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The failure of spectral synthesis on some types of discrete Abelian groups ☆

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Abstract

The problem of spectral synthesis on arbitrary Abelian groups is solved in the negative.

Keywords: Spectral analysis; Spectral synthesis; Discrete Abelian group

1. Introduction

Spectral analysis and spectral synthesis deal with the description of translation invariant function spaces over locally compact Abelian groups. Translation invariant function spaces appear in several different contexts: linear ordinary and partial difference and differential equations with constant coefficients, theory of group representations, classical theory of functional equations, etc. A fundamental problem is to discover the structure of such spaces of functions, or more exactly, to find an appropriate class of basic functions, the building blocks, which serve as "typical elements" of the space, a kind of basis. It turns out that these building blocks are the so-called exponential monomials, which we shall define later. We consider the space $\mathcal{C}(G)$ of all complex valued continuous functions on a locally compact Abelian group G, which is a locally convex topological linear space with respect to the pointwise linear operations (addition, multiplication with scalars) and to the topology of uniform convergence on compact sets. Suppose that a closed linear subspace

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in this space is given, which is translation invariant. This subspace may or may not contain any basic function of the above mentioned form. If it contains, then we say that *spectral analysis holds* for the subspace in question. An exponential monomial in a subspace of this type can be considered as a kind of spectral value together with its multiplicity. A famous and pioneer result of L. Schwartz [4] exhibits the situation by stating that if the group is the real line with the usual topology, then spectral values do exist, that is, any closed translation invariant linear subspace of the function space mentioned above contains an exponential function. Here the multiplicity refers to the highest exponent of the power function which—multiplied by the exponential function—belongs to the subspace, too. The complete description of the subspace means that all the exponential monomials corresponding to the spectral exponentials and their multiplicities characterize the subspace: their linear hull is dense in the subspace. If this happens, then we say that *spectral synthesis holds for the subspace*. Actually this is L. Schwartz's result: any closed translation invariant linear space of continuous functions on the reals is synthesizable from its exponential monomials in this way.

Let a locally compact Abelian group G be given. Continuous homomorphisms of G into the additive topological group of complex numbers, and into the multiplicative topological group of nonzero complex numbers are called *additive*, and *exponential functions*, respectively. A *polynomial* is a finite linear combination of products of additive functions (the empty product being 1) and an *exponential monomial* is a product of a polynomial and an exponential function. Now the problem of spectral analysis, and spectral synthesis can be formulated: is it true, that any nonzero closed, translation invariant linear subspace of the space $\mathcal{C}(G)$ contains an exponential function (spectral analysis), and is it true, that in any subspace of this type the linear hull of all exponential monomials is dense (spectral synthesis)? It is easy to see that we can go one step further: instead of the space of continuous functions with the given topology one can start with other important function spaces, which are translation invariant. For instance, the space of integrable functions is the natural setting in the Wiener–Tauberian theory: different versions of the Wiener–Tauberian theorem can be stated as spectral analysis theorems.

An interesting particular case is presented by discrete Abelian groups. Here the problem seems to be purely algebraic: all complex functions are continuous, and convergence is meant in the pointwise sense. The archetype is the additive group of integers: in this case the closed translation invariant function spaces can be characterized by systems of homogeneous linear difference equations with constant coefficients. It is known that these function spaces are spanned by exponential monomials corresponding to the characteristic values of the equation, together with their multiplicities. In this sense the classical theory of homogeneous linear difference equations with constant coefficients can be considered as spectral analysis and spectral synthesis on the additive group of integers.

The next simplest case is the case of systems of homogeneous linear difference equations with constant coefficients in several variables, or, in other words, spectral analysis and spectral synthesis on free Abelian groups with a finite number of generators. As in this case a structure theorem is available, namely, any group of this type is a direct product of finitely many copies of the additive group of integers, it is not very surprising to have the corresponding—nontrivial—result by M. Lefranc [3]: on finitely generated free Abelian groups spectral analysis and spectral synthesis holds for any closed translation invariant subspace.

