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Weak bimonads and weak Hopf monads

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ABSTRACT

We define a *weak bimonad* as a monad T on a monoidal category $\mathscr{M}$ with the property that the Eilenberg–Moore category $\mathscr{M}^T$ is monoidal and the forgetful functor $\mathscr{M}^T \to \mathscr{M}$ is separable Frobenius. Whenever $\mathscr{M}$ is also Cauchy complete, a simple set of axioms is provided, that characterizes the monoidal structure of $\mathscr{M}^T$ as a weak lifting of the monoidal structure of $\mathscr{M}$. The relation to bimonads, and the relation to weak bimonoids in a braided monoidal category are revealed. We also discuss antipodes, obtaining the notion of weak Hopf monad.

Introduction

Bialgebras (say, over a field) have several equivalent characterizations. One of the most elegant is due to Pareigis, who proved that an algebra A over a field K is a bialgebra if and only if the category of (left or right) A-modules is monoidal and the forgetful functor from the category of A-modules to the category of K-vector spaces is strict monoidal. This fact extends to bialgebras in any braided monoidal category [12].

Pareigis' characterization of a bialgebra was the starting point of Moerdijk's generalization in [14] of bialgebras to monoidal categories possibly without a braiding. He defined a *bimonad* (originally

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called a Hopf monad) as a monad T on a monoidal category $\mathscr{M}$, such that the Eilenberg–Moore category $\mathscr{M}^T$ of T-algebras is monoidal and the forgetful functor $\mathscr{M}^T \to \mathscr{M}$ is strict monoidal. That is, the monoidal structure of $\mathscr{M}$ lifts to $\mathscr{M}^T$. Because liftings of functors (respectively, of natural transformations) are described by 1-cells (respectively, by 2-cells) in the 2-category Mnd(Cat) of monads (in the notation of [18]), Moerdijk's definition says that a monad is a bimonad if and only if the functor induced by the monoidal unit of $\mathscr{M}$, from the terminal category to $\mathscr{M}$, and the functor provided by the monoidal product of $\mathscr{M}$, from $\mathscr{M} \times \mathscr{M}$ to $\mathscr{M}$, both admit the structure of a 1-cell in Mnd(Cat), and the coherence natural isomorphisms in $\mathscr{M}$ are 2-cells in Mnd(Cat). In [11], McCrudden showed that a bimonad is the same as an opmonoidal monad, that is, a monad in the 2-category of monoidal categories, opmonoidal functors and opmonoidal natural transformations. Equivalently, bimonads are the same as monoids in a multicategory of monads on a monoidal category.

Pareigis' characterization of a bialgebra was generalized to a *weak bialgebra* [15,4] by Szlachányi in [21]. He proved that an algebra A over a field K is a weak bialgebra if and only if the category of (left or right) A-modules is monoidal and the forgetful functor from the category of A-modules to the category of K-vector spaces obeys the so-called *separable Frobenius* condition. The latter means that the forgetful functor admits both a monoidal and an opmonoidal structure that satisfy some compatibility relations: see Definition 1.1. These (op)monoidal structures are no longer strict. In particular, the monoidal unit of the category of A-modules is not K as a vector space but a non-trivial retract of A. Also, the monoidal product of two A-modules is not their K-module tensor product but a linear retract of it.

Weak bialgebras can be defined in any braided monoidal category, see [1] and [16], as objects possessing both a monoid and a comonoid structure, subject to compatibility axioms that generalize those in [15] and [4] in the case of a symmetric monoidal category of vector spaces. The resulting category of modules was investigated in [16].

The aim of this paper is to generalize weak bialgebras to monoidal categories possibly without a braiding. Inspired by Szlachányi's characterization of a weak bialgebra, we define a *weak bimonad* as a monad T on a monoidal category $\mathscr{M}$, with extra structure making $\mathscr{M}^T$ monoidal and the forgetful functor $\mathscr{M}^T \to \mathscr{M}$ separable Frobenius.

For a weak bimonad T, the forgetful functor $\mathscr{M}^T \to \mathscr{M}$ is no longer strict monoidal, hence the monoidal structure of the domain category $\mathscr{M}$ does not lift to $\mathscr{M}^T$, and so the monoidal unit and the monoidal product of $\mathscr{M}$ are no longer 1-cells in Mnd(Cat). However, the notion of lifting $\overline{F}: \mathscr{M}^T \to \mathscr{M}'^{T'}$ of a functor $F: \mathscr{M} \to \mathscr{M}'$ was weakened in [3] by replacing commutativity of the diagram of functors

$$
\begin{tikzcd}
\mathscr{M}^T \arrow[r, "\overline{F}"] \arrow[d, "U"'] & \mathscr{M}'^{T'} \arrow[d, "U'"] \\
\mathscr{M} \arrow[r, "F"] & \mathscr{M}'
\end{tikzcd}
$$

by the existence of a split natural monomorphism $i: U'\overline{F} \to FU$. A weak lifting of a natural transformation is defined as a natural transformation between the lifted functors that commutes with the natural monomorphisms i in the evident sense. Weak liftings of functors and of natural transformations in a locally Cauchy complete 2-subcategory of Cat, are related to 1-cells and 2-cells in a 2-category Mnd^{i}(Cat) in [3], extending Mnd(Cat).

In Section 1 we give an interpretation of the axioms of a weak bimonad (on a Cauchy complete monoidal category), similar to the interpretation of a bimonad in [14]. While for a bimonad T the monoidal structure of $\mathscr{M}^T$ is given by lifting of the monoidal structure in the domain category $\mathscr{M}$, for a weak bimonad T the monoidal product in $\mathscr{M}^T$ is a weak lifting of the monoidal product in $\mathscr{M}$, the monoidal unit is a weak lifting of the functor $1 \to \mathscr{M} \xrightarrow{T} \mathscr{M}$, the associativity constraint is a weak lifting of the associativity constraint in $\mathscr{M}$ and the unit constraints are weak liftings of certain morphisms in $\mathscr{M}$ constructed from the other data.

By results in [16], a weak bimonoid in a braided monoidal category can be described as a quantum category over a separable Frobenius base monoid. Extending this result in Section 2, we establish an equivalence between the category of weak bimonads on a Cauchy complete monoidal category $\mathscr{M}$ and the category of bimonads on bimodule categories over separable Frobenius monoids in $\mathscr{M}$.

In Section 3 we show that weak bimonoids in a braided monoidal category (cf. [16,1]) induce weak bimonads. In certain braided monoidal categories the converse can also be proved: if a monoid induces a weak bimonad then it admits the structure of a weak bimonoid.

In Section 4, using the result in Section 2 that any weak bimonad (on a Cauchy complete monoidal category) can be regarded as a bimonad (on another monoidal category), we define a *weak right Hopf monad* to be a weak bimonad such that the associated bimonad is a right Hopf monad in the sense of [5] and [7]; there is a companion result involving weak left Hopf monads and left Hopf monads. A weak bimonoid in a Cauchy complete braided monoidal category is shown to induce a weak right Hopf monad by tensoring with it on the right if and only if it is a weak Hopf monoid in the sense of [1] and [16]; once again there is a companion result with left in place of right.

Notation and conventions. The monoidal categories in this paper are not necessarily strict but in order to simplify our expressions, we omit explicit mention of their coherence isomorphisms wherever possible.

Recall that, for an opmonoidal functor $F:(\mathscr{N},\boxtimes,R)\to(\mathscr{M},\otimes,K)$ with opmonoidal structure $i_{X,Y}:F(X\boxtimes Y)\to FX\otimes FY$ and $i_0:FR\to K$, the diagram

$$\begin{tikzcd}
F(X\boxtimes Y\boxtimes Z) \arrow[r, "i_{X\boxtimes Y,Z}"] \arrow[d, "i_{X,Y\boxtimes Z}"'] & F(X\boxtimes Y)\boxtimes FZ \arrow[d, "i_{X,Y}\boxtimes FZ"] \\
FX\boxtimes F(Y\boxtimes Z) \arrow[r, "FX\boxtimes i_{Y,Z}"'] & FX\boxtimes FY\boxtimes FZ
\end{tikzcd}$$

commutes. We sometimes write $i^{(3)}_{X,Y,Z}$ for the common composite, and we use an analogous notation for monoidal functors.

We say that a category is *Cauchy complete* provided that idempotent morphisms in it split.

1. Weak bimonads and their Eilenberg–Moore category of algebras

The definition of weak bimonad is based on the notion of separable Frobenius functor introduced in [21]:

Definition 1.1. A functor F from a monoidal category $(\mathscr{N},\boxtimes,R)$ to a monoidal category $(\mathscr{M},\otimes,K)$ is said to be *separable Frobenius* when it is equipped with a monoidal structure $p_{X,Y}:FX\otimes FY\to F(X\boxtimes Y)$, $p_0:K\to FR$ and an opmonoidal structure $i_{X,Y}:F(X\boxtimes Y)\to FX\otimes FY$, $i_0:FR\to K$ such that, for all objects X, Y, Z in $\mathscr{N}$, the following diagrams commute:

$$\begin{tikzcd}
FX\otimes F(Y\boxtimes Z) \arrow[r, "FX\otimes i_{Y,Z}"] \arrow[d, "p_{X,Y\boxtimes Z}"'] & FX\otimes FY\otimes FZ \arrow[d, "p_{X,Y}\otimes FZ"] \\
F(X\boxtimes Y\boxtimes Z) \arrow[r, "i_{X\boxtimes Y,Z}"'] & F(X\boxtimes Y)\otimes FZ
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X\boxtimes Y)\otimes FZ \arrow[r, "i_{X,Y}\otimes FZ"] \arrow[d, "p_{X\boxtimes Y,Z}"'] & FX\otimes FY\otimes FZ \arrow[d, "FX\otimes p_{Y,Z}"] \\
F(X\boxtimes Y\boxtimes Z) \arrow[r, "i_{X,Y\boxtimes Z}"'] & FX\otimes F(Y\boxtimes Z)
\end{tikzcd}$$

$$\begin{tikzcd}
& FX\otimes FY \arrow[dr, "p_{X,Y}"] & \\
F(X\boxtimes Y) \arrow[ur, "i_{X,Y}"] \arrow[rr, equal] & & F(X\boxtimes Y)
\end{tikzcd}$$

Example 1.2. (1) Strong monoidal functors are clearly separable Frobenius.

(2) The composite of separable Frobenius functors is separable Frobenius, cf. [8].

