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# Generating functions of Legendre polynomials: A tribute to Fred Brafman

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**Abstract**

In 1951, Brafman derived several "unusual" generating functions of classical orthogonal polynomials, in particular, of Legendre polynomials $P_n(x)$. His result was a consequence of Bailey's identity for a special case of Appell's hypergeometric function of the fourth type. In this paper, we present a generalisation of Bailey's identity and its implication to generating functions of Legendre polynomials of the form $\sum_{n=0}^{\infty} u_n P_n(x) z^n$, where $u_n$ is an Apéry-like sequence, that is, a sequence satisfying $(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}$, where $n \ge 0$ and $u_{-1} = 0$, $u_0 = 1$. Using both Brafman's generating functions and our results, we also give generating functions for rarefied Legendre polynomials and construct a new family of identities for $1/\pi$.


*Keywords:* Legendre polynomial; Brafman's generating function; Hypergeometric series; Clausen's identity; Apéry-like sequence; Modular function; $\pi$

## 1. Introduction

Consider the Legendre polynomials $P_n(x)$,

$$P_n(x) = {}_2F_1\left(\begin{matrix} -n, \ n+1 \\ 1 \end{matrix} \,\middle|\, \frac{1-x}{2}\right) = \left(\frac{x+1}{2}\right)^n {}_2F_1\left(\begin{matrix} -n, \ -n \\ 1 \end{matrix} \,\middle|\, \frac{x-1}{x+1}\right)$$
$$= \sum_{m=0}^{n} \binom{n}{m}^2 \left(\frac{x-1}{2}\right)^m \left(\frac{x+1}{2}\right)^{n-m}, \qquad (1)$$

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where we use a standard notation for the hypergeometric series,

$$ {}_mF_{m-1}\left(\begin{matrix} a_1,\ a_2,\ \ldots,\ a_m \\ b_2,\ \ldots,\ b_m \end{matrix} \,\Big|\, z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_m)_n}{(b_2)_n \cdots (b_m)_n} \frac{z^n}{n!}, $$

and $(a)_n = \Gamma(a+n)/\Gamma(a)$ denotes the Pochhammer symbol (or rising factorial).

The Legendre polynomials can be alternatively given by the generating function

$$ (1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) z^n, $$

but there are other generating functions. One particular family shown below is due to Fred Brafman in 1951, which, as shown in our previous work [9], finds some nice applications in number theory, namely, in constructing new Ramanujan-type formulae for $1/\pi$.

**Theorem A** (*Brafman [5]*). *The following generating function is valid:*

$$ \sum_{n=0}^{\infty} \frac{(s)_n (1-s)_n}{n!^2} P_n(x) z^n = {}_2F_1\left(\begin{matrix} s, 1-s \\ 1 \end{matrix} \,\Big|\, \frac{1-\rho-z}{2}\right) \cdot {}_2F_1\left(\begin{matrix} s, 1-s \\ 1 \end{matrix} \,\Big|\, \frac{1-\rho+z}{2}\right), \tag{2} $$

*where* $\rho = \rho(x,z) := (1-2xz+z^2)^{1/2}$.

Theorem A in the form

$$ \sum_{n=0}^{\infty} \frac{(s)_n (1-s)_n}{n!^2} P_n\left(\frac{X+Y-2XY}{Y-X}\right)(Y-X)^n = {}_2F_1\left(\begin{matrix} s,\ 1-s \\ 1 \end{matrix} \,\Big|\, X\right) \cdot {}_2F_1\left(\begin{matrix} s,\ 1-s \\ 1 \end{matrix} \,\Big|\, Y\right) \tag{3} $$

is derived in [5] as a consequence of Bailey's identity for a special case of Appell's hypergeometric function of the fourth type [2, Section 9.6],

$$ \sum_{m,k=0}^{\infty} \frac{(s)_{m+k}(1-s)_{m+k}}{m!^2 k!^2} \big(X(1-Y)\big)^m \big(Y(1-X)\big)^k = {}_2F_1\left(\begin{matrix} s,\ 1-s \\ 1 \end{matrix} \,\Big|\, X\right) \cdot {}_2F_1\left(\begin{matrix} s,\ 1-s \\ 1 \end{matrix} \,\Big|\, Y\right). \tag{4} $$

We note that by specialising $Y = X$, one recovers a particular case of Clausen's identity [10]:

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2},\ s,\ 1-s \\ 1,\ 1 \end{matrix} \,\Big|\, 4X(1-X)\right) = {}_2F_1\left(\begin{matrix} s,\ 1-s \\ 1 \end{matrix} \,\Big|\, X\right)^2. $$

**Remark 1.** The region where (3) holds is somewhat subtle for real $X$ and $Y$: it is the open region bounded by $X+Y=1$, $Y=X+1$, $Y=X-1$, and the lower branch of the hyperbola $X^2-6XY+Y^2+2X+2Y+1=0$. When $X=Y$, the left-hand side of (3) is understood as the limit as $X \to Y$.

In 1959 Brafman addressed a different type of generating functions; the results wherein were later generalised by Srivastava in [12, Eq. (37)].

**Theorem B** (*Brafman [6], Srivastava [12]*). *For a positive integer $N$, a (generic) sequence $\lambda_0, \lambda_1, \dots$ and a complex number $w$,*

$$\frac{1}{\rho}\sum_{k=0}^{\infty}\lambda_k P_{Nk}\left(\frac{x-z}{\rho}\right)\left(w\frac{z^N}{\rho^N}\right)^k=\sum_{n=0}^{\infty}A_nP_n(x)z^n,$$

*where $\rho=(1-2xz+z^2)^{1/2}$ and*

$$A_n=A_n(w)=\sum_{k=0}^{\lfloor n/N\rfloor}\binom{n}{Nk}\lambda_k w^k.$$

Brafman's original results in [6] concern the cases $N = 1, 2$ and a sequence $\lambda_n$ given as a quotient of Pochhammer symbols (in modern terminology, $\lambda_n$ is called a *hypergeometric term*).