Based on these results the natural question arises: what about other discrete Abelian groups? In his 1965 paper [2] R.J. Elliot presented a theorem on spectral synthesis for arbitrary Abelian groups. However, in 1987 Z. Gajda drew my attention to the fact that the proof of Elliot's theorem had several gaps. On the other hand, in [5] diverse applications of spectral analysis and spectral synthesis in the theory of functional equations have been presented. In [7] spectral analysis for Abelian torsion groups was proved. In 2001 G. Székelyhidi in [8] presented a different approach to the result of Lefranc, and he actually proved that spectral analysis holds on countably generated Abelian groups, further, his method strongly supported the conjecture that spectral analysis—hence also spectral synthesis—might fail to hold on free Abelian groups having no generating set with cardinality less than the continuum. Our main result, Theorem 2 shows that spectral synthesis fails to hold on Abelian groups containing an isomorphic subgroup to $\mathbb{Z}^\omega$, disproving the result in [2]. However, the problem about spectral analysis on arbitrary Abelian groups still remains unsolved.

2. Notation and terminology

Let G be an Abelian group written additively. For any function f on G having values in a set H the *translate of f by the element y in G* is the function $T_y f : G \to H$ defined by the equation

$$T_y f(x) = f(x + y)$$

for each x in G. A set of functions on G is called *translation invariant* if all translates of the functions in the set belong to the set, too. Clearly any intersection of translation invariant sets is translation invariant. For a given set S of complex valued functions on G the intersection of all translation invariant subspaces of $\mathcal{C}(G)$ including S is called *the translation invariant subspace generated by S*.

Using the translation operators defined above we define *difference operators* Δ_y for each y in G in the usual way: given a complex valued function f on G we let

$$\Delta_y f(x) = f(x + y) - f(x)$$

for each x in G. Then $\Delta_y f$ is a complex valued function on G. Symbolically we can write

$$\Delta_y = T_y - T_0,$$

where 0 is the zero element of G. Translation and difference operators are linear operators on the linear space $\mathcal{C}(G)$. Iterates of Δ_y have the obvious meaning. For instance,

$$\Delta_y^2 f(x) = f(x + 2y) - 2f(x + y) + f(x),$$

and

$$\Delta_y^3 f(x) = f(x + 3y) - 3f(x + 2y) + 3f(x + y) - f(x)$$

holds for any complex valued function f on G and for each x, y in G.

Obviously, we can consider G as a locally compact Abelian group equipped with the discrete topology, and $\mathcal{C}(G)$ with the corresponding topology of pointwise convergence (the Tychonoff topology). For a given set S of complex valued functions on G the intersection of all translation invariant closed subspaces of $\mathcal{C}(G)$ including S is called *the variety generated by S*. This is obviously a translation invariant closed subspace of $\mathcal{C}(G)$, the smallest one of these properties, which includes S. In general, a *variety on G* is a closed translation invariant linear subspace of $\mathcal{C}(G)$. If S consists of a single function, say $S = \{f\}$, then the variety generated by S is called *the variety generated by f*. The statement that "the complex valued function g on G belongs to the variety generated by f" means that g is the pointwise limit of a net of functions, each of them being a linear combination of translates of f. Functions in the variety generated by f are exactly the ones which can be approximated in the sense of pointwise convergence by linear combinations of translates of f.

The concepts of additive and exponential functions, polynomials and exponential monomials have their obvious meaning as defined in the preceding section on arbitrary locally compact Abelian groups. This means, that a polynomial on G has the form $x \mapsto P\big(a_1(x), a_2(x), \dots, a_n(x)\big)$, where P is a complex polynomial in n variables and a_k is additive for $k = 1, 2, \dots, n$. We say that this polynomial is of *degree at most N*, if P is of degree at most N. For example, a polynomial of degree at most 1 is a linear combination of additive functions plus a constant, hence it is an additive function plus a constant. The general form of a polynomial of degree at most 2 is the following:

$$p(x) = \sum_{k=1}^{n} \sum_{l=1}^{m} c_{kl}\, a_k(x) b_l(x) + c(x) + d, \tag{1}$$

with some nonnegative integers n, m, additive functions $a_k, b_l, c: G \to \mathbb{C}$ and constants c_{kl}, d. The "leading term" of this polynomial can be obtained by applying difference operators according to the following simple identity:

$$2\, \Delta_y^2\, p(x) = \sum_{k=1}^{n} \sum_{l=1}^{m} c_{kl}\, a_k(y) b_l(y), \tag{2}$$

which can be verified by direct calculation. It follows, that in the representation (1) the additive function c and the constant d are unique, too.