(3) In a monoidal category $(\mathscr{M}, \otimes, K)$ possessing (appropriate) coequalizers preserved by $\otimes$, one may consider the monoidal category ${}_R\mathscr{M}_R$ of bimodules over a monoid R in $\mathscr{M}$. The monoidal product is provided by the R-module tensor product and the monoidal unit is R. Justifying the terminology, the forgetful functor ${}_R\mathscr{M}_R \to \mathscr{M}$ is separable Frobenius if and only if R is a separable Frobenius monoid; that is, a Frobenius monoid in the sense of [19] such that, in addition, composing its comultiplication $R \to R \otimes R$ with its multiplication $R \otimes R \to R$ yields the identity morphism R.

Definition 1.3. A *weak bimonad* on a monoidal category $(\mathscr{M}, \otimes, K)$ is a monad (T, m, u) on $\mathscr{M}$ equipped with a monoidal structure on the Eilenberg–Moore category $\mathscr{M}^T$ and a separable Frobenius structure on the forgetful functor $\mathscr{M}^T \to \mathscr{M}$.

The main aim of this section is to find an equivalent formulation of Definition 1.3 – in the spirit of the descriptions of bimonads in [14] and [11] (there called Hopf monads).

If a monad T possesses a monoidal Eilenberg–Moore category $(\mathscr{M}^T, \boxtimes, (R, r))$ then, for any T-algebras (A, a) and (B, b), there is a T-algebra $(A, a) \boxtimes (B, b)$ that we denote by $(A \,\square\, B, a \,\square\, b)$. (Note that by definition $a \,\square\, b$ is a morphism $T(A \,\square\, B) \to A \,\square\, B$ in $\mathscr{M}$, while $a \boxtimes b$ is a morphism $(TA, m_A) \boxtimes (TB, m_B) \to (A, a) \boxtimes (B, b)$ in $\mathscr{M}^T$; that is, a morphism $TA \,\square\, TB \to A \,\square\, B$ in $\mathscr{M}$. Note also that $A \,\square\, B$ depends not just on A and B but on the algebras (A, a) and (B, b).)

In order to get started, we need the following basic observation:

Proposition 1.4. *Consider a monad (T, m, u) on a monoidal category $(\mathscr{M}, \otimes, K)$ equipped with a monoidal Eilenberg–Moore category $(\mathscr{M}^T, \boxtimes, (R, r))$. If the forgetful functor $U : \mathscr{M}^T \to \mathscr{M}$ admits both a monoidal structure (p, p_0) and an opmonoidal structure (i, i_0) then T is opmonoidal, with τ_0 and $\tau_{X,Y}$ given, respectively, by the composite morphisms*

$$TK \xrightarrow{Tp_0} TR \xrightarrow{r} R \xrightarrow{i_0} K \quad \text{and} \tag{1.1}$$

$$T(X \otimes Y) \xrightarrow{T(u_X \otimes u_Y)} T(TX \otimes TY) \xrightarrow{Tp_{TX,TY}} T(TX \,\square\, TY) \xrightarrow{m_X \square m_Y} TX \,\square\, TY \xrightarrow{i_{TX,TY}} TX \otimes TY. \tag{1.2}$$

Proof. Since U is monoidal, its left adjoint F is opmonoidal. Since U is also opmonoidal, so is $T = UF$. The explicit form of the structure morphisms (1.1) and (1.2) is immediate. $\square$

At this point we can now state one characterization of weak bimonads:

Theorem 1.5. *Let $T = (T, m, u)$ be a monad on a monoidal category $(\mathscr{M}, \otimes, K)$ in which idempotents split. To give T the structure of a weak bimonad is equivalently to give the endofunctor T the structure of an opmonoidal functor (T, τ, τ_0) in such a way that the following conditions hold:*

$$\begin{tikzcd}
T^2(X \otimes TK) \arrow[r, "T\tau_{X,TK}"] & T(TX \otimes T^2K) \arrow[r, "T(TX\otimes m_K)"] & T(TX \otimes TK) \arrow[r, "T(TX\otimes\tau_0)"] & T^2X \arrow[d, "m_X"] \\
T(X \otimes TK) \arrow[u, "Tu_{X\otimes TK}"] \arrow[r, "\tau_{X,TK}"'] & TX \otimes T^2K \arrow[r, "TX\otimes m_K"'] & TX \otimes TK \arrow[r, "TX\otimes\tau_0"'] & TX
\end{tikzcd} \tag{1.3}$$

$$\begin{tikzcd}
T^2(TK\otimes X) \arrow[r, "T\tau_{TK,X}"] & T(T^2K\otimes TX) \arrow[r, "T(m_K\otimes TX)"] & T(TK\otimes TX) \arrow[r, "T(\tau_0\otimes TX)"] & T^2X \arrow[d, "m_X"] \\
T(TK\otimes X) \arrow[u, "Tu_{TK\otimes X}"] \arrow[r, "\tau_{TK,X}"'] & T^2K\otimes TX \arrow[r, "m_K\otimes TX"'] & TK\otimes TX \arrow[r, "\tau_0\otimes TX"'] & TX
\end{tikzcd} \tag{1.4}$$

$$\begin{tikzcd}
X\otimes T(Y\otimes Z) \arrow[r, "X\otimes\tau_{Y,Z}"] & X\otimes TY\otimes TZ \arrow[r, "u_{X\otimes TY}\otimes TZ"] & T(X\otimes TY)\otimes TZ \arrow[r, "\tau_{X,TY}\otimes TZ"] & TX\otimes T^2Y\otimes TZ \arrow[d, "TX\otimes m_Y\otimes TZ"] \\
X\otimes Y\otimes Z \arrow[u, "X\otimes u_{Y\otimes Z}"] \arrow[r, "u_{X\otimes Y\otimes Z}"'] & T(X\otimes Y\otimes Z) \arrow[r, "\tau_{X\otimes Y,Z}"'] & T(X\otimes Y)\otimes TZ \arrow[r, "\tau_{X,Y}\otimes TZ"'] & TX\otimes TY\otimes TZ
\end{tikzcd} \tag{1.5}$$

$$\begin{tikzcd}
T(X\otimes Y)\otimes Z \arrow[r, "\tau_{X,Y}\otimes Z"] & TX\otimes TY\otimes Z \arrow[r, "TX\otimes u_{TY\otimes Z}"] & TX\otimes T(TY\otimes Z) \arrow[r, "TX\otimes\tau_{TY,Z}"] & TX\otimes T^2Y\otimes TZ \arrow[d, "TX\otimes m_Y\otimes TZ"] \\
X\otimes Y\otimes Z \arrow[u, "u_{X\otimes Y}\otimes Z"] \arrow[r, "u_{X\otimes Y\otimes Z}"'] & T(X\otimes Y\otimes Z) \arrow[r, "\tau_{X\otimes Y,Z}"'] & T(X\otimes Y)\otimes TZ \arrow[r, "\tau_{X,Y}\otimes TZ"'] & TX\otimes TY\otimes TZ
\end{tikzcd} \tag{1.6}$$

$$\begin{tikzcd}
T^2(X\otimes Y) \arrow[r, "T\tau_{X,Y}"] \arrow[d, "m_{X\otimes Y}"'] & T(TX\otimes TY) \arrow[r, "\tau_{TX,TY}"] & T^2X\otimes T^2Y \arrow[d, "m_X\otimes m_Y"] \\
T(X\otimes Y) \arrow[rr, "\tau_{X,Y}"'] & & TX\otimes TY
\end{tikzcd} \tag{1.7}$$

We shall spend the rest of the section proving this theorem as well as formulating a further characterization in terms of weak lifting. One half of the theorem we prove immediately:

Proposition 1.6. *For any weak bimonad, Eqs.* (1.3)–(1.7) *hold when the endofunctor is given the opmonoidal structure of Proposition* 1.4.

Proof. (R, r) is the monoidal unit in $\mathscr{M}^T$ and the coherence natural isomorphisms in $\mathscr{M}^T$ are T-algebra morphisms. For any morphisms $f:(A,a)\to(A',a')$ and $g:(B,b)\to(B',b')$ of T-algebras, $f\boxtimes g$ is a morphism of T-algebras, so that

$$\begin{tikzcd}
T(A\,\square\, B) \arrow[r, "T(f\boxtimes g)"] \arrow[d, "a\square b"'] & T(A'\,\square\, B') \arrow[d, "a'\square b'"] \\
A\,\square\, B \arrow[r, "f\boxtimes g"'] & A'\,\square\, B'
\end{tikzcd} \tag{1.8}$$

commutes. By (1.2) and unitality of the T-action $mX \,\square\, mY$, the diagram

$$\begin{tikzcd}
X \otimes Y \arrow[r, "u_X \otimes u_Y"] \arrow[d, "u_{X\otimes Y}"'] & TX \otimes TY \arrow[r, "p_{TX,TY}"] & TX \,\square\, TY \arrow[d, "i_{TX,TY}"] \\
T(X \otimes Y) \arrow[rr, "\tau_{X,Y}"'] & & TX \otimes TY
\end{tikzcd} \tag{1.9}$$

commutes, for any objects X, Y of $\mathscr{M}$. Hence a straightforward computation, using these facts together with the unitality of m and with the opmonoidality of (U, i, i_0), shows that both routes around (1.3) are equal to

$$T(X \otimes TK) \xrightarrow{T(X\otimes Tp_0)} T(X \otimes TR) \xrightarrow{T(X\otimes r)} T(X \otimes R) \xrightarrow{T(u_X\otimes R)} T(TX \otimes R) \xrightarrow{Tp_{TX,R}} T^2X \xrightarrow{m_X} TX.$$

Equality (1.4) is proved symmetrically. In view of (1.9), the bottom path of (1.5) is equal to

$$X \otimes Y \otimes Z \xrightarrow{u_X\otimes u_Y\otimes u_Z} TX \otimes TY \otimes TZ \xrightarrow{p^{(3)}_{TX,TY,TZ}} TX \,\square\, TY \,\square\, TZ \xrightarrow{i^{(3)}_{TX,TY,TZ}} TX \otimes TY \otimes TZ.$$

This expression is checked to be equal also to the upper path of (1.5), by applying (1.9) repeatedly, and using monoidality of (U, p, p_0) and the first property in Definition 1.1 of the separable Frobenius functor U. Equality (1.6) is proved symmetrically, using the second property in Definition 1.1 of the separable Frobenius functor U instead of the first one. Finally, by (1.2), by naturality of i and p, by (1.8), and by unitality of m, we deduce that $(m_X \otimes m_Y) \circ \tau_{TX,TY} = i_{TX,TY} \circ (m_X \,\square\, m_Y) \circ Tp_{TX,TY}$. Hence (1.7) follows by the third property in Definition 1.1 of the separable Frobenius functor U and associativity of the action $m_X \,\square\, m_Y : T(TX \,\square\, TY) \to TX \,\square\, TY$. $\square$

Lemma 1.7. *Let (T, τ, τ_0) be an opmonoidal endofunctor of a monoidal category $(\mathscr{M}, \otimes, K)$, and (T, m, u) a monad on $\mathscr{M}$, and suppose that Eq. (1.3) holds. Then the morphism*

$$\sqcap := \big(TK \xrightarrow{u_{TK}} T^2K \xrightarrow{\tau_{K,TK}} TK \otimes T^2K \xrightarrow{TK\otimes m_K} TK \otimes TK \xrightarrow{TK\otimes\tau_0} TK\big)$$

is idempotent, and the diagram

$$\begin{tikzcd}
T^2K \arrow[r, "T\sqcap"] \arrow[d, "m_K"'] & T^2K \arrow[r, "m_K"] & TK \arrow[d, "\sqcap"] \\
TK \arrow[rr, "\sqcap"'] & & TK
\end{tikzcd} \tag{1.10}$$

commutes.

