTRODUCTION

For $k \geqslant 2$, let A^k be the 0-1 matrix with k columns that has as rows all possible distinct 0-1 vectors except for the k unit vectors and the zero vector. Thus A^k has $2^k - k - 1$ rows. Define the *complete binary matroid of order k* to be the matroid specified by the binary standard representation matrix $[I \mid A^k]$, where I is an identity matrix of order $2^k - k - 1$. Denote this binary matroid by L_k. Let L_1 be the matroid consisting of just one loop, and declare it to be the smallest complete binary matroid. Thus for $k \geqslant 1$, L_k is the largest binary matroid of corank k that has no coloops and no coparallel elements. The complete binary matroids are exactly the duals of the binary projective spaces.

Let M be a binary matroid on a set E. Denote by $P(M)$ the polytope of the cycles (= disjoint unions of circuits) of M, i.e.,

$$P(M) = \mathrm{conv}\{\chi^C \in \mathbf{R}^E | C \text{ is a cycle of } M\}, \tag{1.1}$$

where χ^C denotes the incidence (or characteristic) vector of C. Note that $0 \in P(M)$, since the empty set is considered to be a cycle. Furthermore, each polytope $P(L_k)$ is a simplex with 2^k vertices and with easily specified facets; see Section 2.

In this paper we show that the facets of the polytopes $P(L_k)$ may be used to deduce easily some, and in principle all, facets of the cycle polytopes of general binary matroids M. For this reason we call the polytopes $P(L_k)$ *master polytopes*. Specifically, we describe two methods by which facets of $P(M)$ can be deduced from the facets of certain $P(L_k)$.

The first method relies on projection and deduces *all* facets of a given binary matroid M with corank k from the facets of $P(L_k)$. This result is elementary, and the procedure is not computationally efficient, except for certain special cases.

The second method is a lifting procedure which produces a *subset* of the facets of $P(M)$ from certain $P(L_j)$, $j \leqslant k$. To describe the latter process we define a minor N of M to be a *maximal complete contraction minor* if N is complete and obtainable from M by contractions only, and is maximal with respect to these two conditions. It is not difficult to determine whether a given minor N is a maximal complete contraction minor. With similar ease one can find, for each element e of M, at least one maximal complete contraction minor containing e. Evidently, the polytope $P(N)$ of any maximal complete contraction minor N of M is equal to $P(L_j)$ for some $j \leqslant k$. We show that for every maximal complete contraction minor N of M, every facet of $P(N)$ can be lifted to a facet of $P(M)$ by a surprisingly simple formula. This construction supplies a sufficient number of facets of $P(M)$ to establish the Hirsch property for $P(M)$. [We note that D. Naddef (personal communication) recently proved the Hirsch conjecture for all 0-1 polytopes, and thus in particular for the case at hand.]

We omit a detailed review of prior results, since it may be found in Barahona and Grötschel (1986) or Grötschel and Truemper (1989). However, we do include a summarizing list of the connected binary matroids M for which all facets of $P(M)$ are known, and/or for which the weighted cycle optimization problem has been solved:

(1) Graphic M [Orlova and Dorfman (1972), Edmonds and Johnson (1973), Hadlock (1975)].

(2) Cographic M, but without $\mathcal{M}(K_5)^*$ minor [Barahona (1983); $\mathcal{M}(K_5)$ is the polygon matroid of K_5, the complete graph on five nodes, and the asterisk denotes the dual].

(3) M has no F_7^*, $\mathcal{M}(K_5)^*$, R_{10} minor [Seymour (1981), Barahona and Grötschel (1986); F_7 is the Fano matroid, and R_{10} is the binary matroid associated with the 5 by 10 matrix whose columns are the ten 0-1 vectors with three 1's and two 0's].

(4) M can be built up by 2-sums and Y-sums where each of the initial building blocks does not have an F_7 or F_7^* minor or belongs to an arbitrary but finite class of binary matroids [Grötschel and Truemper (1989); the terms 2-*sum* and Y-*sum* refer to certain rank 1 and rank 2 compositions of binary matroids].

Note that the class defined under item (4) properly includes those of items (1)–(3).

By Tutte (1958) and Seymour (1980) no matroid M of the class defined under item (4) can contain a 3-connected minor N that properly contains F_7 or F_7^* unless N is a minor of a matroid in the finite class. Thus one may reasonably claim that to date almost nothing has been published about the structure of cycle polytopes $P(M)$ where M is nonregular.

A few results of Barahona and Grötschel (1986) will be repeatedly invoked. The *trivial inequalities*

$$0 \leqslant x_e \leqslant 1 \quad \text{for all} \quad e \in E \tag{1.2}$$

are valid for $P(M)$, as are the equations

$$\begin{aligned} x_e &= 0 && \text{for all coloops } e \in E, \\ x_e - x_f &= 0 && \text{for all coparallel elements } e, f \in E. \end{aligned} \tag{1.3}$$

Indeed, the latter equations define the affine hull of $P(M)$, which implies the following result.