In this work we extend Bailey's identity (4) to more general Apéry-like sequences $u_0, u_1, u_2, \dots$ which satisfy the second order recurrence relation

$$(n+1)^2u_{n+1}=(an^2+an+b)u_n-cn^2u_{n-1}$$
$$\text{for } n=0,1,2,\dots,\ u_{-1}=0,\ u_0=1, \tag{5}$$

for given $a$, $b$ and $c$.

Our first result concerns the generating function of $u_n$:

**Theorem 1.** *For the solution $u_n$ of the recurrence equation (5) define*

$$g(X,Y)=\frac{X(1-aY+cY^2)}{(1-cXY)^2}. \tag{6}$$

*Then in a neighbourhood of $X = Y = 0$,*

$$\left\{\sum_{n=0}^{\infty}u_nX^n\right\}\left\{\sum_{n=0}^{\infty}u_nY^n\right\}=\frac{1}{1-cXY}\sum_{n=0}^{\infty}u_n\sum_{m=0}^{n}\binom{n}{m}^2g(X,Y)^mg(Y,X)^{n-m}. \tag{7}$$

We remark that the generating function $F(X)=\sum_{n=0}^{\infty}u_nX^n$ for a sequence satisfying (5) is a unique, analytic-at-the-origin solution of the differential equation

$$\big(\theta^2-X(a\theta^2+a\theta+b)+cX^2(\theta+1)^2\big)F(X)=0, \quad \text{where } \theta=\theta_X:=X\frac{\partial}{\partial X}. \tag{8}$$

The hypergeometric term $u_n=(s)_n(1-s)_n/n!^2$ corresponds to a special degenerate case $c = 0$ and $a = 1, b = s(1-s)$ in (5). Therefore, Bailey's identity (4) corresponds to the particular choice $c = 0$ in Theorem 1.

Theorem 1 also generalises Clausen-type formulae given in [8] which arise as specialisation $Y = X$; see Section 2 for details.

Following Brafman's derivation of Theorem A in [5] we deduce the following generalised generating functions of Legendre polynomials.

**Theorem 2.** *For the solution $u_n$ of the recurrence equation (5), the following identity is valid in a neighbourhood of $X = Y = 0$:*

$$\sum_{n=0}^{\infty} u_n P_n\left(\frac{(X+Y)(1+cXY)-2aXY}{(Y-X)(1-cXY)}\right)\left(\frac{Y-X}{1-cXY}\right)^n$$
$$= (1-cXY)\left\{\sum_{n=0}^{\infty} u_n X^n\right\}\left\{\sum_{n=0}^{\infty} u_n Y^n\right\}. \tag{9}$$

Finally, combining the results of Theorems 2 and B we construct two new generating functions of rarefied Legendre polynomials.

**Theorem 3.** *The following identities are valid in a neighbourhood of $X = Y = 1$:*

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!^2} P_{2n}\left(\frac{(X+Y)(1-XY)}{(X-Y)(1+XY)}\right)\left(\frac{X-Y}{1+XY}\right)^{2n}$$
$$= \frac{1+XY}{2}\, {}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}\\ 1\end{matrix} \,\middle|\, 1-X^2\right) {}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}\\ 1\end{matrix} \,\middle|\, 1-Y^2\right), \tag{10}$$

*and*

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{n!^2} P_{3n}\left(\frac{X+Y-2X^2Y^2}{(X-Y)\sqrt{1+4XY(X+Y)}}\right)\left(\frac{X-Y}{\sqrt{1+4XY(X+Y)}}\right)^{3n}$$
$$= \frac{\sqrt{1+4XY(X+Y)}}{3}\, {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3}\\ 1\end{matrix} \,\middle|\, 1-X^3\right) {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3}\\ 1\end{matrix} \,\middle|\, 1-Y^3\right). \tag{11}$$

As application of Theorems 2 and 3, we outline proofs of identities of Ramanujan type for $1/\pi$ experimentally observed by Sun in [13], as well as of several new ones; this is addressed in Section 5. In Section 2 we discuss arithmetic sequences that solve the recursion (5). Our proofs of Theorems 1–3 are given in Sections 3 and 4.

## 2. Apéry-like sequences

Although our Theorems 1 and 2 are true for generic $(a, b, c)$ in (5), there are fourteen (up to normalisation) non-degenerate examples when the sequence $u_n$ satisfies (5) and takes *integral* values. These were first listed by Zagier in [14] (see also [1]), and the generating functions of all these sequences are known to have a *modular* parametrisation. Table 1 indicate the related data for the sequences; the first four examples are hypergeometric ($c = 0$), the next four are known as Legendrian examples ($a^2 - 4c = 0$), while the remaining six cases are so-called "sporadic" examples in the terminology of [14]. Note that for the hypergeometric examples, Theorem 2 reduces precisely to special cases of Theorem A investigated in [9].