Proof. In the diagram

$$
\begin{tikzcd}
TK \arrow[r, "\sqcap"] \arrow[d, "u_{TK}"'] & TK \arrow[rr, "u_{TK}"] & & T^2K \arrow[d, "\tau_{K,TK}"] \\
T^2K \arrow[r, "\tau_{K,TK}"] \arrow[d, "\tau_{K,TK}"'] & TK \otimes T^2K \arrow[rr, "TK\otimes T\sqcap"] \arrow[d, "TK\otimes\tau_{K,TK}"] & & TK \otimes T^2K \arrow[d, "TK\otimes m_K"] \\
TK \otimes T^2K \arrow[r, "\tau_{K,K}\otimes T^2K"] \arrow[dr, equal] & TK \otimes TK \otimes T^2K \arrow[r, "TK\otimes TK\otimes m_K"] \arrow[d, "TK\otimes\tau_0\otimes T^2K"] & TK \otimes TK \otimes TK \arrow[r, "TK\otimes TK\otimes\tau_0"] \arrow[d, "TK\otimes\tau_0\otimes TK"] & TK \otimes TK \arrow[d, "TK\otimes\tau_0"] \\
& TK \otimes T^2K \arrow[r, "TK\otimes m_K"'] & TK \otimes TK \arrow[r, "TK\otimes\tau_0"'] & TK
\end{tikzcd}
$$

the squares at the bottom and the region at the top commute by naturality, the triangle and the square above it commute since (T, τ, τ_0) is opmonoidal, and the remaining region is seen to commute by taking $X = K$ in Eq. (1.3) and then tensoring on the left by TK. The composite of the top path is $\sqcap\sqcap$, and that of the bottom path is $\sqcap$.

As for commutativity of the displayed diagram, in the following diagram

$$
\begin{tikzcd}
TK \arrow[d, "u_{TK}"'] & T^2K \arrow[l, "m_K"'] \arrow[r, "T\sqcap"] \arrow[d, "u_{T^2K}"] & T^2K \arrow[dr, "m_K"] & \\
T^2K \arrow[d, "\tau_{K,TK}"'] & T^3K \arrow[d, "\tau_{K,T^2K}"] & & TK \arrow[d, "u_{TK}"] \\
TK \otimes T^2K \arrow[ddd, equal] & TK \otimes T^3K \arrow[l, "TK\otimes Tm_K"'] \arrow[r, "TK\otimes T^2\sqcap"] \arrow[d, "TK\otimes m_{TK}"] & TK \otimes T^3K \arrow[d, "TK\otimes m_{TK}"] \arrow[dr, "TK\otimes Tm_K"] & T^2K \arrow[d, "\tau_{K,TK}"] \\
& TK \otimes T^2K \arrow[r, "TK\otimes T\sqcap"] \arrow[d, "TK\otimes\tau_{K,TK}"] \arrow[dd, equal, bend right=60] & TK \otimes T^2K \arrow[dr, "TK\otimes m_K"] & TK \otimes T^2K \arrow[d, "TK\otimes m_K"] \\
& TK \otimes TK \otimes T^2K \arrow[r, "TK\otimes TK\otimes m_K"] \arrow[d, "TK\otimes\tau_0\otimes T^2K"] & TK \otimes TK \otimes TK \arrow[r, "TK\otimes TK\otimes\tau_0"] \arrow[d, "TK\otimes\tau_0\otimes TK"] & TK \otimes TK \arrow[d, "TK\otimes\tau_0"] \\
TK \otimes T^2K \arrow[rr, bend right=40, "TK\otimes m_K"'] & TK \otimes T^2K \arrow[r, "TK\otimes m_K"] & TK \otimes TK \arrow[r, "TK\otimes\tau_0"] & TK
\end{tikzcd}
$$

the large regions at the top commute by naturality, the pentagonal region in the middle commutes by the case $X = K$ of Eq. (1.3), and remaining regions commute by naturality, associativity of m, and the opmonoidal functor axioms. $\square$

Lemma 1.8. *Consider a weak bimonad (T, m, u) on a monoidal category $(\mathscr{M}, \otimes, K)$, with opmonoidal structure τ_0 in* (1.1) *and τ in* (1.2)*. Then the idempotent morphism*

$$\sqcap := \left(TK \xrightarrow{u_{TK}} T^2K \xrightarrow{\tau_{K,TK}} TK \otimes T^2K \xrightarrow{TK\otimes m_K} TK \otimes TK \xrightarrow{TK\otimes\tau_0} TK\right)$$

factorizes through an epimorphism $TK \to R$ and a section of it, where R denotes the object in $\mathscr{M}$ underlying the monoidal unit (R, r) of $\mathscr{M}^T$.

Proof. The desired epimorphism is constructed as

$$P := (TK \xrightarrow{Tp_0} TR \xrightarrow{r} R) \tag{1.11}$$

with a section

$$I := (R \xrightarrow{u_R} TR \xrightarrow{\tau_{K,R}} TK \otimes TR \xrightarrow{TK\otimes r} TK \otimes R \xrightarrow{TK\otimes i_0} TK), \tag{1.12}$$

where (p, p_0) denotes the monoidal structure and (i, i_0) denotes the opmonoidal structure of the forgetful functor $U : \mathscr{M}^T \to \mathscr{M}$. $\square$

Lemma 1.9. *For a weak bimonad T and any T-algebras (A, a) and (B, b), the idempotent morphism*

$$E_{A,B} := \left(A \otimes B \xrightarrow{u_{A\otimes B}} T(A \otimes B) \xrightarrow{\tau_{A,B}} TA \otimes TB \xrightarrow{a\otimes b} A \otimes B\right)$$

is equal to $i_{A,B} \circ p_{A,B}$, where τ is the natural transformation (1.2)*, (p, p_0) denotes the monoidal structure and (i, i_0) denotes the opmonoidal structure of the forgetful functor $U : \mathscr{M}^T \to \mathscr{M}$.*

Proof. This is immediate by (1.2) and unitality of the T-actions a, b and $m_A \,\square\, m_B$. $\square$

The 2-category Mnd(Cat) of monads was extended in [3] to a 2-category Mnd^i(Cat), as follows. The objects of Mnd^i(Cat) are the monads in Cat. The 1-cells from a monad (T, m, u) on a category $\mathscr{C}$ to a monad (T', m', u') on $\mathscr{C}'$, are pairs consisting of a functor $V : \mathscr{C} \to \mathscr{C}'$ and a natural transformation $\psi : T'V \to VT$, such that the diagram below on the left commutes, while the 2-cells $(V, \psi) \to (W, \varphi)$ are natural transformations $\omega : V \to W$ such that the diagram on the right commutes.

$$\begin{tikzcd}
T'T'V \arrow[r, "T'\psi"] \arrow[d, "m'V"'] & T'VT \arrow[r, "\psi T"] & VTT \arrow[d, "Vm"] \\
T'V \arrow[rr, "\psi"'] & & VT
\end{tikzcd}
\qquad
\begin{tikzcd}
T'T'V \arrow[r, "T'\psi"] & T'VT \arrow[r, "T'\omega T"] & T'WT \arrow[r, "\varphi T"] & WTT \arrow[d, "Wm"] \\
T'V \arrow[u, "T'u'V"] \arrow[r, "\psi"'] & VT \arrow[rr, "\omega T"'] & & WT
\end{tikzcd}$$

There is a variant, Mnd^p(Cat), of this 2-category which has the same objects and 1-cells but in which a 2-cell $(V, \psi) \to (W, \varphi)$ is a natural transformation $\omega : V \to W$ such that $\varphi \circ T'\omega = Wm \circ \varphi T \circ u'WT \circ \omega T \circ \psi$.

For a monad T on $\mathscr{C}$ and a monad T' on $\mathscr{C}'$, we say that a functor $\overline{V} : \mathscr{C}^T \to \mathscr{C}'^{T'}$ is a *weak lifting* of a functor $V : \mathscr{C} \to \mathscr{C}'$ if there exists a split natural monomorphism $i : U'\overline{V} \to VU$ (where $U : \mathscr{C}^T \to \mathscr{C}$ and $U' : \mathscr{C}'^{T'} \to \mathscr{C}'$ are the forgetful functors). Associated to a 1-cell (V, ψ) in Mnd^i(Cat), there is an idempotent natural transformation $VU \to VU$. Evaluated on a T-algebra (A, a), it is the morphism $Va \circ \psi A \circ u'VA : VA \to VA$. Whenever it splits (that is, it factorizes through some natural epimorphism $VU \to V_0$ and a section), the resulting functor $V_0 : \mathscr{C}^T \to \mathscr{C}'$ has a lifting to a functor $\overline{V} : \mathscr{C}^T \to \mathscr{C}'^{T'}$, which is clearly a weak lifting of V. Conversely, every weak lifting $\overline{V} : \mathscr{C}^T \to \mathscr{C}'^{T'}$ of a functor $V : \mathscr{C} \to \mathscr{C}'$ arises in this way from a unique 1-cell (V, ψ) in $\mathsf{Mnd}^i(\mathsf{Cat})$ such that the corresponding idempotent natural transformation splits: see [3, Theorem 4.4].