THEOREM 1.4 [Seymour (1981), Barahona and Grötschel (1986)]. *The dimension of $P(M)$ is equal to the number of coparallel classes of M.*

By (1.3) and Theorem 1.4 we only need to investigate cycle polytopes $P(M)$ where M has no coloops and no coparallel elements. We shall do this from now on; thus, all polytopes $P(M)$ have (full) dimension $|E|$. A *triad* is a cocircuit with three elements.

THEOREM 1.5 [Barahona and Grötschel (1986)]. *The diameter of $P(M)$ is at most equal to the maximum number of disjoint circuits in E.*

THEOREM 1.6 [Barahona and Grötschel (1986)].

(a) *If e is not contained in a triad of M, then $x_e \geq 0$ and $x_e \leq 1$ define facets of $P(M)$.*

(b) *If M has no F_7^* minor, then for any triad $\{e, f, g\}$ the inequalities*

$$
(1.7)\qquad
\begin{aligned}
x_e + x_f + x_g &\leq 2,\\
x_e - x_f - x_g &\leq 0,\\
-x_e + x_f - x_g &\leq 0,\\
-x_e - x_f + x_g &\leq 0,
\end{aligned}
$$

define facets of $P(M)$.

THEOREM 1.8 [Barahona and Grötschel (1986)]. *Let*

$$(1.9)\qquad \sum_{j \in E} a_j x_j \leq \alpha$$

define a facet of $P(M)$, and let C be a cycle of M. Then

$$(1.10)\qquad \sum_{j \in E \setminus C} a_j x_j - \sum_{j \in C} a_j x_j \leq \alpha - \sum_{j \in C} a_j$$

also defines a facet of $P(M)$.

Theorem 1.8 is clearly equivalent to the observation that $x \in P(M)$ implies $y \in P(M)$, where y is defined by

$$(1.11) \qquad y_j = \begin{cases} x_j, & j \in E \setminus C, \\ 1 - x_j, & j \in C. \end{cases}$$

To show that indeed $y \in P(M)$, let x be the convex combination $\Sigma_D \lambda_D \chi^D$, where the summation is over all cycles D of M. Then $y = \Sigma_D \lambda_D \chi^{D \triangle C}$, and hence y is in $P(M)$. Here $\triangle$ denotes the symmetric difference, i.e.,

$$D \triangle C = (D \cup C) \setminus (D \cap C).$$

For a full dimensional polytope P, all inequalities defining a certain facet of P are positive multiples of each other. To have a unique representation $a^T x \leqslant \alpha$ of the facets of $P(M)$ we proceed as follows. We number the elements of M as $1, 2, \dots, n$, then demand that the absolute value of the nonzero coefficient with lowest index of a facet defining inequality be equal to 1. This way we can refer to *the* inequality of a facet, as we shall do from now on.

We also define a binary relation on the set of facet defining inequalities of a given cycle polytope $P(M)$ as follows: Two inequalities are *related* if one is of the form (1.9) and the other of the form (1.10), for some vector a, some scalar α, and some cycle C of M. It is easily verified that this relation is an equivalence relation, and we thus have *facet inequality equivalence classes*. Sometimes the cycles needed for derivation of all members of an equivalence class from a given representative are readily available or easily determined. In that case we shall implicitly describe the equivalence class by listing just one representative. For example, instead of the complete listing of the equivalence class of (1.7), one actually need only write down one representative, say the inequality $x_e + x_f + x_g \leqslant 2$.

Finally, a brief comment about the matroid terminology seems appropriate. We follow Welsh (1976), so in particular the prefix "co" dualizes a term. However, our use of *addition* (*expansion*), which denotes the inverse of deletion (contraction), is different. Either case is covered by *extension*. Relabeling of groundsets of matroids will be of no consequence, so for this reason we consider two isomorphic matroids to be equal. This convention does not affect the use of "maximal complete contraction minor," which refers to a specific minor produced by a particular sequence of contractions.

2. THE CYCLE POLYTOPES OF COMPLETE BINARY MATROIDS

Let L_k, $k \geq 1$, be the complete binary matroids of the Introduction, i.e., L_1 is the matroid consisting of just one loop, and L_k, $k \geq 2$, is the binary matroid defined by $[I \mid A^k]$, where I is the identity matrix of order $2^k - k - 1$, and A^k is the matrix with k columns that has as rows all possible distinct 0-1 vectors except for the k unit vectors and the zero vector. In particular,

$$A^2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \text{and} \quad A^3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \tag{2.1}$$

so L_2 is the matroid consisting of just one triad, while L_3 is the Fano dual F_7^*. In general, we have, for $k \geq 2$,

$$A^{k+1} = \left[\begin{array}{c|c} A^k & 0 \\ \hline A^k & \mathbf{1} \\ \hline I & \mathbf{1} \end{array}\right], \tag{2.2}$$