We need the concept of bi-additive functions. The function $B: G \times G \to \mathbb{C}$ is called *bi-additive*, if the functions $x \mapsto B(x, y)$ and $x \mapsto B(y, x)$ are additive for each fixed y in G. It is called *symmetric* if $B(x, y) = B(y, x)$ for all x, y in G. It is easy to check that if $B: G \times G \to \mathbb{C}$ is bi-additive, $c: G \to \mathbb{C}$ is additive, d is a complex number and $f(x) = B(x, x) + c(x) + d$, then we have the following generalization of (2):

$$2\Delta_y^2 f(x) = B(y, y)$$

for each x, y in G. In particular, it follows that B, c and d are unique in the given representation of f. On the other hand, the argument we used in the preceding paragraph can be extended to polynomials of higher degree: if p is a polynomial of the form

$$p(x) = \sum_{k=0}^{N} P_k\big(a_1(x), a_2(x), \dots, a_n(x)\big), \tag{3}$$

where P_k is a homogeneous complex polynomial of degree k in n variables, and $a_1, a_2, \ldots, a_n : G \to \mathbb{C}$ are additive functions, then a representation of p in the form $p(x) = B(x,x) + c(x) + d$ with a symmetric, bi-additive function $B : G \times G \to \mathbb{C}$, an additive function $c : G \to \mathbb{C}$ and a constant d is possible only with $P_k(a_1(x), a_2(x), \ldots, a_n(x)) = 0$ for $k = 3, 4, \ldots, N$, $P_2(x) = B(x,x)$, $P_1(x) = c(x)$ and $P_0(x) = d$ for each x in G. For more about polynomials on Abelian groups see, e.g., [5].

Here we recall a remarkable property of a translation invariant linear function space: if it contains an exponential monomial pm with a nonzero polynomial p, where m is an exponential, then it contains m, too (see, e.g., [5, Theorem 3.4.8, p. 43]).

3. The failure of spectral synthesis

The following theorem is of fundamental importance.

Theorem 1. *Let G be an Abelian group. If there exists a symmetric bi-additive function $B : G \times G \to \mathbb{C}$ such that the variety V generated by the function $x \mapsto B(x,x)$ is of infinite dimension, then spectral synthesis fails to hold for V.*

Proof. Let $f(x) = B(x,x)$ for all x in G. By the equation

$$f(x+y) = B(x+y, x+y) = B(x,x) + 2B(x,y) + B(y,y) \tag{4}$$

we see that the translation invariant subspace generated by f is generated by the functions 1, f and all the additive functions of the form $x \mapsto B(x,y)$, where y runs through G. Hence our assumption on B is equivalent to the condition that there are infinitely many functions of the form $x \mapsto B(x,y)$ with y in G, which are linearly independent. This also implies that there is no positive integer n such that B can be represented in the form

$$B(x,y) = \sum_{k=1}^{n} a_k(x) b_k(y),$$

where $a_k, b_k : G \to \mathbb{C}$ are additive functions $(k = 1, 2, \ldots, n)$. Indeed, the existence of a representation of this form would mean that the number of linearly independent additive functions of the form $x \mapsto B(x,y)$ is at most n.

It is clear that any translate of f, hence any function g in V satisfies

$$\Delta_y^3 g(x) = 0 \tag{5}$$

for all x, y in G: this can be checked directly for f. Hence any exponential m in V satisfies the same equation, which implies

$$m(x)\big(m(y) - 1\big)^3 = 0$$

for all x, y in G, and this means that m is identically 1. By the last remark of the preceding section it follows that any exponential monomial in V is a polynomial. By the results in [1] (see also [5]) and by (5) g can be uniquely represented in the following form:

$$g(x) = A(x,x) + c(x) + d$$

for all x in G, where $A : G \times G \to \mathbb{C}$ is a symmetric bi-additive function, $c : G \to \mathbb{C}$ is additive and d is a complex number. Here "uniqueness" means that the "monomial terms" $x \mapsto A(x, x)$, $x \mapsto c(x)$ and d are uniquely determined, as we pointed out in the preceding section. In particular, any polynomial p in V has a similar representation, which means that it can be written in the form