A natural transformation $\overline{\omega} : \overline{V} \to \overline{W}$ between weakly lifted functors is said to be a *weak i-lifting* of a natural transformation $\omega : V \to W$ provided that $\omega U \circ i = i \circ U'\overline{\omega}$. By [3, Proposition 4.3], a natural transformation has a weak i-lifting if and only if it is a 2-cell in $\mathsf{Mnd}^i(\mathsf{Cat})$. Symmetrically, $\overline{\omega}$ is said to be a *weak p-lifting* of ω provided that $p \circ \omega U = U'\overline{\omega} \circ p$, in terms of a natural retraction p of i. By [3, Proposition 4.3], a natural transformation has a weak p-lifting if and only if it is a 2-cell in $\mathsf{Mnd}^p(\mathsf{Cat})$.

Theorem 1.10. *Let (T, m, u) be a monad on a monoidal category $(\mathscr{M}, \otimes, K)$, where $(T, \tau, \tau_0) : (\mathscr{M}, \otimes, K) \to (\mathscr{M}, \otimes, K)$ is an opmonoidal functor. Then Eqs.* (1.3)–(1.7) *hold if and only if the following conditions are satisfied*:

(i) *The functor $1 \xrightarrow{K} \mathscr{M} \xrightarrow{T} \mathscr{M}$ and the natural transformation $T^2K \xrightarrow{m_K} TK \xrightarrow{\sqcap} TK$ constitute a 1-cell $1 \to T$ in $\mathsf{Mnd}^i(\mathsf{Cat})$.*

(ii) *The functor $\mathscr{M} \times \mathscr{M} \xrightarrow{\otimes} \mathscr{M}$ and the natural transformation $T(\bullet \otimes \bullet) \xrightarrow{\tau} T(\bullet) \otimes T(\bullet)$ constitute a 1-cell $T \times T \to T$ in $\mathsf{Mnd}^i(\mathsf{Cat})$.*

(iii) *The natural transformations*

$$\begin{tikzcd}
\mathscr{M} \times \mathscr{M} \arrow[r, "\mathscr{M} \times T"] & \mathscr{M} \times \mathscr{M} \arrow[d, "\otimes"] \\
\mathscr{M} \arrow[u, "\mathscr{M} \times K"] \arrow[r, "\mathscr{M}"'] \arrow[ur, phantom, "\Downarrow \mathscr{M} \times \tau_0"] & \mathscr{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{M} \times \mathscr{M} \arrow[r, "T \times \mathscr{M}"] & \mathscr{M} \times \mathscr{M} \arrow[d, "\otimes"] \\
\mathscr{M} \arrow[u, "K \times \mathscr{M}"] \arrow[r, "\mathscr{M}"'] \arrow[ur, phantom, "\Downarrow \tau_0 \times \mathscr{M}"] & \mathscr{M}
\end{tikzcd}$$

are 2-cells in $\mathsf{Mnd}^i(\mathsf{Cat})$.

(iv) *The idempotent natural transformations $E_{TX,TY}$ and $E^{(3)}_{TX,TY,TZ}$ which are respectively the composites*

$$TX \otimes TY \xrightarrow{u_{TX \otimes TY}} T(TX \otimes TY) \xrightarrow{\tau_{TX,TY}} T^2X \otimes T^2Y \xrightarrow{m_X \otimes m_Y} TX \otimes TY \quad \textit{and}$$

$$TX \otimes TY \otimes TZ \xrightarrow{u_{TX \otimes TY \otimes TZ}} T(TX \otimes TY \otimes TZ) \xrightarrow{\tau^{(3)}_{TX,TY,TZ}} T^2X \otimes T^2Y \otimes T^2Z$$
$$\xrightarrow{m_X \otimes m_Y \otimes m_Z} TX \otimes TY \otimes TZ,$$

make the following diagram commute:

$$\begin{tikzcd}
TX \otimes TY \otimes TZ \arrow[r, "E_{TX,TY} \otimes TZ"] \arrow[d, "TX \otimes E_{TY,TZ}"'] \arrow[dr, "E^{(3)}_{TX,TY,TZ}"] & TX \otimes TY \otimes TZ \arrow[d, "TX \otimes E_{TY,TZ}"] \\
TX \otimes TY \otimes TZ \arrow[r, "E_{TX,TY} \otimes TZ"'] & TX \otimes TY \otimes TZ
\end{tikzcd}$$

for any objects X, Y, Z in $\mathscr{M}$.

Proof. Using the definition of 1-cells in $\mathsf{Mnd}^i(\mathsf{Cat})$, assertion (i) is seen to be equivalent to the equation $\sqcap \circ m_K \circ T\sqcap = \sqcap \circ m_K$ which by Lemma 1.7 holds whenever Eq. (1.3) does. Assertion (ii) is clearly equivalent to (1.7). Condition (iii) depends on the 1-cells in $\mathsf{Mnd}^i(\mathsf{Cat})$ constructed in parts (i) and (ii). Now $\tau_0 \circ \sqcap = \tau_0$ by opmonoidality of T, and then the two conditions in (iii) are equivalent to (1.3) and (1.4). Similarly the two conditions in (iv) are equivalent to (1.5) and (1.6). □

Our next aim is to prove the other half of Theorem 1.5, which we state as:

Proposition 1.11. *Consider a monad (T, m, u) on a monoidal category $(\mathcal{M}, \otimes, K)$ and an opmonoidal structure (τ, τ_0) on the functor T. Assume that the identities* (1.3)–(1.7) *hold and that the following idempotent morphisms split*:

$$\sqcap := \big(TK \xrightarrow{u_{TK}} T^2K \xrightarrow{\tau_{K,TK}} TK \otimes T^2K \xrightarrow{TK\otimes m_K} TK \otimes TK \xrightarrow{TK\otimes\tau_0} TK\big) \tag{1.13}$$

and

$$E_{A,B} := \big(A \otimes B \xrightarrow{u_{A\otimes B}} T(A \otimes B) \xrightarrow{\tau_{A,B}} TA \otimes TB \xrightarrow{a\otimes b} A \otimes B\big), \tag{1.14}$$

for any T-algebras (A, a) and (B, b). Then T is a weak bimonad.

Proof. We prove this claim by constructing a monoidal structure on $\mathcal{M}^T$ weakly lifting that of $\mathcal{M}$, and by showing that with respect to this monoidal structure the forgetful functor $U : \mathcal{M}^T \to \mathcal{M}$ is separable Frobenius.

By Theorem 1.10 (ii), $(\otimes, \tau)$ is a 1-cell in $\mathsf{Mnd}^i(\mathsf{Cat})$ (equivalently, in $\mathsf{Mnd}^p(\mathsf{Cat})$) and so induces a weak lifting $\boxtimes : \mathcal{M}^T \times \mathcal{M}^T \to \mathcal{M}^T$; that is, a functor $\boxtimes$ equipped with natural transformations

$$\begin{tikzcd}
\mathcal{M}^T \times \mathcal{M}^T \arrow[r, "\boxtimes"] \arrow[d, "U\times U"'] \arrow[dr, phantom, "p\Uparrow\ \ \Downarrow i"] & \mathcal{M}^T \arrow[d, "U"] \\
\mathcal{M} \times \mathcal{M} \arrow[r, "\otimes"'] & \mathcal{M}
\end{tikzcd}$$

such that $p \circ i$ is the identity natural transformation. Explicitly, for T-algebras (A, a) and (B, b), the to-be-tensor product $(A, a) \boxtimes (B, b) = (A \square B, a \square b)$ is given by splitting the idempotent (1.14) to obtain $A \square B$, via maps $i_{A,B} : A \square B \to A \otimes B$ and $p_{A,B} : A \otimes B \to A \square B$, and then $a \square b$ is the composite

$$T(A \square B) \xrightarrow{Ti_{A,B}} T(A \otimes B) \xrightarrow{\tau_{A,B}} TA \otimes TB \xrightarrow{a\otimes b} A \otimes B \xrightarrow{p_{A,B}} A \square B.$$

By coassociativity of τ, the associativity isomorphism in $\mathcal{M}$ is an invertible 2-cell both in $\mathsf{Mnd}^i(\mathsf{Cat})$ and $\mathsf{Mnd}^p(\mathsf{Cat})$. So it weakly lifts to an associativity isomorphism for $\boxtimes$ such that the following diagrams, with the associativity isomorphisms on the vertical arrows, commute.

$$\begin{tikzcd}
(A \square B) \square C \arrow[r, "i_{A\square B,C}"] \arrow[d] & (A \square B) \otimes C \arrow[r, "i_{A,B}\otimes C"] & (A \otimes B) \otimes C \arrow[d] \\
A \square (B \square C) \arrow[r, "i_{A,B\square C}"'] & A \otimes (B \square C) \arrow[r, "A\otimes i_{B,C}"'] & A \otimes (B \otimes C)
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "p_{A,B}\otimes C"] \arrow[d] & (A \square B) \otimes C \arrow[r, "p_{A\square B,C}"] & (A \square B) \square C \arrow[d] \\
A \otimes (B \otimes C) \arrow[r, "A\otimes p_{B,C}"'] & A \otimes (B \square C) \arrow[r, "p_{A,B\square C}"'] & A \square (B \square C)
\end{tikzcd}$$

That is, p and i satisfy the associativity conditions that will be needed to make U a monoidal and an opmonoidal functor. The pentagon identity for $\boxtimes$ follows from that for $\otimes$ and commutativity of either of the diagrams above.