We remark that our Theorem 2 for the Legendrian cases (entries (e), (h), (i), and (j) in Table 1) follows from Theorem A applied to hypergeometric instances (A)–(D) and Theorem B with choice $N = 1$; this is because the Legendrian and hypergeometric cases are related by a *binomial transform*. Moreover, entries (a) and (c) as well as (a) and (g) are also related by similar transforms and so are connected by Theorem B; for example, the first pair is related by the identity

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j}^3 = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}.$$

Table 1
Arithmetic solutions of (5).

| # in [1] | # in [14] | $(a, b, c)$ | $u_n$ |
|---|---|---|---|
| (A) | #11 | (16, 4, 0) | $\binom{2n}{n}^2$ |
| (B) | #14 | (27, 6, 0) | $\binom{2n}{n}\binom{3n}{n}$ |
| (C) | #20 | (64, 12, 0) | $\binom{2n}{n}\binom{4n}{2n}$ |
| (D) | | (432, 60, 0) | $\binom{3n}{n}\binom{6n}{3n}$ |
| (e) | #19 | $(32, 12, 16^2)$ | $16^n \sum_{k=0}^{n}(-1)^k \binom{-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{n-k}^2$ |
| (h) | #25 | $(54, 21, 27^2)$ | $27^n \sum_{k=0}^{n}(-1)^k \binom{-\frac{2}{3}}{k}\binom{-\frac{1}{3}}{n-k}^2$ |
| (i) | #26 | $(128, 52, 64^2)$ | $64^n \sum_{k=0}^{n}(-1)^k \binom{-\frac{3}{4}}{k}\binom{-\frac{1}{4}}{n-k}^2$ |
| (j) | | $(864, 372, 432^2)$ | $432^n \sum_{k=0}^{n}(-1)^k \binom{-\frac{5}{6}}{k}\binom{-\frac{1}{6}}{n-k}^2$ |
| (a) | #5, **A** | (7, 2, −8) | $\sum_{k=0}^{n}\binom{n}{k}^3$ |
| (b) | #9, **D** | (11, 3, −1) | $\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{n}$ |
| (c) | #8, **C** | (10, 3, 9) | $\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}$ |
| (d) | #10, **E** | (12, 4, 32) | $\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}$ |
| (f) | #7, **B** | (9, 3, 27) | $\sum_{k=0}^{\lfloor n/3 \rfloor}(-1)^k 3^{n-3k}\binom{n}{3k}\frac{(3k)!}{(k!)^3}$ |
| (g) | #13, **F** | (17, 6, 72) | $\sum_{k=0}^{n}\binom{n}{k}(-1)^k 8^{n-k}\sum_{j=0}^{k}\binom{k}{j}^3$ |

We also recall that if $f(x), g(x)$ are the generating functions of two sequences related by a binomial transform, then

$$g(x) = \frac{1}{1-x} f\left(\frac{x}{x-1}\right),$$

which we implicitly use in Section 5.

The following general Clausen-type formula was shown in [8].

**Proposition 1.** *For the solution $u_n$ of the recurrence equation (5),*

$$\left\{\sum_{n=0}^{\infty} u_n X^n\right\}^2 = \frac{1}{1-cX^2}\sum_{n=0}^{\infty} u_n \binom{2n}{n}\left(\frac{X(1-aX+cX^2)}{(1-cX^2)^2}\right)^n. \tag{12}$$

Because $g(X,X) = X(1-aX+cX^2)/(1-cX^2)^2$ for the function $g(X,Y)$ defined in (6) and

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n},$$

identity (12) follows from taking $Y = X$ in Theorem 1. However, Proposition 1 is the result which suggested us the form of Theorem 1.

Discussions of why the six sporadic examples are arithmetically important, as well as details of modular parametrisations of the corresponding generating functions $\sum_{n=0}^{\infty} u_n X^n$ can be found in [1,7,8,14]. Our new series for $1/\pi$ in Section 5 are consequences of the above knowledge and our Theorem 2.

## 3. Generalised Bailey's identity

We begin by proving our main theorem, which generalises Bailey's identity.

**Proof of Theorem 1.** First, define the two-variable generating function

$$H(x,y) := \sum_{n=0}^{\infty} u_n \sum_{m=0}^{n}\binom{n}{m}^2 x^m y^{n-m} \tag{13}$$

and the linear differential operator

$$\begin{aligned}\Delta_{x,y} := {} & (c(x^2+6xy+y^2) - a(x+y) + 1)\left(x\frac{\partial^2}{\partial x^2} + y\frac{\partial^2}{\partial y^2}\right) \\ & + 4xy(2c(x+y)-a)\frac{\partial^2}{\partial x\,\partial y} + (c(5x^2+14xy+y^2) - a(3x+y) + 1)\frac{\partial}{\partial x} \\ & + (c(x^2+14xy+5y^2) - a(x+3y) + 1)\frac{\partial}{\partial y} + 2(c(x+y) - b).\end{aligned} \tag{14}$$

Applying the operator (14) to (13) and rearranging the summation over monomials, we find that (after a lot of elementary algebra)

$$\Delta_{x,y}H = 2\sum_n\big((n+1)^2u_{n+1} - (an^2+an+b)u_n + cn^2u_{n-1}\big)\sum_m\binom{n}{m}^2 x^m y^{n-m} = 0 \tag{15}$$

because of the recurrence equation (5).

Secondly, the one-variable differential operator

$$\begin{aligned}D_X := {} & X(1-aX+cX^2)\frac{\partial^2}{\partial X^2} + (1-2aX+3cX^2)\frac{\partial}{\partial X} + (cX-b) \\ := {} & X^{-1}\big(\theta_X^2 - X(a\theta_X^2 + a\theta_X + b) + cX^2(\theta_X+1)^2\big)\end{aligned}$$

annihilates the series $F(X) := \sum_{n=0}^{\infty} u_n X^n$ by (8), therefore

$$(D_X + D_Y)\big(F(X)F(Y)\big) = 0. \tag{16}$$

On the other hand, we find after some work that

$$(1 - cXY)(D_X + D_Y)\left(\frac{1}{1 - cXY} H\big(g(X,Y), g(Y,X)\big)\right) = \big(\Delta_{x,y} H(x,y)\big)\Big|_{x=g(X,Y),\, y=g(Y,X)},$$

and the latter vanishes by (15). Comparing this result with (16) we conclude that both $F(X)F(Y)$ and $H(g(X,Y), g(Y,X))/(1 - cXY)$ satisfy the same second order linear partial differential equation $(D_X + D_Y)G(X,Y) = 0$. By straightforward verification, these two (analytic at the origin) solutions agree as functions of $X$ when $Y = 0$; we claim that they in fact coincide, and Theorem 1 follows.