where $\mathbf{1}$ is a column vector containing only 1's. Every cycle of L_k corresponds to an Eulerian column submatrix (i.e., each row of such a matrix has an even number of 1's) of $[I|A^k]$, and conversely. A straightforward induction argument, using (2.2), proves that every nonempty cycle C of L_k has cardinality 2^{k-1}. This implies that every nonempty cycle C of L_k is actually a circuit, and that the inequality

$$\sum_j x_j \leq 2^{k-1} \tag{2.3}$$

is valid for $P(L_k)$. Indeed, (2.3) defines a facet of $P(L_k)$. This follows from the fact that $P(L_k)$ is full dimensional (see Theorem (1.4)) and that for every cycle C of L_k, except the empty one, the vector $x = \chi^C$ satisfies (2.3) with equality. Observe that L_k has 2^k cycles; so $P(L_k)$ is a full dimensional polytope with 2^k vertices in $\mathbf{R}^{2^k-1}$, which implies that $P(L_k)$ is a simplex. The 2^k facet defining inequalities of $P(L_k)$ are obtained by viewing (2.3) as an instance of (1.9), and by deriving $2^k - 1$ instances of (1.10) using the $2^k - 1$ nonempty cycles of L_k. In passing we note that these observations, which are summarized in the next theorem, also follow from the fact that the L_k are duals of binary projective spaces.

THEOREM 2.4. *For any $k \geqslant 1$, the complete binary matroid L_k has $2^k - 1$ elements and $2^k - 1$ nonempty cycles. Each such cycle C is a circuit, and $|C| = 2^{k-1}$. The cycle polytope $P(L_k)$ is a full dimensional simplex with 2^k vertices. The facet defining inequalities of $P(L_k)$ constitute just one equivalence class, and the inequality*

$$\sum_j x_j \leqslant 2^{k-1} \tag{2.5}$$

is a representative.

We now relate the cycle polytopes $P(L_k)$ to cycle polytopes $P(M)$, where M is a binary matroid without coloops and without coparallel elements.

In general, let $\tilde{M}$ be a binary matroid with M as minor, say $M = \tilde{M}/X \setminus Y$. Then any cycle of M is of the form $C \setminus X$, where C is a cycle of $\tilde{M}$ satisfying $C \cap Y = \varnothing$, and $P(M)$ is obtained from $P(\tilde{M})$ by setting $x_e = 0$, for all $e \in Y$, and by projecting out the components x_e, $e \in X$. If for some a^i, α^i, and J the inequalities $(a^i)^T x \leqslant \alpha^i$, $i \in J$, define $P(\tilde{M})$, then Fourier-Motzkin elimination of the variables x_e, $e \in X$, plus addition of the constraints $x_e = 0$, $e \in Y$, produces a system defining $P(M)$.

We apply these observations as follows. Let M be a given binary matroid of corank $m \geqslant 1$ without coloops and without coparallel elements. If $m = 1$, then $M = L_1$ since otherwise M has a coloop or coparallel elements. If $m \geqslant 2$, then M contains only loops, or the matrix B of any standard representation matrix $[I \mid B]$ of M is a row submatrix of A^m, where $[I \mid A^m]$ defines L_m. Thus M is a contraction minor of L_m, i.e., $M = L_m/X$ for some X, and by projecting out the components x_e, $e \in X$, we can obtain $P(M)$ from $P(L_m)$.

Since we have a complete description of the facet defining inequalities for $P(L_m)$ by Theorem 2.4, we can sometimes explicitly compute a complete description for $P(M)$ for special cases of m and X. As a demonstration, let us look at the case $|X| = 1$, say $X = \{e\}$. Fourier-Motzkin elimination of the variable x_e from the system

$$\begin{aligned} &\sum_j x_j \leqslant 2^{m-1}, \\ &\sum_{j \notin C} x_j - \sum_{j \in C} x_j \leqslant 0 \qquad \text{for all circuits } C \text{ of } L_m \end{aligned} \tag{2.6}$$

for $P(L_m)$ results in the system

(2.7.1)
$$\sum_{j \notin C} x_j \leqslant 2^{m-2} \qquad \text{for all circuits } C \text{ of } L_m \text{ with } e \in C,$$

(2.7.2)
$$\sum_{j \notin C_1 \cup C_2} x_j - \sum_{j \in C_1 \cap C_2} x_j \leqslant 0 \qquad \text{for all circuits } C_1, C_2 \text{ of } L_m \text{ with } e \in C_1 \text{ and } e \notin C_2$$

for $P(L_m/e)$. From (2.7) we can deduce the following result by straightforward arguments.

THEOREM 2.8. *For $m \geqslant 2$, the system of facet defining inequalities for $P(L_m/e)$ can be divided into 2^{m-1} equivalence classes, where each class corresponds to a circuit C of L_m with $e \in C$, and where the inequality*

$$\sum_{j \notin C} x_j \leqslant 2^{m-2}$$

is a representative.

The fact that $P(M)$ can be derived from $P(L_k)$, where k is the corank of M, is theoretically appealing but computationally of very little use for two reasons. First, the number of elements of L_k may be exponential in the number of elements of M. Second, no computationally efficient procedure is known for carrying out the projection in general.