Next we construct the unit object for the monoidal category $\mathscr{M}^T$. By Theorem 1.10 (i), $(TK, \sqcap \circ m_K)$ is a 1-cell in $\mathsf{Mnd}^i(\mathsf{Cat})$ and so gives an object (R, r) of $\mathscr{M}^T$. Explicitly, R is obtained by splitting the idempotent $\sqcap$ via maps $I : R \to TK$ and $P : TK \to R$, and r is the composite

$$TR \xrightarrow{TI} T^2K \xrightarrow{m_K} TK \xrightarrow{P} R.$$

The unit constraints for $\mathscr{M}^T$ are constructed by applying Theorem 1.10 (iii). The 2-cells of $\mathsf{Mnd}^i(\mathsf{Cat})$ therein induce morphisms $\varrho_A : A \square R \to A$ and $\lambda_A : R \square A \to A$ of T-algebras, natural in the T-algebra (A, a). Explicitly, ϱ_A and λ_A are given by the composites

$$A \square R \xrightarrow{i_{A,R}} A \otimes R \xrightarrow{A\otimes I} A \otimes TK \xrightarrow{A\otimes\tau_0} A,$$

$$R \square A \xrightarrow{i_{R,A}} R \otimes A \xrightarrow{I\otimes A} TK \otimes A \xrightarrow{\tau_0\otimes A} A, \tag{1.15}$$

respectively. Since $A \square R$ was constructed by splitting the idempotent $E_{A,R}$, to show that ϱ_A is invertible, it will suffice to show that

$$A \otimes R \xrightarrow{E_{A,R}} A \otimes R \xrightarrow{A\otimes I} A \otimes TK \xrightarrow{A\otimes\tau_0} A$$

is the epimorphism part of a splitting for $E_{A,R}$. We claim that the other half of the splitting can be taken to be

$$A \xrightarrow{u_A} TA \xrightarrow{\tau_{A,K}} TA \otimes TK \xrightarrow{a\otimes P} A \otimes R.$$

By commutativity of the following diagram, one composite yields the identity morphism on A.

$$\begin{tikzcd}
A \arrow[r, "u_A"] \arrow[ddddr, "u_A"'] \arrow[dddddrrr, equal, bend right] & TA \arrow[r, "\tau_{A,K}"] \arrow[d, "u_{TA}"] & TA\otimes TK \arrow[r, "a\otimes P"] & A\otimes R \arrow[d, "u_{A\otimes R}"] \\
& T^2A \arrow[r, "T\tau_{A,K}"] \arrow[ddd, "m_A"] & T(TA\otimes TK) \arrow[r, "T(a\otimes P)"] \arrow[d, "\tau_{TA,TK}"'] & T(A\otimes R) \arrow[d, "\tau_{A,R}"] \\
& & T^2A\otimes T^2K \arrow[r, "Ta\otimes TP"] \arrow[dd, "m_A\otimes m_K"'] & TA\otimes TR \arrow[d, "a\otimes r"] \\
& & & A\otimes R \arrow[d, "A\otimes I"] \\
& TA \arrow[r, "\tau_{A,K}"] & TA\otimes TK \arrow[r, "a\otimes\sqcap"] \arrow[dr, "a\otimes\tau_0"'] & A\otimes TK \arrow[d, "A\otimes\tau_0"] \\
& & & A
\end{tikzcd}$$

The top two squares on the right-hand side commute by naturality. The third (pentagonal) region below them commutes by the associativity of a, definition of r and (1.10). The other pentagonal region on its left commutes by (1.7) and the region on the left of that commutes by the unitality condition on a monad. The triangle at the bottom commutes by the (straightforward) fact that $\tau_0 \circ \sqcap = \tau_0$. The bottom-left region commutes by the opmonoidality of T and unitality of a.

Composition in the opposite order yields $E_{A,R}$ by commutativity of

$$
\begin{tikzcd}
& A \otimes R \arrow[rrrr, equal] \arrow[d, "A\otimes I"'] & & & & A \otimes R \arrow[dddddl, "E_{A,R}"] \\
& A \otimes TK \arrow[urrrr, "A\otimes P"] \arrow[rrr, "E_{A,TK}"'] \arrow[dl, bend right, "u_{A\otimes TK}"'] \arrow[dddd, "E_{A,TK}"] & & & A \otimes TK \arrow[ddddl, "A\otimes\bar{\sqcap}"'] \arrow[dddd, "A\otimes P"] & \\
T(A \otimes TK) \arrow[d, "\tau_{A,TK}"'] & & & & & \\
TA \otimes T^2K \arrow[dr, bend right, "a\otimes m_K"'] & & & & & \\
& A \otimes TK \arrow[d, "A\otimes\tau_0"'] & & & & \\
& A \arrow[r, "A\otimes u_K"] \arrow[dr, bend right, "u_A"'] & A \otimes TK \arrow[r, "E_{A,TK}"] & A \otimes TK \arrow[r, "A\otimes P"] & A \otimes R & \\
& & TA \arrow[r, "\tau_{A,K}"'] & TA \otimes TK \arrow[u, bend right, "a\otimes TK"'] & &
\end{tikzcd}
$$

The leftmost vertical path is equal to $(A \otimes \tau_0) \circ (A \otimes I) \circ E_{A,R}$ since by the definition of r, $\tau_0 \circ I \circ r = \tau_0 \circ m_K \circ TI$. The regions surrounded by the curved arrows commute by (1.14). The triangle at the top commutes by the definitions of I and P. The concave quadrangle below it commutes by naturality of E, since P is a morphism of T-algebras by (1.10). The large polygon at the bottom-left involves the morphism

$$\bar{\sqcap} := (\tau_0 \otimes TK) \circ E_{TK,TK} \circ (TK \otimes u_K) = (\tau_0 \otimes TK) \circ (m_K \otimes TK) \circ \tau_{TK,K} \circ u_{TK},$$

where the last equality follows by (1.14). In order to see that this polygon commutes, note that associativity of a together with (1.3) implies

$$E_{A,TK} \circ (A \otimes \tau_0 \otimes TK) \circ (E_{A,TK} \otimes TK) = (A \otimes \tau_0 \otimes TK) \circ E^{(3)}_{A,TK,TK}.$$

Using the first one of the equivalent forms of $\bar{\sqcap}$ above, this implies commutativity of the bottom-left polygon. In order to see that the triangle on its right commutes, use the second form of $\bar{\sqcap}$. By (1.4) and opmonoidality of T, it obeys $\tau_0 \circ m_K \circ T\bar{\sqcap} = \tau_0 \circ m_K$ which implies $\sqcap \circ \bar{\sqcap} = \sqcap$ hence commutativity of the triangle in question.

The case of λ is similar. We record here the explicit forms of ϱ_A^{-1} and λ_A^{-1} as

$$A \xrightarrow{u_A} TA \xrightarrow{\tau_{A,K}} TA \otimes TK \xrightarrow{a\otimes P} A \otimes R \xrightarrow{p_{A,R}} A \,\square\, R,$$

$$A \xrightarrow{u_A} TA \xrightarrow{\tau_{K,A}} TK \otimes TA \xrightarrow{P\otimes a} R \otimes A \xrightarrow{p_{R,A}} R \,\square\, A, \tag{1.16}$$

respectively.

To conclude that $\mathscr{M}^T$ is a monoidal category, we only need to prove that the triangle condition holds. This follows by functoriality of weak lifting because both morphisms

$$(A \,\square\, R) \,\square\, B \longrightarrow A \,\square\, (R \,\square\, B) \xrightarrow{A \square \lambda_B} A \,\square\, B \quad \text{and} \quad (A \,\square\, R) \,\square\, B \xrightarrow{\varrho_A \square B} A \,\square\, B$$

are weak i-liftings of $(A \otimes \tau_0) \otimes B : (A \otimes TK) \otimes B \to A \otimes B$, for any T-algebras A and B.

It remains to show that the forgetful functor $U : \mathscr{M}^T \to \mathscr{M}$ is separable Frobenius. We already have the binary parts of the monoidal and opmonoidal structures, in the form of morphisms $p_{A,B} : A \otimes B \to A \,\square\, B$ and $i_{A,B} : A \,\square\, B \to A \otimes B$. We already proved that they satisfy the associativity, respectively, coassociativity conditions. A counit i_0 for U, so that (U, i, i_0) becomes an opmonoidal functor, is constructed as the composite

$$R \xrightarrow{I} TK \xrightarrow{\tau_0} K$$

and the counit laws then reduce to Eqs. (1.15) defining λ and ϱ. The unit p_0 for the monoidal structure of U will be the composite

$$K \xrightarrow{u_K} TK \xrightarrow{P} R.$$

One of the unit laws follows by commutativity of the diagram below; the other is similar and left to the reader.

$$\begin{tikzcd}
A \arrow[r, "A\otimes u_K"] \arrow[d, "u_A"'] & A \otimes TK \arrow[r, "A\otimes P"] \arrow[d, "u_{A\otimes TK}"'] & A \otimes R \arrow[r, "p_{A,R}"] \arrow[d, "u_{A\otimes R}"'] & A \,\square\, R \arrow[dd, "i_{A,R}"] \\
TA \arrow[r, "T(A\otimes u_K)"] \arrow[d, "\tau_{A,K}"'] & T(A \otimes TK) \arrow[r, "T(A\otimes P)"] \arrow[d, "\tau_{A,TK}"'] & T(A \otimes R) \arrow[d, "\tau_{A,R}"'] & \\
TA \otimes TK \arrow[r, "TA\otimes Tu_K"] \arrow[dr, equal] & TA \otimes T^2K \arrow[r, "TA\otimes TP"] \arrow[d, "TA\otimes m_K"] & TA \otimes TR \arrow[r, "a\otimes r"] & A \otimes R \arrow[d, "A\otimes I"] \\
& TA \otimes TK \arrow[rr, "a\otimes\sqcap"] \arrow[d, "TA\otimes\tau_0"'] & & A \otimes TK \arrow[d, "A\otimes\tau_0"] \\
& TA \arrow[rr, "a"'] & & A
\end{tikzcd}$$

The four squares in the top left corner commute by naturality; the large region in the top right corner by definition of i and p; the triangle by one of the unit laws for a monad, the region to its right by definition of r and (1.10), and the bottom region by the equation $\tau_0 \circ \sqcap = \tau_0$ once again. The left/bottom path yields an identity morphism by opmonoidality of T and unitality of a.

The separability condition $p_{A,B} \circ i_{A,B} = A \,\square\, B$ holds by construction. As for the Frobenius conditions in Definition 1.1, by (1.14) we have $E_{A,B\square C} = (A \otimes p_{B,C}) \circ E^{(3)}_{A,B,C} \circ (A \otimes i_{B,C})$, and now the first Frobenius condition follows by Theorem 1.10 (iv); the other Frobenius condition is proved similarly. $\square$

2. Weak bimonads vs. bimonads over a separable Frobenius base

The aim of this section is to study the category of weak bimonads on a given Cauchy complete monoidal category $\mathscr{M}$. As a main result, we prove that it is equivalent to an appropriate category of bimonads on bimodule categories over separable Frobenius monoids in $\mathscr{M}$.

Recall that if T is a bimonad then $\mathcal{M}^T$ can be given a monoidal structure so that the forgetful functor $U:\mathcal{M}^T \to \mathcal{M}$ is strict monoidal. Conversely, any monad T for which $U:\mathcal{M}^T \to \mathcal{M}$ is strong monoidal can be made into a bimonad; and these two processes are, in a suitable sense, mutually inverse. Similarly, if $g:T \to T'$ is a morphism of bimonads and $\mathcal{M}^T$ and $\mathcal{M}^{T'}$ are made monoidal as above, then the induced functor $g^*:\mathcal{M}^{T'} \to \mathcal{M}^T$ is strict monoidal; and conversely if $g:T \to T'$ is a morphism of monads for which the induced functor g^* is opmonoidal, compatibly with the forgetful functors, then g can be made into a morphism of bimonads.