To verify the claim, consider the function

$$G(X,Y) := F(X)F(Y) - \frac{H(g(X,Y), g(Y,X))}{1 - cXY},$$

which is analytic at the origin, is annihilated by $D_X + D_Y$, and satisfies $G(X,0) = 0$. The latter condition implies that in the power series

$$G(X,Y) = \sum_{m,k} v_{m,k} X^m Y^k = \sum_{m,k=0}^{\infty} v_{m,k} X^m Y^k$$

we have $v_{m,0} = 0$ for all $m$. Applying $D_X + D_Y$ to the series, we obtain

$$\sum_{m,k}\big((m+1)^2 v_{m+1,k} - (am^2 + am + b)v_{m,k} + cm^2 v_{m-1,k} + (k+1)^2 v_{m,k+1} - (ak^2 + ak + b)v_{m,k} + ck^2 v_{m,k-1}\big)X^m Y^k = 0. \tag{17}$$

Now, assuming that $v_{m,k} = 0$ for all $m$ and all $k \le k'$ and substituting $k = k'$ into (17), we readily see that $v_{m,k'+1} = 0$ for all $m$. It thus follows by induction on $k$ that $v_{m,k} = 0$ for all $m$ and $k$, that is, $G$ is identically zero. □

## 4. Generating functions of Legendre polynomials

Theorem 1 paves way for an easy proof of our next result.

**Proof of Theorem 2.** The application of (7) follows the lines of deducing Brafman's formula (2) from Bailey's reduction formula (4): using representation (1) for Legendre polynomials, write

$$\sum_{n=0}^{\infty} u_n P_n(x) z^n = \sum_{n=0}^{\infty} u_n \sum_{m=0}^{n} \binom{n}{m}^2 \left(\frac{z(x-1)}{2}\right)^m \left(\frac{z(x+1)}{2}\right)^{n-m}$$

and choose $X$ and $Y$ in (7) to satisfy

$$\begin{aligned} \frac{z(x-1)}{2} &= g(X,Y) = \frac{X(1 - aY + cY^2)}{(1 - cXY)^2}, \\ \frac{z(x+1)}{2} &= g(Y,X) = \frac{Y(1 - aX + cX^2)}{(1 - cXY)^2}. \end{aligned} \tag{18}$$

One easily solves (18) with respect to $x$ and $z$:

$$x=\frac{(X+Y)(1+cXY)-2aXY}{(Y-X)(1-cXY)}, \qquad z=\frac{Y-X}{1-cXY},$$

and identity (9) follows. □

By taking $N=2$ (a case considered by Brafman), $\lambda_k=(\frac{1}{2})_k^2/k!^2$, and $w=1$ in Theorem B, we obtain

**Proposition 2.**

$$\frac{1}{\rho}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k^2}{k!^2}P_{2k}\left(\frac{x-z}{\rho}\right)\left(\frac{z}{\rho}\right)^{2k}=\sum_{n=0}^{\infty}v_nP_n(x)\left(\frac{z}{4}\right)^n, \tag{19}$$

*where*

$$v_n=4^n\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\frac{(\frac{1}{2})_k^2}{k!^2}=\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}.$$

A different choice of $N=3$, $\lambda_k=(\frac{1}{3})_k(\frac{2}{3})_k/k!^2$, and $w=-1$ in Theorem B results in

**Proposition 3.**

$$\frac{1}{\rho}\sum_{k=0}^{\infty}\frac{(\frac{1}{3})_k(\frac{2}{3})_k}{k!^2}P_{3k}\left(\frac{x-z}{\rho}\right)\left(-\frac{z}{\rho}\right)^{3k}=\sum_{n=0}^{\infty}w_nP_n(x)\left(\frac{z}{3}\right)^n, \tag{20}$$

*where*

$$w_n=\sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k3^{n-3k}\binom{n}{3k}\frac{(3k)!}{k!^3}.$$

We are now in a position to prove Theorem 3.

**Proof of Theorem 3.** Write identity (19) in the form

$$\sum_{n=0}^{\infty}v_nP_n(x)z^n=\frac{1}{\rho_2}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k^2}{k!^2}P_{2k}\left(\frac{x-4z}{\rho_2}\right)\left(\frac{4z}{\rho_2}\right)^{2k}, \tag{21}$$

where $\rho_2=\rho_2(x,z):=(1-8xz+16z^2)^{1/2}$, and apply Theorem 2 to the left-hand side of (21) and the sequence $v_n=u_n^{(\mathrm{d})}$ to get

$$\left\{\sum_{n=0}^{\infty}v_nX^n\right\}\left\{\sum_{n=0}^{\infty}v_nY^n\right\}=\sum_{k=0}^{\infty}\binom{2k}{k}^2P_{2k}\left(\frac{(1-4X-4Y)(X+Y-8XY)}{(Y-X)(1-4X-4Y+32XY)}\right)\times\frac{(X-Y)^{2k}}{(1-4X-4Y+32XY)^{2k+1}}. \tag{22}$$

To each of the factors on the left-hand side we can further apply

$$\sum_{n=0}^{\infty}v_nX^n={}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\1\end{matrix}\,\middle|\,16X(1-4X)\right)$$

to reduce (22) to a hypergeometric form. Finally, making the change of variables $X \mapsto (1-X)/8$, $Y \mapsto (1-Y)/8$ we arrive at (10).