In the following two sections we develop a lifting procedure for constructing facets of $P(M)$ from the facets of certain $P(L_j)$, $j \leqslant k$. The method typically does not produce a complete description of $P(M)$, but it is computationally simple and efficient, in contrast to the above projection method.

3. PARALLEL AND COPARALLEL LIFTING

In this section we establish some auxiliary results, which we then use in Section 4 to prove the main facet lifting theorem. Specifically, we relate elementary matroidal extension operations to facet lifting as follows.

THEOREM 3.1. *Let N be a binary matroid on a groundset E that has no coloops and no coparallel elements. Suppose that $\sum_{j \in E} a_j x_j \leqslant \alpha$ defines a facet of $P(N)$, and let $Q := \{j \mid a_j \neq 0\}$. Let R and S be (possible empty) subsets of the groundset E of N such that*

$$\begin{gathered} R \text{ is independent in } N, \\ R \cap Q = R \cap S = \varnothing, \\ Q \setminus S \neq \varnothing. \end{gathered} \tag{3.2}$$

Extend N to a binary matroid M as follows. First expand N by a set T of coparallel elements where each $t \in T$ is coparallel to precisely one $s \in S$ and conversely. Next add an element z such that $R \cup T \cup \{z\}$ is a circuit of the resulting matroid M. Then $\sum_{j \in E} a_j x_j \leqslant \alpha$ defines a facet of $P(M)$.

Proof. It is helpful to express the assumptions of the theorem in terms of a standard representation matrix for M. Let e be an arbitrary element of $Q \setminus S$. Pick a basis for M that contains R and T, but not e or z. This is possible by (3.2) and by the fact that N has no coloops. The nonbasic part of the representation matrix is then

$$\begin{array}{c|ccc|c|} & e & S_1 & & z \\ \hline S_2 & & & & 0 \\ & & B & & \\ \cline{5-5} R & & & & 1 \\ \hline T & & D & & 1 \\ \hline \end{array}, \tag{3.3}$$

where $S = S_1 \cup S_2$, and where the indexing of the rows corresponds in the obvious way to the indexing of the unit vectors of the omitted identity matrix. Furthermore, the submatrix B corresponds to N, since $N = M/T \setminus \{z\}$. Each row of D is parallel to a row of B indexed by S_2, or is a unit vector with 1 in a column indexed by S_1. The matrix B as well as the entire matrix contain no zero row or unit vector row, and no two rows are identical, since N has no coloops and no coparallel elements, and since $R \cap S = \varnothing$. Thus M has no coloops and no coparallel elements. We now prove the claim of the theorem.

First we show that we may assume that the right hand side α of the facet defining inequality for $P(N)$ is nonzero. If $\alpha = 0$, we produce a facet defining inequality $b^T x \leqslant \beta$, $\beta \neq 0$, of the same equivalence class using an appropriate cycle C of N and the process of (1.9) and (1.10). The assumptions of the theorem are not affected by this change. Assuming that the new inequality defines a facet of $P(M)$, we then transform that inequality back to $\sum_{j \in E} a_j x_j \leqslant 0$ using the unique cycle C' of M for which $C' \setminus T = C$, and conclude that $\sum_{j \in E} a_j x_j \leqslant \alpha$ defines a facet of $P(M)$. Thus we may indeed suppose that α is nonzero.

Since $\dim(P(N)) = |E|$ and since $\sum_{j \in E} a_j x_j \leqslant \alpha$ with $\alpha \neq 0$ defines a facet of $P(N)$, there are $|E|$ cycles of N whose incidence vectors satisfy this inequality with equality and are linearly independent in $\mathbf{R}^E$. Let W be a (nonsingular) $|E| \times |E|$ matrix with such incidence vectors as rows. We may assume that the elements of E are ordered in such a way that:

$$\text{(3.4)} \qquad \begin{array}{rccc} & e & R & S \\ W = [& F \mid & G \mid & H \;]. \end{array}$$

From any row $[f \mid g \mid h]$ of W, partitioned as in (3.4), we can derive two incidence vectors of cycles of M that satisfy $\sum_{j \in E} a_j x_j \leqslant \alpha$ with equality,

$$\text{(3.5)} \qquad \begin{array}{ccccc} e & R & S & T & z \\ {[}\, f \mid & g \mid & h \mid & h \mid & 0 \,], \\ e & R & S & T & z \\ {[}\, f \mid & \bar{g} \mid & h \mid & \bar{h} \mid & 1 \,], \end{array}$$

where the bar denotes complement, e.g., $\bar{g} = \mathbf{1}^T - g$. The two vectors of (3.5) clearly do represent cycles of M [just use (3.3) for verification], and they satisfy $\sum_{j \in E} a_j x_j \leqslant \alpha$ with equality, since $Q \cap R = \varnothing$ by (3.2). From (3.4) and (3.5) we obtain the following real matrix $\tilde{W}$, each of whose rows is the characteristic vector of a cycle of M satisfying $\sum_{j \in E} a_j x_j \leqslant \alpha$ with equality:

$$\text{(3.6)} \qquad \begin{array}{cc} & \begin{array}{ccccc} e & R & S & T & z \end{array} \\ \tilde{W} = & \left[\begin{array}{c|c|c|c|c} F & G & H & H & 0 \\ \hline F & \bar{G} & H & \bar{H} & \mathbf{1} \end{array} \right]. \end{array}$$