For the entire section, we introduce the following notation. We work in a monoidal category $\mathcal{M}$, with monoidal product $\otimes$ and monoidal unit K. For a weak bimonad T, the monad structure is denoted by $m:T^2 \to T$ and $u:\mathcal{M} \to T$. The opmonoidal structure of T is denoted by $\tau_{X,Y}:T(X\otimes Y) \to TX \otimes TY$ and $\tau_0:TK \to K$. The forgetful functor $\mathcal{M}^T \to \mathcal{M}$ is called U. The monoidal unit of $\mathcal{M}^T$ is denoted by (R,r). By the separable Frobenius property of U, R is a separable Frobenius monoid in $\mathcal{M}$. Its monoid structure is denoted by $(\mu:R\otimes R \to R, \eta:K \to R)$ and for the comonoid structure we write $(\delta:R \to R\otimes R, \varepsilon:R \to K)$. (See their explicit expressions in terms of (m,u) and (τ,τ_0) below.) For the monoidal category of R-bimodules, the forgetful functor is denoted by $V:{}_R\mathcal{M}_R \to \mathcal{M}$. We use the notation $\sqcap$ introduced in (1.13), E in (1.14) and $E^{(3)}$ in Theorem 1.10 (iv). For other (weak) bimonads T', $\widetilde{T}$, etc., we use the same symbols introduced for T, distinguished by prime, tilde, etc.

Our starting point is the following result due to Szlachányi.

Theorem 2.1. *(See [21, Theorem 2.2 and Lemma 6.2].) Any separable Frobenius functor U, from a monoidal category $\mathcal{N}$ with unit R, to a Cauchy complete monoidal category $\mathcal{M}$, factorizes through the forgetful functor ${}_{UR}\mathcal{M}_{UR} \to \mathcal{M}$ via a strong monoidal functor $\mathcal{N} \to {}_{UR}\mathcal{M}_{UR}$.*

In particular, for a weak bimonad T on a Cauchy complete monoidal category $\mathcal{M}$, the forgetful functor $U:\mathcal{M}^T \to \mathcal{M}$ factorizes through a strong monoidal functor $\widetilde{U}$ from $\mathcal{M}^T$ to the bimodule category ${}_R\mathcal{M}_R$ for the monoidal unit R of $\mathcal{M}^T$ and the forgetful functor $V:{}_R\mathcal{M}_R \to \mathcal{M}$. Explicitly, the monoid structure of R comes out as

$$\mu := (R\otimes R \xrightarrow{E_{R,R}} R\otimes R \xrightarrow{R\otimes I} R\otimes TK \xrightarrow{R\otimes\tau_0} R), \qquad \eta := (K \xrightarrow{u_K} TK \xrightarrow{P} R) \tag{2.1}$$

and its comonoid structure is given by

$$\delta := (R \xrightarrow{R\otimes u_K} R\otimes TK \xrightarrow{R\otimes P} R\otimes R \xrightarrow{E_{R,R}} R\otimes R), \qquad \varepsilon := (R \xrightarrow{I} TK \xrightarrow{\tau_0} K). \tag{2.2}$$

By (1.15), $\widetilde{U}$ takes a T-algebra (A,a) to the $R\otimes\bullet\otimes R$-algebra (A,ϱ_A) with the structure morphism

$$\varrho_A = (R\otimes A\otimes R \xrightarrow{E^{(3)}_{R,A,R}} R\otimes A\otimes R \xrightarrow{I\otimes A\otimes I} TK\otimes A\otimes TK \xrightarrow{\tau_0\otimes A\otimes\tau_0} A), \tag{2.3}$$

where $TK \overset{P}{\twoheadrightarrow} R \overset{I}{\rightarrowtail} TK$ denotes a chosen splitting of the idempotent morphism $\sqcap$ of (1.13). (Recall that $R\otimes\bullet\otimes R$-algebras (M,ϱ_M) are in bijection with R-bimodules $(M, \alpha_M:M\otimes R \to M, \beta_M:R\otimes M \to M)$ via the correspondence $\varrho_M = \alpha_M \circ (\beta_M\otimes R) = \beta_M \circ (R\otimes\alpha_M)$.)

Next we compare the monadicity properties of the functors in the factorization in Theorem 2.1.

Lemma 2.2. *For a separable Frobenius monoid R in a Cauchy complete monoidal category $\mathscr{M}$ with forgetful functor $V : {}_R\mathscr{M}_R \to \mathscr{M}$, any V-contractible pair is a split coequalizer pair.* (*For the terminology we refer to* [2].)

Proof. Consider a Cauchy complete category $\mathscr{C}$ and an adjunction $L \dashv V : \mathscr{C} \to \mathscr{M}$ in which the counit $n : LV \to 1$ is split by a natural monomorphism $\bar{n}$. Under these assumptions, any V-contractible pair is a split coequalizer pair. Indeed, if for some morphisms $\mu, \nu : M \to N$ in $\mathscr{C}$, the first diagram in

$$VM \underset{V\nu}{\overset{V\mu}{\underset{\xi}{\leftleftarrows\!\!\rightrightarrows}}} VN, \qquad M \underset{\nu}{\overset{\mu}{\underset{nM\circ L\xi\circ\bar{n}N}{\leftleftarrows\!\!\rightrightarrows}}} N$$

is a contractible pair, then so is the second one. Hence by Cauchy completeness of $\mathscr{C}$, the coequalizer of μ and ν exists.

We conclude by applying this observation to the adjunction $R \otimes \bullet \otimes R \dashv V : {}_R\mathscr{M}_R \to \mathscr{M}$, whose counit is given by the $R \otimes \bullet \otimes R$-action $\varrho_M : R \otimes M \otimes R \to M$, for any object (M, ϱ_M) of ${}_R\mathscr{M}_R$, hence by separable Frobenius property of R it is split by $(R \otimes \varrho_M \otimes R) \circ (\delta \circ \eta \otimes M \otimes \delta \circ \eta)$. $\square$

Proposition 2.3. *Let R be a separable Frobenius monoid in a Cauchy complete monoidal category $\mathscr{M}$. Then for a Cauchy complete category $\mathscr{C}$, a functor $W : \mathscr{C} \to {}_R\mathscr{M}_R$ is monadic if and only if its composite with the forgetful functor $V : {}_R\mathscr{M}_R \to \mathscr{M}$ is monadic.*

Proof. This is proved by applying Beck's theorem [2, Theorem 3.14].

Assume first that W is monadic. Then it is immediate by monadicity of V that VW has a left adjoint and that it is conservative. It remains to show that the Beck condition holds. For a VW-split coequalizer pair (α, β) in $\mathscr{C}$, $(W\alpha, W\beta)$ is in particular V-contractible. Hence by Lemma 2.2 it is a split coequalizer pair (evidently preserved by V). Then by monadicity of W, there exists the coequalizer of α and β and it is preserved by W, so also by VW.

Conversely, assume that VW is monadic. Since VW is conservative by assumption, so is W. As for the Beck condition, a W-split coequalizer pair (α, β) is also a VW-split coequalizer pair. Hence by monadicity of VW, its coequalizer exists and it is preserved by VW. Since V is faithful, this implies that W preserves the coequalizer of α and β. Thus we need only to check that W has a left adjoint. This holds by a standard adjoint-lifting argument [2], made particularly simple here since the relevant coequalizers are split. In more detail, let L be the left adjoint of VW, with counit $n : LVW \to 1$ and unit $u : 1 \to VWL$. Consider the mate of $V\varrho W$ for $\varrho : R \otimes V(\bullet) \otimes R \to {}_R\mathscr{M}_R$ under the adjunction $L \dashv VW$; that is, the morphism

$$\lambda_X := \big(L(R \otimes X \otimes R) \xrightarrow{L(R\otimes u_X\otimes R)} L(R \otimes VWLX \otimes R) \xrightarrow{LV\rho_{WLX}} LVWLX \xrightarrow{n_{LX}} LX\big) \tag{2.4}$$

for any object X of $\mathscr{M}$. Whenever the coequalizer of $L\varrho_M, \lambda_M : L(R \otimes M \otimes R) \to LM$ exists for any object (M, ϱ_M) of ${}_R\mathscr{M}_R$, it defines a left adjoint for the lifting W of VW; see [2]. By the separable Frobenius property of R, the morphism λ_X is split by the natural monomorphism $\lambda_{R\otimes X\otimes R} \circ L(\delta \circ \eta \otimes X \otimes \delta \circ \eta)$. Since the diagram

$$\begin{tikzcd}[column sep=huge]
L(R \otimes M \otimes R) \arrow[r, shift left, "L\varrho_M"] \arrow[r, shift right, "\lambda_M"'] & LM \arrow[d, equal] \\
LM \arrow[u, "\lambda_{R\otimes M\otimes R}\circ L(\delta\circ\eta\otimes M\otimes\delta\circ\eta)"] \arrow[r, shift left, "\lambda_M\circ L((R\otimes\varrho_M\otimes R)\circ(\delta\circ\eta\otimes M\otimes\delta\circ\eta))"] \arrow[r, shift right, "LM"'] & LM
\end{tikzcd}$$

is serially commutative, and the coequalizer of the bottom pair exists by Cauchy completeness of $\mathscr{C}$, it follows that it is also a coequalizer of the top pair, defining a left adjoint of W. $\square$

Remark 2.4. Proposition 2.3 implies a relation between weak bimonads on a Cauchy complete monoidal category $\mathscr{M}$ and bimonads on ${}_R\mathscr{M}_R$, for some separable Frobenius monoid R in $\mathscr{M}$. Namely, for a weak bimonad T, the separable Frobenius forgetful functor $U:\mathscr{M}^T \to \mathscr{M}$ factorizes through a strong monoidal functor $\widetilde{U}:\mathscr{M}^T \to {}_R\mathscr{M}_R$ for a separable Frobenius monoid R, and the forgetful functor $V: {}_R\mathscr{M}_R \to \mathscr{M}$; see Theorem 2.1. By Proposition 2.3, $\widetilde{U}$ is also monadic hence together with its left adjoint $\widetilde{L}$, it induces a bimonad $\widetilde{T} := \widetilde{U}\widetilde{L}$ on ${}_R\mathscr{M}_R$, whose Eilenberg–Moore category is equivalent to $\mathscr{M}^T$; see [9]. Conversely, for a bimonad $\widetilde{T}$ on a bimodule category ${}_R\mathscr{M}_R$ over a separable Frobenius monoid R, the composite U of the forgetful functor $\widetilde{U}:({}_R\mathscr{M}_R)^{\widetilde{T}} \to {}_R\mathscr{M}_R$ and the forgetful functor $V: {}_R\mathscr{M}_R \to \mathscr{M}$ is separable Frobenius; cf. Example 1.2. It is also monadic by Proposition 2.3, hence together with its left adjoint L, it induces a weak bimonad $T := UL$ on $\mathscr{M}$ such that $\mathscr{M}^T$ is equivalent to $({}_R\mathscr{M}_R)^{\widetilde{T}}$. What is more, by uniqueness of a left adjoint up to natural isomorphism, T and $V\widetilde{T}(R \otimes \bullet \otimes R)$ differ by an opmonoidal isomorphism of monads (or in fact they can be chosen equal).