For the second identity in Theorem 3, write (20) as

$$\sum_{n=0}^{\infty} w_n P_n(x) z^n = \frac{1}{\rho_3} \sum_{k=0}^{\infty} \frac{(\frac{1}{3})_k (\frac{2}{3})_k}{k!^2} P_{3k}\left(\frac{x-3z}{\rho_3}\right)\left(-\frac{3z}{\rho_3}\right)^{3k}, \tag{23}$$

where $\rho_3 = \rho_3(x,z) := (1-6xz+9z^2)^{1/2}$. Then apply Theorem 2 to the left-hand side of (23) and the sequence $w_n = u_n^{(\mathrm{f})}$, use

$$\sum_{n=0}^{\infty} w_n X^n = \frac{1}{1-9X}\, {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3} \\ 1\end{matrix} \,\middle|\, -\frac{27X(1-9X+27X^2)}{(1-9X)^3}\right),$$

and make the change of variables $X \mapsto (X-1)/(9X)$, $Y \mapsto (Y-1)/(9Y)$ in the resulting identity. This gives us (11). □

## 5. Formulae for $1/\pi$

We briefly recall our general strategy in [9] for proving identities for $1/\pi$.

Suppose that we have an *arithmetic* sequence $u_n$ satisfying (5), and denote by

$$F(t) := \sum_{n=0}^{\infty} u_n t^n$$

the corresponding generating function and by

$$G(t) := \sum_{n=0}^{\infty} u_n n t^n = t\,\frac{\mathrm{d}F}{\mathrm{d}t}$$

its derivative. Then there exists a *modular* function $t(\tau)$ on a congruence subgroup of $SL_2(\mathbb{Z})$ such that $F(t(\tau))$ is a weight 1 modular form on the subgroup. In particular, for a quadratic irrationality $\tau_0$ with $\operatorname{Im}\tau_0 > 0$, the value $t(\tau_0)$ is an algebraic number and, under some technical conditions on $|t(\tau_0)|$, there is a *Ramanujan-type series* [11] of the form

$$aF^2(t(\tau_0)) + 2bF(t(\tau_0))G(t(\tau_0)) = \frac{c}{\pi}, \tag{24}$$

where $a$, $b$ and $c$ are certain (effectively computable) algebraic numbers.

Suppose furthermore that we have a functional identity of the form

$$\sum_{n=0}^{\infty} u_n P_{\ell n}(x) z^n = \gamma F(\alpha) F(\beta), \tag{25}$$

where $\ell \in \{1,2,3\}$, and $\alpha$, $\beta$ and $\gamma$ are algebraic functions of $x$ and $z$ (note that Theorems 2 and 3 are a source of such identities). Computing the $z$-derivative of both sides of (25) results in

$$\sum_{n=0}^{\infty} u_n n P_{\ell n}(x) z^n = \gamma_0 F(\alpha)F(\beta) + \gamma_1 F(\alpha)G(\beta) + \gamma_2 G(\alpha)F(\beta), \tag{26}$$

where $\gamma_0$, $\gamma_1$ and $\gamma_2$ are again algebraic functions of $x$ and $z$. We now take algebraic $x = x_0$ and $z = z_0$, from the convergence domain, in both equalities (25) and (26) such that the corresponding quantities $\alpha = \alpha(x_0, z_0)$ and $\beta = \beta(x_0, z_0)$ are values of the modular function $t(\tau)$ at quadratic irrationalities: $\alpha = t(\tau_0)$, and $\beta = t(\tau_0/N)$ or $1 - t(\tau_0/N)$ for an integer $N > 1$. Using the corresponding modular equation of degree $N$, we can always express $F(\beta)$ and $G(\beta)$ by means of $F(\alpha)$ and $G(\alpha)$ only:

$$F(\beta) = \mu_0 F(\alpha) \quad \text{and} \quad G(\beta) = \lambda_0 F(\alpha) + \lambda_1 G(\alpha) + \frac{\lambda_2}{\pi F(\alpha)}, \tag{27}$$

where $\mu_0, \lambda_0, \lambda_1$, and $\lambda_2$ are algebraic (with $\lambda_2 = 0$ when $\beta = t(\tau_0/N)$). Substituting relations (27) into (25) and (26), and choosing the algebraic numbers $A$ and $B$ appropriately, we find that $\sum_{n=0}^{\infty} u_n (A + Bn) P_{\ell n}(x_0) z_0^n$ is an algebraic multiple of the left-hand side of (24); in other words,

$$\sum_{n=0}^{\infty} u_n (A + Bn) P_{\ell n}(x_0) z_0^n = \frac{C}{\pi} \tag{28}$$

where $A$, $B$ and $C$ are algebraic numbers.

In practice, all the algebraic numbers involved are very cumbersome, so that the computations happen to be quite involved. Because any identity of the form (28) is uniquely determined by the choice of quadratic irrationality $\tau_0$ and degree $N > 1$, these two quantities serve as natural data for the identity. Below we provide computational details for some examples only; however we have done all the required computations for each of our illustrative identities.

### 5.1. Sun's identities

Here we show that all identities from groups IV and V in [13] can be routinely proven by the techniques we have developed.