We now show that the columns of $\tilde{W}$ are linearly independent. Subtract the column submatrix of $\tilde{W}$ indexed by S from that indexed by T. Then, using cofactor expansion via the nonsingular submatrix $W = [F \mid G \mid H]$ of the top left corner of $\tilde{W}$, the columns of the matrix $[\bar{H} - H \mid \mathbf{1}]$ are linearly independent if and only if this is so for $\tilde{W}$. In $[\bar{H} - H \mid 1]$, subtract the last column from all others, then divide each column except the last one by -2. If the columns of the resulting matrix $[H \mid 1]$ are linearly dependent, then there exist a vector c and a scalar γ, not both 0, such that $Hc = \gamma 1$. Indeed, $\gamma \neq 0$, since H is a column submatrix of W and thus its columns are linearly independent. We may presume $\gamma = \alpha$ due to scaling, so for the equation system $Wy = \alpha 1$ we now have the solutions $y = a$ and $y = c^1$, where c^1 is derived from c by augmentation of 0's. But $a_e \neq 0$ and $c_e^1 = 0$, which contradicts the nonsingularity of W. Thus the columns of $[H \mid 1]$, and hence of $[\bar{H} - H \mid 1]$ and of $\tilde{W}$, are linearly independent. This implies that $\sum_{j \in E} a_j x_j \leqslant \alpha$ defines a facet of $P(M)$. ∎

COROLLARY 3.7. *For $k \geqslant 2$, let M be a binary matroid with a standard representation matrix whose nonbasic part is of the form*

$$\begin{array}{c|c|c|} & Y & z \\ \hline X & A^k & 0 \\ \hline T & D & 1 \\ \hline \end{array},$$

where D is a proper row submatrix of the matrix

$$\left[\begin{array}{c} A^k \\ \hline I \end{array}\right].$$

Then $L_k = M/T \backslash \{z\}$, and each facet defining inequality of $P(L_k)$ also defines a facet of $P(M)$.

Proof. Apply Theorem 3.6 with $N = L_k$, $Q = X \cup Y$, $R = \varnothing$, and with $S \subset X \cup Y$ appropriately selected according to the rows of $\left[\begin{array}{c} A^k \\ \hline I \end{array}\right]$ present in D. ∎

The next result is a useful observation. We omit its easy proof.

PROPOSITION 3.8. *Let N be a binary matroid on a ground set E. Suppose that $a^T x \leqslant \alpha$ defines a nontrivial facet of $P(N)$, and let f be any element of E. Construct a matroid M on the ground set E' from N by adding an element*

f' that is parallel to f. Define $\bar{a} \in \mathbf{R}^{E'}$ by

$$\bar{a}_e := a_e \qquad \textit{for all} \quad e \in E,$$
$$\bar{a}_{f'} := -|a_f|.$$

Then $\bar{a}^T x \leqslant \alpha$ defines a facet of $P(M)$.

When Theorem 1.8 is combined with a recursive application of the lifting results (Theorem 3.1 and Proposition 3.8), quite a number of interesting facet defining inequalities can be obtained from the known classes of such inequalities.

4. LIFTING FACETS FROM COMPLETE CONTRACTION MINORS

In this section we establish the main result. We prove that for every maximal complete contraction minor N of a given binary matroid M, every facet of $P(N)$ can be trivially lifted to become a facet of $P(M)$. In the proof of this result we invoke the following two lemmas, where for convenience from now on we consider two matrices to be equal if one of them can be derived from the other one by a permutation of rows and columns.

We first examine the recognition problem for maximal complete contraction minors.

For some $k \geqslant 2$, let L_{k+1} be a contraction minor of a binary matroid M, and let some N in turn be a contraction minor of L_{k+1} (and thus of M). Assume N is equal to L_k. Select disjoint sets X_0, X_1, and X_2 such that X_0 is a basis of N, $X_0 \cup X_1$ is a basis of L_{k+1}, and $X_0 \cup X_1 \cup X_2$ is a basis of M. Partition the remaining nonbasic elements of M into sets $Y_0, \{e\}, Y_2$ so that Y_0 contains the nonbasic elements of N, $Y_0 \cup \{e\}$ contains those of L_{k+1}, and Y_2 is the set of the remaining nonbasic elements of M. Then the corresponding representation matrix $[I \mid B]$ for M has by (2.2) the form