$$p(x) = \sum_{k=1}^{n} \sum_{l=1}^{m} c_{kl} a_k(x) b_l(x) + c(x) + d = p_2(x) + c(x) + d$$

with some positive integers n, m, additive functions $a_k, b_l, c : G \to \mathbb{C}$ and constants c_{kl}, d. Suppose that p_2 is not identically zero. By assumption, p is the pointwise limit of a net formed by linear combinations of translates of f, that means, by functions of the form (4). Linear combinations of functions of the form (4) can be written as

$$\varphi(x) = cB(x, x) + A(x) + D,$$

with some additive function $A : G \to \mathbb{C}$, and constants c, D. Any net formed by these functions has the form

$$\varphi_\gamma(x) = c_\gamma B(x, x) + A_\gamma(x) + D_\gamma.$$

By pointwise convergence

$$\lim_\gamma \frac{1}{2} \Delta_y^2 \varphi_\gamma(x) = \frac{1}{2} \Delta_y^2 p(x) = p_2(y)$$

follows for all x, y in G. On the other hand,

$$\lim_\gamma \frac{1}{2} \Delta_y^2 \varphi_\gamma(x) = B(y, y) \lim_\gamma c_\gamma,$$

holds for all x, y in G, hence the limit $\lim_\gamma c_\gamma = c$ exists and is different from zero, which gives $B(x, x) = \frac{1}{c} p_2(x)$ for all x in G, and this is impossible.

We infer that any exponential monomial φ in V is actually a polynomial of degree at most 1, which satisfies

$$\Delta_y^2 \varphi(x) = 0 \tag{6}$$

for each x, y in G, hence any function in the closed linear hull of the exponential monomials in V satisfies this equation. However f does not satisfy (6), hence the linear hull of the exponential monomials in V is not dense in V. $\square$

Now we are in the position to present our main result. Here $\mathbb{Z}^\omega$ denotes the (non-complete) direct sum of countably many copies of the additive group of integers, or, in other words, the set of all finitely supported $\mathbb{Z}$-valued functions on the nonnegative integers.

Theorem 2. *Spectral synthesis fails to hold on any Abelian group with torsion free rank at least ω.*

Proof. First we show that there exists a symmetric bi-additive function $B:\mathbb{Z}^\omega \times \mathbb{Z}^\omega \to \mathbb{C}$ with the property that there are infinitely many linearly independent functions of the form $x \mapsto B(x, y)$, where y is in $\mathbb{Z}^\omega$. For any nonnegative integer n let p_n denote the projection of the direct sum $\mathbb{Z}^\omega$ onto the nth copy of $\mathbb{Z}$. This means that for any x in $\mathbb{Z}^\omega$ the number $p_n(x)$ is the coefficient of the characteristic function of the singleton $\{n\}$ in the unique representation of x. It is clear that the functions p_n are additive and linearly independent for different choices of n. Let

$$B(x, y) = \sum_n p_n(x) p_n(y)$$

for each x, y in $\mathbb{Z}^\omega$. The sum is finite for any fixed x, y, and obviously B is symmetric and bi-additive. On the other hand, if χ_k is the characteristic function of the singleton $\{k\}$, then we have

$$B(x, \chi_k) = \sum_n p_n(x) p_n(\chi_k) = p_k(x),$$

hence the functions $x \mapsto B(x, \chi_k)$ are linearly independent for different nonnegative integers k.

The next step is to show that if G is an Abelian group, H is a subgroup of G and $B:H \times H \to \mathbb{C}$ is a symmetric, bi-additive function, then B extends to a symmetric bi-additive function on $G \times G$. Then the extension obviously satisfies the property given in Theorem 1 and our statement follows. On the other hand, the existence of the desired extension is proved in [6, Theorem 2].

The proof is complete. □

In the light of this theorem Lefranc's result is the best possible for free Abelian groups: spectral synthesis holds exactly on the finitely generated ones. However, the following problem arises: is it true that if spectral synthesis fails to hold on an Abelian group, then its torsion free rank is at least ω?

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