We begin by differentiating the identities in Theorem 3. In each of (10) and (11), let $F(t)$ denote the respective ${}_2F_1$ hypergeometric function and $G(t) := t\,\mathrm{d}F/\mathrm{d}t$. Furthermore, let $\widetilde{F}(t) = F(1 - t^2)$ in (10) and $\widetilde{F}(t) = F(1 - t^3)$ in (11), as well as $\widetilde{G}(t) = G(1 - t^2)$ and $\widetilde{G}(t) = G(1 - t^3)$, respectively. Then, standard partial differentiation techniques yield the derivatives

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\frac12)_n^2}{n!^2}\, n\, P_{2n}\left(\frac{(X+Y)(1-XY)}{(X-Y)(1+XY)}\right)\left(\frac{X-Y}{1+XY}\right)^{2n} \\
&\quad = \frac{1+XY}{2(1+X+Y-XY)(1-X-Y-XY)}\big(XY(1-XY)\widetilde{F}(X)\widetilde{F}(Y) \\
&\qquad - Y^2(1+X^2)\widetilde{F}(X)\widetilde{G}(Y) - X^2(1+Y^2)\widetilde{F}(Y)\widetilde{G}(X)\big)
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\frac13)_n(\frac23)_n}{n!^2}\, n\, P_{3n}\left(\frac{X+Y-2X^2Y^2}{(X-Y)\sqrt{1+4XY(X+Y)}}\right)\left(\frac{X-Y}{\sqrt{1+4XY(X+Y)}}\right)^{3n} \\
&\quad = \frac{\sqrt{1+4XY(X+Y)}}{(1-X-Y-2XY)((1+2XY)^2+(1+X+Y)(X+Y-2XY))} \\
&\qquad \times \big(2XY(X+Y-XY(X^2+Y^2))\widetilde{F}(X)\widetilde{F}(Y) \\
&\qquad - Y^3(1+2X^2(3Y+X))\widetilde{F}(X)\widetilde{G}(Y) - X^3(1+2Y^2(3X+Y))\widetilde{F}(Y)\widetilde{G}(X)\big).
\end{aligned} \tag{30}$$

All group IV identities in [13] correspond to the form (10). The arguments of the hypergeometric functions on the right-hand side of (10) take the form $t(\tau_0)$ and $t(\tau_0/N)$ (or $1-t(\tau_0/N)$ in case (IV1)), where

$$\text{(IV1) } \tau_0 = \frac{i\sqrt{5/3}+1}{4},\ N=2;\ \text{(IV2) } \tau_0 = \frac{3i\sqrt{5}+5}{4},\ N=5;\ \text{and (IV3) } \tau_0 = \frac{i\sqrt{85}+5}{4},\ N=5.$$

It is further hypothesised in [13] that group IV contains all such series with rational parameters. Our analysis shows that the identities (IV5)–(IV18) all have $\tau_0$ of the form $\sqrt{-pq/8}$ and $N=p$, where $p$ and $q$ are odd primes and the class number of the quadratic field $\mathbb{Q}(\tau_0)$ is 4. It transpires that, whenever this is satisfied, we produce a rational series, and $p,q$ can only be taken from the seemingly exhaustive list $\{3,5,7,13,17,19\}$. Thus our analysis lends weight to this observation.

Identity (V1) in [13] is of the form (11) and may be similarly analysed and proven. In this case we in fact have $t(3\tau_0)=t(15\tau_1)$, where $t(\tau_0)=\alpha$, $t(\tau_1)=\beta$ and $\tau_0=(i\sqrt{91}+3)/6$.

The only remaining case, identity (IV4), is particularly pretty and lends itself as an example for our analysis. It states

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!^2} \frac{8n+1}{6^n} P_{2n}\left(\frac{5}{2\sqrt{6}}\right) = \frac{10\sqrt{2}}{3\pi}. \tag{31}$$

The left-hand side corresponds to the choice $X=(4\sqrt{3}+7)(5\sqrt{2}-7)$, $Y=\sqrt{2}-1$, $\tau_0=3i/(2\sqrt{2})$, and $N=3$. So $\alpha=1-X^2$ and $\beta=1-Y^2$ in the notation of (25). Using the degree 3 modular equation and multiplier for $s=1/2$, we deduce that

$$\begin{aligned} F(\alpha) &= \frac{1-\sqrt{2}+\sqrt{6}}{3} F(\beta), \\ G(\alpha) &= \frac{172\sqrt{6}+243\sqrt{3}-298\sqrt{2}-421}{3} F(\beta) \\ &\quad + \bigl(235\sqrt{6}+332\sqrt{3}-407\sqrt{2}-575\bigr) G(\beta). \end{aligned}$$

With the help of (29) and the above relations, identity (IV4) is reduced to

$$\left(\frac{20}{3}-5\sqrt{2}\right) F^2(\beta) + \left(20-\frac{40\sqrt{2}}{3}\right) F(\beta)G(\beta) = \frac{10\sqrt{2}}{3\pi}.$$

Another computation relates $F(\beta)$ and $G(\beta)$ to $F(1-\beta)$ and $G(1-\beta)$ (more details, again, are found in [9]), which enables us to apply Clausen's identity. (IV4) thus holds because Clausen's identity produces a form equivalent to the Ramanujan-type series [3, Eq. (4.1)]

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{n!^3} \bigl(3-2\sqrt{2}+(8-5\sqrt{2})n\bigr)\bigl(2\sqrt{2}-2\bigr)^{3n} = \frac{1}{\pi}.$$

The other cases can be done similarly but the algebra is formidable. For instance, in (IV7), using the notation of (10), we have

$$\begin{Bmatrix} X \\ Y \end{Bmatrix} = -171 \mp 120\sqrt{2} \pm 98\sqrt{3} \pm 76\sqrt{5} + 70\sqrt{6} + 54\sqrt{10} - 44\sqrt{15} \mp 31\sqrt{30}.$$

**Remark 2.** In [9] we produced "companion series" which involve derivatives of $P_n(x)$ in the summand. We note here that the series for $1/\pi$ in this work also admit companion series; as an example, a companion to (IV4) is

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!^2} \frac{1}{6^n} \left[ P_{2n}\left(\frac{5}{2\sqrt{6}}\right) + 8\sqrt{6}\, n\, P_{2n-1}\left(\frac{5}{2\sqrt{6}}\right) \right] = \frac{14\sqrt{2}}{3\pi}.$$