(4.1)
$$B = \begin{array}{c|c|c|c|}
 & Y_0 & e & Y_2 \\ \hline
X_0 & A^k & \multicolumn{2}{c|}{0} \\ \hline
X_1 & \begin{array}{c} A^k \\ \hline I \end{array} & 1 & 0 \\ \hline
X_2 & \multicolumn{3}{c|}{0\text{-}1} \\ \hline
\end{array}$$

Consider now any other standard representation matrix for M where X_0 is basic and Y_0 nonbasic. Any such matrix can be obtained from $[I \mid B]$ by a sequence of pivots within the submatrix of B indexed by $X_1 \cup X_2$ and $\{e\} \cup Y_2$. Examine the first pivot of such a sequence. If the pivot element is in a column indexed by Y_2, then in the new representation matrix $[I \mid B']$ the matrix B' has the same structure as B of (4.1) except that the index sets X_2 and Y_2 have been changed. The same conclusion holds if the pivot element is a 1 of the explicitly shown $\mathbf{1}$ subvector of the column indexed by e. Thus only one case remains, where an $x \in X_2$ indexes the pivot row and e indexes the pivot column. A routine examination of cases reveals that the new B' is of the form

$$(4.2) \qquad B' = \begin{array}{c} \\ X_0 \\ \\ X_1' \\ \\ f \\ X_2' \end{array} \begin{array}{c|c} Y_0 & Y_2' \\ \hline A^k & 0 \\ \hline A^k & \\ \cline{1-1} I & \mathbf{1}\cdot d^T \\ \cline{1-1} 0 & \\ \hline \multicolumn{2}{c}{\text{0-1}} \end{array},$$

where $X_1' \cup \{F\} = X_1 \cup \{e\}$, $X_2' \cup Y_2' = X_2 \cup Y_2$, and where d is nonzero and is the row subvector of B indexed by x and Y_2. Equally simple arguments prove that the next pivot leads to a new matrix $[I \mid B'']$ where B'' has the structure of B or B', so by induction B and B' are the only two matrices obtainable in any sequence of pivots, up to relabelling of rows and columns as described above. We thus have established the following result.

LEMMA 4.3. *Let N be a contraction minor of a binary matroid M. Assume that N is equal to some L_k, $k \geqslant 2$. Let E_0 be the groundset of N, X_0 be a basis of N, and $Y_0 = E_0 \setminus X_0$. Extend X_0 to an arbitrary basis X of M. Then M has L_{k+1} as a contraction minor that in turn has N as a contraction minor, if and only if $\bar{B}$ of the standard representation matrix $[I \mid \bar{B}]$ defined from X is equal to B of* (4.1) *or B' of* (4.2).

Let M be a binary matroid, and suppose we know of a complete contraction minor N that is equal to, say, L_k. Then we can test in polynomial time whether or not N is a *maximal* complete contraction minor as follows. If $k = 1$, we only need to check whether the single element of N is contained in a triad. If $k \geqslant 2$, then we select an arbitrary basis X of M that contains a basis X_0 of N, but that does not contain any other element of N. In polynomial time we can check whether or not $\bar{B}$ of the standard representation $[I \mid \bar{B}]$ defined by X is a matrix B of type (4.1) or a matrix B' of type (4.2), and then may use Lemma 4.3 to answer the question. Indeed, by repeated application of Lemma 4.3, we obtain a polynomial algorithm that locates a maximal complete contraction minor L_{k+l}, $l \geqslant 0$, of M that in turn has N as a minor. In particular, for any element of M, we can find in polynomial time a maximal complete contraction minor containing it.

The next lemma relies on a particular way of arranging the rows of A^{k+l} of the complete binary matroid L_{k+l} when $k, l \geqslant 2$. Suppose we partition the columns of A^{k+l} into a column set Y_0 of cardinality k, then label the remaining l columns by $z_1, z_2, \ldots, z_l$. Next we sort the rows of A^{k+l} so that any two rows become adjacent whenever the subvector indexed by $z_1, z_2, \ldots, z_l$ of one row is equal to that of the other row. It is not difficult to see that A^{k+l} then has the following form, where

$$A := A^k \quad \text{and} \quad \hat{A} := \left[\frac{A^k}{I}\right],$$

and where $d^1, d^2, \ldots, d^n$ are the rows of A^l:

$$
(4.4)\qquad
\begin{array}{c}
\\ \cdots \\ X_0 \\ \cdots \\ Z_1 \\ \cdots \\ \vdots \\ \cdots \\ Z_l \\ \cdots \\ Z_{l+1} \\ \cdots \\ z_{l+1} \\ \cdots \\ \vdots \\ \cdots \\ Z_{l+n} \\ \cdots \\ z_{l+n} \\ \cdots
\end{array}
\begin{array}{c|c|c|c|c}
 & Y_0 & z_1 & \cdots & z_l \\ \hline
 & A & \multicolumn{3}{c}{0} \\ \hline
 & \hat{A} & 1 & \multicolumn{2}{c}{0} \\ \hline
 & \vdots & \multicolumn{3}{c}{\ddots} \\ \hline
 & \hat{A} & \multicolumn{2}{c|}{0} & 1 \\ \hline
 & \hat{A} & \multicolumn{3}{c}{\mathbf{1}\cdot(d^1)^T} \\ \cline{2-2}
 & 0 & \multicolumn{3}{c}{} \\ \hline
 & \vdots & \multicolumn{3}{c}{\vdots} \\ \hline
 & \hat{A} & \multicolumn{3}{c}{\mathbf{1}\cdot(d^n)^T} \\ \cline{2-2}
 & 0 & \multicolumn{3}{c}{}
\end{array}
\;.
$$

We are now ready for the next, rather technical lemma.