Remark 2.5. Consider a weak bimonad T on a Cauchy complete monoidal category $\mathscr{M}$. By (the proof of) Proposition 2.3, the left adjoint $\widetilde{L}$ of the strong monoidal functor $\widetilde{U}:\mathscr{M}^T \to {}_R\mathscr{M}_R$ (occurring in the factorization of the forgetful functor $U:\mathscr{M}^T \to \mathscr{M}$) is constructed by choosing a splitting $LM \twoheadrightarrow \widetilde{L}(M, \varrho_M) \rightarrowtail LM$ of the idempotent natural transformation

$$\lambda_M \circ L\big((R \otimes \varrho_M \otimes R) \circ (\delta \circ \eta \otimes M \otimes \delta \circ \eta)\big) : LM \to LM, \tag{2.5}$$

for any object (M, ϱ_M) of ${}_R\mathscr{M}_R$, where λ_M is as in (2.4). Applying U, this yields a split idempotent natural transformation $ULV = TV \overset{q}{\twoheadrightarrow} U\widetilde{L} = V\widetilde{T} \overset{j}{\rightarrowtail} TV = ULV$. What is more, also as a monad, $\widetilde{T} = \widetilde{U}\widetilde{L}$ is a weak q-lifting of T hence by [3, Proposition 3.7], the Eilenberg–Moore categories $\mathscr{M}^T$ and $({}_R\mathscr{M}_R)^{\widetilde{T}}$ are in fact isomorphic. Explicitly, there is an isomorphism $\Xi:\mathscr{M}^T \to ({}_R\mathscr{M}_R)^{\widetilde{T}}$, taking a T-algebra (A, a) to the $R \otimes \bullet \otimes R$-algebra A described in (2.3), with a $\widetilde{T}$-algebra structure provided by the unique morphism $\widetilde{a}:\widetilde{T}A \to A$ for which $\widetilde{a} \circ q_A = a$. On the morphisms, Ξ acts as the identity map. The inverse of Ξ takes an object $((A, \varrho_A), \widetilde{a})$ of $({}_R\mathscr{M}_R)^{\widetilde{T}}$ to the T-algebra A, with structure morphism $TA \xrightarrow{q_A} V\widetilde{T}A \xrightarrow{\widetilde{a}} A$, and it also acts on the morphisms as an identity map. In particular, also the Eilenberg–Moore categories $({}_R\mathscr{M}_R)^{\widetilde{T}}$ and $\mathscr{M}^{V\widetilde{T}(R\otimes\bullet\otimes R)}$ are isomorphic, for any bimonad $\widetilde{T}$ on a bimodule category ${}_R\mathscr{M}_R$ over a separable Frobenius monoid R.

The final aim of this section is to extend the correspondence in Remark 2.4 between weak bimonads on one hand, and bimonads over separable Frobenius base monoids on the other hand, to an equivalence of categories.

Definition 2.6. A *morphism of weak bimonads* on a monoidal category $\mathscr{M}$ is defined as an opmonoidal morphism of monads; that is, as a natural transformation $g: T \to T'$ which is opmonoidal in the sense that, for any objects X and Y in $\mathscr{M}$,

$$\begin{tikzcd}
T(X\otimes Y) \arrow[r, "g_{X\otimes Y}"] \arrow[d, "\tau_{X,Y}"'] & T'(X\otimes Y) \arrow[d, "\tau'_{X,Y}"] \\
TX\otimes TY \arrow[r, "g_X\otimes g_Y"'] & T'X\otimes T'Y
\end{tikzcd}
\qquad
\begin{tikzcd}
TK \arrow[r, "g_K"] \arrow[d, "\tau_0"'] & T'K \arrow[d, "\tau'_0"] \\
K \arrow[r, equal] & K
\end{tikzcd}$$

and which is a morphism of monads in the sense that, for any object X in $\mathcal{M}$,

$$\begin{tikzcd} T^2X \arrow[r, "Tg_X"] \arrow[d, "m_X"'] & TT'X \arrow[r, "g_{T'X}"] & T'^2X \arrow[d, "m'_X"] \\ TX \arrow[rr, "g_X"'] & & T'X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, equal] \arrow[d, "u_X"'] & X \arrow[d, "u'_X"] \\ TX \arrow[r, "g_X"'] & T'X. \end{tikzcd}$$

Weak bimonads on $\mathcal{M}$ (as objects) and their morphisms (as arrows) constitute a category $\mathsf{Wbm}(\mathcal{M})$, which contains the category of bimonads on $\mathcal{M}$ as a full subcategory.

Example 2.7. Any monoid R in a monoidal category $\mathcal{M}$, induces a monad $R \otimes \bullet \otimes R$ on $\mathcal{M}$. If R is a separable Frobenius monoid in a Cauchy complete monoidal category $\mathcal{M}$, then $R \otimes \bullet \otimes R$ is a weak bimonad; see Example 1.2 (3). Its opmonoidal structure is provided by the maps

$$R \otimes R \xrightarrow{\varepsilon\circ\mu} K \quad \text{and} \quad R \otimes X \otimes Y \otimes R \xrightarrow{R\otimes X\otimes\delta\circ\eta\otimes Y\otimes R} R \otimes X \otimes R \otimes R \otimes Y \otimes R,$$

for any objects X, Y in $\mathcal{M}$, where (μ, η) and (δ, ε) denote the monoid and comonoid structures of R, respectively. A morphism $\gamma : R \to R'$ of separable Frobenius monoids in $\mathcal{M}$ induces a morphism of weak bimonads $\gamma \otimes \bullet \otimes \gamma : R \otimes \bullet \otimes R \to R' \otimes \bullet \otimes R'$.

Lemma 2.8. *Consider any morphism $g : T \to T'$ of weak bimonads on a Cauchy complete monoidal category $\mathcal{M}$, with monoidal units R, respectively R', of the Eilenberg–Moore categories $\mathcal{M}^T$ and $\mathcal{M}^{T'}$. There is a unique isomorphism $\gamma : R \to R'$ of separable Frobenius monoids such that the following diagram of functors commutes,*

$$\begin{tikzcd} ({}_{R'}\mathcal{M}_{R'})^{\widetilde{T}'} \cong \mathcal{M}^{T'} \arrow[rr, "g^*"] \arrow[d, "\widetilde{U}'"'] & & \mathcal{M}^T \cong ({}_R\mathcal{M}_R)^{\widetilde{T}} \arrow[d, "\widetilde{U}"] \\ {}_{R'}\mathcal{M}_{R'} \arrow[rr, "\gamma^*"] \arrow[dr, "V'"'] & & {}_R\mathcal{M}_R \arrow[dl, "V"] \\ & \mathcal{M} & \end{tikzcd} \tag{2.6}$$

where the bimonads $\widetilde{T}$ and $\widetilde{T}'$ are associated to the weak bimonads T and T' as in Remark 2.4.

Proof. By Lemma 1.8, for weak bimonads T and T', the associated idempotent morphisms $\sqcap$ and $\sqcap'$ in (1.13) split through R and R', respectively; thus there are epi-mono pairs $TK \overset{P}{\twoheadrightarrow} R \overset{I}{\rightarrowtail} TK$ and $T'K \overset{P'}{\twoheadrightarrow} R' \overset{I'}{\rightarrowtail} T'K$. Using the fact that a morphism $g : T \to T'$ of weak bimonads is an opmonoidal natural transformation as well as a morphism of monads, one checks that for any objects X, Y in $\mathcal{M}$, the two diagrams on the left

$$\begin{tikzcd}
TK \arrow[r, "g_K"] \arrow[d, "\sqcap"'] & T'K \arrow[d, "\sqcap'"] \\
TK \arrow[r, "g_K"'] & T'K
\end{tikzcd}
\qquad
\begin{tikzcd}
TX \otimes TY \arrow[r, "g_X \otimes g_Y"] \arrow[d, "E_{TX,TY}"'] & T'X \otimes T'Y \arrow[d, "E'_{T'X,T'Y}"] \\
TX \otimes TY \arrow[r, "g_X \otimes g_Y"'] & T'X \otimes T'Y
\end{tikzcd}
\qquad
\begin{tikzcd}
TK \arrow[r, "g_K"] & T'K \arrow[d, "P'"] \\
R \arrow[u, "I"] \arrow[r, "\gamma"'] & R'
\end{tikzcd}
\tag{2.7}$$

commute and so the morphism γ defined by the diagram on the right is compatible both with the monoid and with the comonoid structures of R and R', written out explicitly in (2.1) and (2.2). That is, γ is a morphism of separable Frobenius monoids. By [16, Proposition A.3], any morphism of Frobenius monoids is an isomorphism hence so is γ. It obviously renders commutative the lower triangle in (2.6). It renders commutative also the upper square by commutativity of the following diagram, for any T'-algebra (A, a).

$$\begin{tikzcd}
R \otimes A \otimes R \arrow[r, "u_{R\otimes A\otimes R}"] \arrow[d, "\gamma\otimes A\otimes\gamma"'] & T(R \otimes A \otimes R) \arrow[r, "\tau^{(3)}_{R,A,R}"] \arrow[d, "T(\gamma\otimes A\otimes\gamma)"] & TR \otimes TA \otimes TR \arrow[r, "TR\otimes g_A\otimes TR"] \arrow[d, "T\gamma\otimes TA\otimes T\gamma"] & TR \otimes T'A \otimes TR \arrow[d, "r\otimes a\otimes r"] \\
R' \otimes A \otimes R' \arrow[r, "u_{R'\otimes A\otimes R'}"] \arrow[d, "u'_{R'\otimes A\otimes R'}"'] & T(R' \otimes A \otimes R') \arrow[r, "\tau^{(3)}_{R',A,R'}"] \arrow[dl, "g_{R'\otimes A\otimes R'}"'] & TR' \otimes TA \otimes TR' \arrow[dl, "g_{R'}\otimes g_A\otimes g_{R'}"'] & R \otimes A \otimes R \arrow[dl, "\gamma\otimes A\otimes\gamma"'] \arrow[d, "\varepsilon\otimes A\otimes\varepsilon"] \\
T'(R' \otimes A \otimes R') \arrow[r, "\tau'^{(3)}_{R',A,R'}"'] & T'R' \otimes T'A \otimes T'R' \arrow[r, "r'\otimes a\otimes r'"'] & R' \otimes A \otimes R' \arrow[r, "\varepsilon'\otimes A\otimes\varepsilon'"'] & A
\end{tikzcd}
\tag{2.8}$$