*5.2. New series for* $1/\pi$

Using (10) and the theory developed in [9] and outlined in the beginning of this section, we can produce series for $1/\pi$ at will. The following two are among the neatest:

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!^2} (2+15n)\, P_{2n}\left(\frac{3\sqrt{3}}{5}\right) \left(\frac{2\sqrt{2}}{5}\right)^{2n} = \frac{15}{\pi}, \tag{32}$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!^2}\, n\, P_{2n}\left(\frac{45}{17\sqrt{7}}\right) \left(\frac{4\sqrt{14}}{17}\right)^{2n} = \frac{68}{21\pi}. \tag{33}$$

For the first formula, $\tau_0 = i\sqrt{3}/2$ and $N = 3$, while for the second, $\tau_0 = i\sqrt{7}/2$ and $N = 7$. Note that as the these are precisely the 3rd and 7th singular values of the complete elliptic integral $K$, we may prove each series directly without resorting to a Ramanujan-type series. To demonstrate that the choice of $\tau_0$ is not confined to the singular values, here is another example corresponding to $\tau_0 = i\sqrt{3}/2$ and $N = 2$:

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!^2} (5-\sqrt{6}+20n) P_{2n}\left(\frac{17}{15}\right) \left(\frac{217-88\sqrt{6}}{25}\right)^{n} = \frac{3(4+\sqrt{6})}{2\pi}.$$

Similarly, in (11), we can take $\tau_0 = 2i/3$ and $N = 2$, therefore

$$\alpha = \frac{3\big(465+413\sqrt{3}-3\sqrt{30254\sqrt{3}-13176}\big)}{5324} \quad \text{and} \quad \beta = \frac{3(3-\sqrt{3})}{4}.$$

The algebraic numbers involved in (11) simplify remarkably, and aided by (30), we produce the new series

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{n!^2} (1+9n) P_{3n}\left(\frac{4}{\sqrt{10}}\right) \left(\frac{1}{3\sqrt{10}}\right)^{3n} = \frac{\sqrt{15+10\sqrt{3}}}{\pi\sqrt{2}}, \tag{34}$$

whose truth is equivalent to the following series for $1/\pi$,

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n(\frac{1}{2})_n(\frac{2}{3})_n}{n!^3} \big(1+(5+\sqrt{3})n\big) \left(\frac{3(7\sqrt{3}-12)}{2}\right)^{n} = \frac{2+\sqrt{3}}{\pi}.$$

Finally, note that each term in the sums of (32)–(34) is *rational*.

*5.3. New series for* $1/\pi$ *with Apéry-like sequences*

As one of the consequences of Theorem 2, we exhibit here some new series of the form

$$\sum_{n=0}^{\infty} u_n (A + Bn) P_n(x_0) z_0^n = \frac{C}{\pi}, \tag{35}$$

where $u_n$ satisfies (5). As such series are not the main goal of this project but rather curiosities, we will only list the relevant $\tau_0$, $N$ and the final result.

We start with entry (a) of Table 1. Denoting the sequence by $u_n^{(\mathrm{a})}$ (and other entries in the table are denoted similarly), we have the generating function

$$\sum_{n=0}^{\infty} u_n^{(\mathrm{a})} x^n = \frac{1}{1-2x}\, {}_2F_1\left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix} \,\middle|\, \frac{27x^2}{(1-2x)^3}\right).$$

Therefore, combined with Theorem 2, we can analyse (35) for $u_n^{(\mathrm{a})}$ as we did in [9]. Indeed, taking $\tau_0 = 2i\sqrt{2/3}$ and $N = 2$, we have

$$\sum_{n=0}^{\infty} u_n^{(\mathrm{a})} \big(7 - 2\sqrt{3} + 18n\big) P_n\left(\frac{1+\sqrt{3}}{\sqrt{6}}\right) \left(\frac{2-\sqrt{3}}{2\sqrt{6}}\right)^n = \frac{27 + 11\sqrt{3}}{\pi\sqrt{2}}.$$

This is in fact equivalent to the classical series

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n (\frac{1}{2})_n (\frac{2}{3})_n}{n!^3} \frac{1+6n}{2^n} = \frac{3\sqrt{3}}{\pi}.$$

Next, for entry (b), there is no simple hypergeometric generating function. Nevertheless, using the results from [7] we pick $\tau_0 = 2i\sqrt{2/5}$, $N = 2$, and obtain

$$\sum_{n=0}^{\infty} u_n^{(\mathrm{b})} \big(16 - 5\sqrt{10} + 60n\big) P_n\left(\frac{5\sqrt{2} + 17\sqrt{5}}{45}\right) \left(\frac{5\sqrt{2} - 3\sqrt{5}}{5}\right)^n = \frac{135\sqrt{2} + 81\sqrt{5}}{\pi\sqrt{2}}.$$

For entry (c), the generating function of $u_n^{(\mathrm{c})}$ is

$$\sum_{n=0}^{\infty} u_n^{(\mathrm{c})} x^n = \frac{1}{1+3x}\, {}_2F_1\left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix} \,\middle|\, \frac{27x(1-x)^2}{(1+3x)^3}\right).$$

The sequence gives, incidentally, the $(2n)$th moment of the distance to the origin in a uniform 3-step walk in the plane [4]. Again, Theorem 2 applies; as an example, for $\tau_0 = i$, $N = 3$, and using the same $1/\pi$ series as for (34), we have

$$\sum_{n=0}^{\infty} u_n^{(\mathrm{c})} \big(7 - 3\sqrt{3} + 22n\big) P_n\left(\frac{\sqrt{14\sqrt{3} - 15}}{3}\right) \left(\frac{\sqrt{2\sqrt{3} - 3}}{9}\right)^n = \frac{9(9 + 4\sqrt{3})}{2\pi}.$$