LEMMA 4.5. *For a given $l \geqslant 2$, let $d^1, d^2, \dots, d^n$ be the rows of A^l. For a given $k \geqslant 2$, define*

$$A := A^k \quad \textit{and} \quad \hat{A} := \left[\frac{A^k}{I}\right].$$

Arrange the matrix A^{k+l} of the complete binary matroid L_{k+l} as depicted in (4.4), and let $E_0 = X_0 \cup Y_0$. Then the inequality

$$\sum_{i \in E_0} x_j \leqslant 2^{k-1} \tag{4.6}$$

is valid for $P(L_{k+l})$. Collect as rows in a matrix W all incidence vectors x of cycles of L_{k+l} satisfying (4.6) with equality. Correspondingly index the columns of W by the elements of the groundset of L_{k+l} in the obvious way. Then the columns of any column submatrix U of W that contains all columns of E_0, are linearly independent if and only if for each $i \in \{1,2,\dots,n+l\}$, at least one column of W indexed by some $y_i \in Z_i \cup \{z_i\}$ does not belong to U.

Proof. The inequality (4.6) is obviously valid for $P(L_{k+l})$. Any cycle of L_{k+l} is uniquely specified by subsets $\tilde{Y}_0 \subseteq Y_0$ and $\tilde{Z} \subseteq \{z_1, \dots, z_l\}$. By Theorem 2.4, the incidence vector x of any such cycle satisfies (4.6) with equality if and only if $\tilde{Y}_0 \neq \varnothing$. Indeed, by (4.4) any such incidence vector has the form

$$\begin{array}{ccccccc} E_0 & Z_1 & z_1 & & Z_{n+l} & z_{n+l} \\ \left[\; g \right. & | \; h^1 & | \; \beta^1 & | \; \cdots & | \; h^{n+l} & \left. | \; \beta^{n+l} \right] \end{array}, \tag{4.7}$$

where for $i = 1,2,\dots,n+l$, $h^i = g$ if $\beta^i = 0$, and $h^i = \bar{g}$ (recall that the bar denotes complement) if $\beta^i = 1$. The rows (4.7) of W can be grouped so that all rows with the same $\beta^1, \dots, \beta^{n+l}$ values are adjacent. Then W consists of row blocks of the form

$$\begin{array}{ccccccc} E_0 & Z_1 & z_1 & & Z_{n+l} & z_{n+l} \\ \left[\; G \right. & | \; H^1 & | \; b^1 & | \; \cdots & | \; H^{n+l} & \left. | \; b^{n+l} \right] \end{array}, \tag{4.8}$$

where for $i = 1,2,\dots,n+l$, $H^i = G$ if $b^i = 0$, and $H^i = \overline{G}$ if $b^i = 1$. By Theorem 2.4, G is nonsingular, since its rows consist of the characteristic vectors of all nonempty cycles of L_k. The $(2^{k+l}-1)-(2^l-1)$ rows of W are linearly independent by Theorem 2.4, since L_{k+l} is complete. Thus we must delete at least 2^l-1 of the $2^{k+l}-1$ columns of W to obtain a submatrix U all whose columns are linearly independent. There exists such a U that contains all columns of E_0 and that has exactly $(2^{k+l}-1)-(2^l-1)$ columns. For any $i \in \{1,2,\dots,n+l\}$, consider the column submatrix of W defined by $E_0 \cup Z_i \cup \{z_i\}$. Deletion of duplicate rows reduces that column submatrix to

$$F = \begin{array}{c} \begin{array}{ccc} E_0 & Z_i & z_i \end{array} \\ \left[\begin{array}{c|c|c} G & G & 0 \\ \hline G & \bar{G} & 1 \end{array}\right] \end{array}.$$

Clearly the latter matrix has dependent columns. Suppose for an arbitrary $y \in Z_i \cup \{z_i\}$, we delete the column indexed by y from F. Then we get a matrix, say $\tilde{F}$, whose rows are precisely all incidence vectors x of cycles of a matroid M of Corollary 3.7 satisfying $\sum_{j \in E_0} x_j = 2^{k-1}$. By that corollary the columns of $\tilde{F}$ are linearly independent. By induction (or better, by a simple matroid argument about the real matroid represented by W) it is then easily seen that deletion of at least one arbitrarily selected column $y_i \in Z_i \cup \{z_i\}$, $i = 1,2,\dots,n+l$, is necessary and sufficient to reduce W to a U all whose columns are linearly independent, provided no column of E_0 is deleted. ■

Combination of Corollary 3.7 and Lemmas 4.3 and 4.5 produces the main result of this section, which generalizes Theorem 1.6.