For entry (d), we can take $\tau_0 = i\sqrt{3}/2$, $N = 3$, and produce the new series

$$\sum_{n=0}^{\infty} u_n^{(\mathrm{d})} \big(4 - 2\sqrt{6} + 15n\big) P_n\left(\frac{24 - \sqrt{6}}{15\sqrt{2}}\right) \left(\frac{4 - \sqrt{6}}{10\sqrt{3}}\right)^n = \frac{6(7 + 3\sqrt{6})}{\pi}.$$

For entry (f), we found after some searching that by using $\tau_0 = 1 + i\sqrt{7}/3$ and $N = 2$,

$$\sum_{n=0}^{\infty} u_n^{(\mathrm{f})} \big(7 - \sqrt{21} + 14n\big) P_n\left(\frac{\sqrt{21}}{5}\right) \left(\frac{7\sqrt{21} - 27}{90}\right)^n = \frac{5\sqrt{7}\sqrt{7\sqrt{21} + 27}}{4\pi\sqrt{2}}.$$

As for the last sporadic example (g), we take $\tau_0 = 2i/\sqrt{3}$ and $N = 2$ to generate the compact-looking series

$$\sum_{n=0}^{\infty} u_n^{(g)} n P_n\left(\frac{5}{3\sqrt{3}}\right)\left(\frac{1}{6\sqrt{3}}\right)^n = \frac{9\sqrt{3}}{2\pi}.$$

As stated earlier, the Legendrian entries are binomial transforms of the hypergeometric entries in Table 1, therefore the $1/\pi$ series for them are comparatively easy to find; we list one example for each entry below:

$$\sum_{n=0}^{\infty} u_n^{(e)}(8n-1) P_n\left(\frac{26}{15\sqrt{3}}\right)\left(\frac{\sqrt{3}}{80}\right)^n = \frac{15\sqrt{3}}{2\pi\sqrt{2}},$$

$$\sum_{n=0}^{\infty} u_n^{(h)}(125n+42) P_n\left(\frac{463}{182\sqrt{6}}\right)\left(-\frac{\sqrt{3}}{90\sqrt{2}}\right)^n = \frac{546\sqrt{3}}{25\pi},$$

$$\sum_{n=0}^{\infty} u_n^{(i)}(363n+109) P_n\left(\frac{746}{425\sqrt{3}}\right)\left(-\frac{17}{2048\sqrt{3}}\right)^n = \frac{7600\sqrt{2}}{33\pi\sqrt{11}},$$

and

$$\sum_{n=0}^{\infty} u_n^{(j)}\left(\frac{2n+1}{2457} - \frac{139}{4875\sqrt{17^3}}\right) P_n\left(\frac{2456}{2457}\right)\left(\frac{4081-57\sqrt{17^3}}{359424}\right)^n = \frac{\sqrt{7}\sqrt{4081\sqrt{17}+16473}}{17^3\cdot 250\pi\sqrt{2}}.$$

The corresponding data for the identities are as follows: $\tau_0 = i\sqrt{3}$, $N = 3$; $\tau_0 = i\sqrt{2}$, $N = 2$; $\tau_0 = i\sqrt{3}$, $N = 2$; and $\tau_0 = 1 + i\sqrt{7}$, $N = 2$, respectively.

## 6. Concluding remarks

We briefly outline the genesis of Theorems 1–3. While working on the project [9], it became clear that generating functions of type (10) and (11) should exist. Our confidence was boosted by examples like (31) in [13]. We learned, after coming across Theorem B, that generating functions of $P_{\ell n}(x)$ could be obtained by generating functions of $P_n(x)$ multiplied by an arithmetic sequence. We then studied Brafman's proof of Theorem A using Bailey's identity (4), at which point it dawned on us that a more general form of the identity was needed to encompass not just hypergeometric, but arithmetic sequences. Inspired by the form of (12), we empirically discovered Theorem 1 which meets this goal and also contains (12) as a special case. Therefore, the significance of "arithmeticity" has been a major driving force towards Theorem 3.

In conclusion, we expect that our Theorem 1 can be generalised even further to include the general form of Bailey's transform [2, Section 9.6] and Clausen's identity [10], both of which depend on more than one parameter $s$. This could possibly imply new generating functions of Jacobi and other orthogonal polynomials. There would be, however, no arithmetic significance in such generalisations, as the sequences $u_n$ involved would no longer admit binomial expressions.

Our main motivation for the present paper is the remarkable work of Fred Brafman on generating functions of Legendre polynomials, and more generally, orthogonal polynomials.

Fred Brafman was born on July 10, 1923 in Cincinnati, Ohio. He attended Lebanon High School (Ohio) from 1936 to 1940, then spent a year at Greenbrier Military School (Jr. College) before enrolling in the Engineering School at the University of Michigan in September 1941. He received a Bachelor of Science in Engineering (in Electrical Engineering) degree in 1943 and then a Bachelor of Science in Mathematics degree from Michigan in 1946. Brafman entered the graduate programme in Mathematics in the fall of 1946 and compiled an outstanding academic record. He received an AM degree in 1947 and a Ph.D. in February 1951 from the University of Michigan under the supervision of E.D. Rainville. After completion of his Ph.D., he was hired by the Wayne State University, by the Southern Illinois University, and then by the University of Oklahoma. Brafman had an invitation to visit the Institute for Advanced Studies (Princeton),[1] which was not materialised because of his ultimate death on February 4, 1959 in Oklahoma. He solely authored ten mathematical papers, all about special (orthogonal) polynomials; the works [5,6] are his first and last publications, respectively.

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