THEOREM 4.9. *Let M be a binary matroid that has no coloops and no coparallel elements. Suppose a matroid N with groundset E_0 is a complete contraction minor that is equal to, say, L_k. Then the following statements are equivalent*:

(i) *N is a maximal complete contraction minor of M.*

(ii) *Every facet defining inequality of $P(N)$ also defines a facet of $P(M)$.*

(iii) *At least one facet defining inequality of $P(N)$ defines a facet of $P(M)$.*

(iv) *The facet defining inequality*

$$\sum_{j \in E_0} x_j \leqslant 2^{k-1} \tag{4.10}$$

of $P(N)$ also defines a facet of $P(M)$.

Proof. The equivalence of (ii), (iii), and (iv) follows from the equivalence result of Theorem 2.4 and the fact that for any cycle C of $N = M/T$, there is a cycle C' of M such that $C = C' \backslash T$. Thus we only need to show (i) $\Leftrightarrow$ (iv).

(i) $\Rightarrow$ (iv): The case $k = 1$ follows immediately from Theorem 1.6, part (a), but for completeness we include the short proof by Barahona and Grötschel (1986). Let $E_0 = \{e\}$. By (i), e is not contained in any triad of M, and thus $M \backslash e$ has no coloops and no coparallel elements. Then by Theorem 1.4 [which, incidently, also has a short proof; see Barahona and Grötschel (1986)], $\dim(P(M)) - 1 = \dim(P(M \backslash e)) = \dim(\{x \in P(M) \mid x_e = 0\})$. Thus $x_e \geqslant 0$ defines a facet of $P(M)$, and via the construction of (1.9) and (1.10), $x_e \leqslant 1$ is also a facet defining inequality of $P(M)$. Thus suppose $k \geqslant 2$. If $\operatorname{corank}(M) \leqslant \operatorname{corank}(N) + 1$, then $M = N$, or M is a matroid of Corollary 3.7. In either case (iv) holds. If $\operatorname{corank}(M) - \operatorname{corank}(N) = l \geqslant 2$, then Lemma 4.3 and (i) imply that M is a contraction minor of some $L_{k+l}/\{y_1, y_2, \ldots, y_{l+n}\}$, where L_{k+l} is defined by A^{k+l} of (4.4), where $y_i \in Z_i \cup \{z_i\}$, $i = 1, 2, \ldots, l + n$, and where the submatrix A of A^{k+l} corresponds to N. Lemma 4.5 then supplies the desired conclusion, since a matrix U of that lemma must be the matrix where each row is the incidence vector x of a cycle of M for which $\sum_{j \in E_0} x_j = 2^{k-1}$.

(iv) $\Rightarrow$ (i): If the complete contraction minor N of M is not maximal, then M has an L_{k+1} as a contraction minor which in turn has N as a contraction minor. If E_1 is the groundset of L_{k+1}, then $E_1 \setminus E_0$ is a cycle of L_{k+1}, and by Theorem 2.4 and the fact that L_{k+1} is a contraction minor, the two inequalities

$$\sum_{j \in E_1} x_j \leqslant 2^k,$$

$$\sum_{j \in E_0} x_j - \sum_{j \in E_1 \setminus E_0} x_j \leqslant 0$$

define facets of $P(L_{k+1})$ and are valid inequalities for $P(M)$. We obtain $\sum_{j \in E_0} x_j \leqslant 2^{k-1}$ if we multiply the two inequalities by $\frac{1}{2}$ and add the resulting inequalities. But then $\sum_{j \in E_0} x_j \leqslant 2^{k-1}$ cannot define a facet of the full dimensional $P(M)$, which contradicts (iv). ∎

If a matroid has no coloops and no coparallel elements, then contraction of all elements save one always produces L_1. Thus every element of a binary M without coloops and without coparallel elements is contained in some maximal contraction minor L_k of M, for some $k \geqslant 1$. Indeed, by the observations following Lemma 4.3, at least one such maximal minor L_k can be found in polynomial time for a given element e of M. Thus by Theorems 2.4 and

4.9 we can find in polynomial time 2^k inequalities $(a^i)^T x \leqslant \alpha^i$ for $P(M)$ where the supports of the a^i are all equal, and where any a^i contains exactly $2^k - 1$ nonzeros, including one nonzero in the position indexed by e. With these observations one can prove the Hirsch conjecture for the cycle polytopes $P(M)$ of arbitrary binary matroids M by straightforward arguments. That well-known conjecture states that every d-dimensional polyhedron with f facets has diameter at most equal to $f - d$. As already stated in the Introduction, D. Naddef (private communication) recently proved the Hirsch conjecture for *all* polytopes with $\{0, 1\}$ extreme points, and thus in particular for the case at hand.

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