The two squares in the upper left corner commute by naturality and the square below them commutes by the opmonoidality of g. The triangles in the bottom row commute since g is a monad morphism, and since γ is a comonoid morphism, respectively. The remaining region commutes by commutativity of the following diagram,

$$\begin{tikzcd}
TR \arrow[r, "TI"] \arrow[d, "TI"'] \arrow[rrrr, bend left, "r"] \arrow[ddd, bend right, "T\gamma"'] & TTK \arrow[rr, "m_K"] \arrow[d, phantom, "(1.10)"] & & TK \arrow[r, "P"] & R \arrow[d, "I"] \arrow[ddd, bend left, "\gamma"] \\
TTK \arrow[d, "Tg_K"'] \arrow[r, phantom, "(2.7)"] & TTK \arrow[ul, "TP"'] \arrow[rrr, "m_K"] \arrow[d, "Tg_K"] & & & TK \arrow[d, "g_K"] \\
TT'K \arrow[d, "TP'"'] & TT'K \arrow[r, "g_{T'K}"] \arrow[dl, "TP'"'] & T'T'K \arrow[rr, "m'_K"] \arrow[dl, "T'P'"'] \arrow[r, phantom, "(1.10)"] & & T'K \arrow[d, "P'"] \\
TR' \arrow[r, "g_{R'}"'] & T'R' \arrow[r, "T'I'"'] \arrow[rrr, bend right, "r'"'] & T'T'K \arrow[r, "m'_K"'] & T'K \arrow[r, "P'"'] & R'
\end{tikzcd}
\tag{2.9}$$

where the undecorated region in the middle commutes since g is a morphism of monads. It remains to show that γ is unique with the stated property. The upper part of the diagram in (2.6) commutes if and only if, for any T'-algebra (A,a) and the corresponding T-algebra $(A, a\circ g_A)$, the $R\otimes\bullet\otimes R$-action ϱ_A and the $R'\otimes\bullet\otimes R'$-action ϱ'_A in (2.3) obey $\varrho_A=\varrho'_A\circ(\gamma\otimes A\otimes\gamma)$; see (2.8). Applying it in the case where (A,a) is the monoidal unit (R',r') of $\mathcal{M}^{T'}$ and composing the resulting identity on the right with $\eta\otimes\eta'\otimes R$, we deduce that $\gamma=(R'\otimes\varepsilon)\circ E_{R',R}\circ(\eta'\otimes R)$. Thus γ renders commutative the last diagram in (2.7) by the forms of P' and I in (1.11) and (1.12), the form of $E_{R',R}$ in (1.14), and naturality. $\square$

On the other hand, any isomorphism of separable Frobenius monoids clearly induces a strict monoidal isomorphism between the categories of bimodules, which can be seen as a particular case of the following:

Lemma 2.9. *If $g:T\to T'$ is a morphism of weak bimonads on a Cauchy complete monoidal category $\mathcal{M}$ then the induced functor $g^*:\mathcal{M}^{T'}\to\mathcal{M}^T$ is strong monoidal and* (2.6) *is a commutative diagram of separable Frobenius monoidal functors.*

Proof. If T and T' are bimonads then the monoidal structures on $\mathcal{M}^T$ and $\mathcal{M}^{T'}$ are lifted from that on $\mathcal{M}$, and so clearly $g^*:\mathcal{M}^{T'}\to\mathcal{M}^T$ is strong (in fact strict) monoidal. If T is only a weak bimonad then the monoidal structure on $\mathcal{M}^T$ is only weakly lifted from that on $\mathcal{M}$, but it is lifted, up to equivalence, from that on ${}_R\mathcal{M}_R$. Thus if $g:T\to T'$ is a morphism of weak bimonads, then the isomorphism $\gamma:R\to R'$ of the previous lemma induces a strict monoidal isomorphism ${}_{R'}\mathcal{M}_{R'}\to{}_R\mathcal{M}_R$, and now the strong monoidal structure on the composite $\mathcal{M}^{T'}\to{}_{R'}\mathcal{M}_{R'}\to{}_R\mathcal{M}_R$ lifts to a strong monoidal structure on $g^*:\mathcal{M}^{T'}\to\mathcal{M}^T$. Explicitly, this is given by

$$\gamma_{A,B}:=\big(A\,\Box\,B\xrightarrow{i_{A,B}}A\otimes B\xrightarrow{p'_{A,B}}A\,\Box'\,B\big),$$

and $\gamma:R\to R'$. $\square$

We now turn to our category of bimonads on categories of bimodules over separable Frobenius monoids in $\mathcal{M}$.

Definition 2.10. For a monoidal category $\mathcal{M}$, the category $\mathsf{Sfbm}(\mathcal{M})$ is defined to have *objects* which are pairs $(R,\widetilde{T})$, consisting of a separable Frobenius monoid R in $\mathcal{M}$ and a bimonad $\widetilde{T}$ on ${}_R\mathcal{M}_R$. *Morphisms* $(R,\widetilde{T})\to(R',\widetilde{T}')$ are pairs (γ,Γ), consisting of an isomorphism $\gamma:R\to R'$ of separable Frobenius monoids (inducing a strong monoidal isomorphism $\gamma^*:{}_{R'}\mathcal{M}_{R'}\to{}_R\mathcal{M}_R$), and a morphism of bimonads $\Gamma:\widetilde{T}\to\gamma^*\widetilde{T}'(\gamma^*)^{-1}$ (that is, an opmonoidal morphism of monads, in the sense explicated in Definition 2.6).

There is an evident functor $\Psi:\mathsf{Sfbm}(\mathcal{M})\to\mathsf{Wbm}(\mathcal{M})$ defined as follows. The object map is given by associating to a pair $(R,\widetilde{T})$ the weak bimonad induced by the composite of the forgetful functors $\widetilde{U}:({}_R\mathcal{M}_R)^{\widetilde{T}}\to{}_R\mathcal{M}_R$ and $V:{}_R\mathcal{M}_R\to\mathcal{M}$, as described in Remark 2.4; explicitly, $\Psi(R,\widetilde{T})X=V\widetilde{T}(R\otimes X\otimes R)$. A morphism $(\gamma,\Gamma):(R,\widetilde{T})\to(R',\widetilde{T}')$ in $\mathsf{Sfbm}(\mathcal{M})$ gives rise to a commutative diagram of functors

$$\begin{tikzcd}
\mathcal{M}^{\Psi(R',\widetilde{T}')}\cong({}_{R'}\mathcal{M}_{R'})^{\widetilde{T}'} \arrow[rr, "\Gamma^*"] \arrow[d, "\widetilde{U}'"'] & & ({}_R\mathcal{M}_R)^{\widetilde{T}}\cong\mathcal{M}^{\Psi(R,\widetilde{T})} \arrow[d, "\widetilde{U}"] \\
{}_{R'}\mathcal{M}_{R'} \arrow[rr, "\gamma^*"] \arrow[dr, "V'"'] & & {}_R\mathcal{M}_R \arrow[dl, "V"] \\
& \mathcal{M} &
\end{tikzcd}$$

and so in particular to a morphism of monads $g : \Psi(R, \widetilde{T}) \to \Psi(R', \widetilde{T}')$, explicitly, $g_X : V\widetilde{T}(R \otimes X \otimes R) \to V'\widetilde{T}'(R' \otimes X \otimes R')$ is given by

$$\begin{tikzcd} V\widetilde{T}(R\otimes X\otimes R) \arrow[r, "V\Gamma_{R\otimes X\otimes R}"] & V\gamma^*\widetilde{T}'(\gamma^*)^{-1}(R\otimes X\otimes R) \arrow[d, equal] & \\ & V'\widetilde{T}'(\gamma^*)^{-1}(R\otimes X\otimes R) \arrow[r, "V'\widetilde{T}'(\gamma\otimes X\otimes\gamma)"] & V'\widetilde{T}'(R'\otimes X\otimes R') \end{tikzcd}$$

and it is opmonoidal since V, V', and $\widetilde{T}'$ are opmonoidal functors, and Γ and $\gamma \otimes X \otimes \gamma : (\gamma^*)^{-1}(R \otimes X \otimes R) \to R' \otimes X \otimes R'$ are opmonoidal natural transformations.

Theorem 2.11. *If $\mathscr{M}$ is a Cauchy complete monoidal category, the functor $\Psi : \mathsf{Sfbm}(\mathscr{M}) \to \mathsf{Wbm}(\mathscr{M})$ is an equivalence of categories.*

Proof. First we show that Ψ is fully faithful. Suppose then that objects $(R, \widetilde{T})$ and $(R', \widetilde{T}')$ of $\mathsf{Sfbm}(\mathscr{M})$ are given. We must show that any morphism $g : \Psi(R, \widetilde{T}) \to \Psi(R', \widetilde{T}')$ of weak bimonads is induced by a unique morphism $(\gamma, \Gamma) : (R, \widetilde{T}) \to (R', \widetilde{T}')$ in $\mathsf{Sfbm}(\mathscr{M})$. The existence of a unique isomorphism $\gamma : R \to R'$ of monoids, inducing an isomorphism $\gamma^* : {}_{R'}\mathscr{M}_{R'} \to {}_R\mathscr{M}_R$ of categories rendering commutative (2.6), is given by Lemma 2.8. By Lemma 2.9, g induces a strong monoidal functor $g^* : \mathscr{M}^{T'} \to \mathscr{M}^T$. By commutativity of the upper square in (2.6) as a diagram of strong monoidal functors, it is necessarily of the form Γ^* for a unique opmonoidal monad morphism $\Gamma : \widetilde{T} \to \gamma^*\widetilde{T}'(\gamma^*)^{-1}$. This proves that Ψ is fully faithful. It is essentially surjective on objects by Remark 2.4. □

Bimonads are monads in the 2-category OpMon of monoidal categories, opmonoidal functors and opmonoidal natural transformations; cf. [11]. That is, they can be regarded as 0-cells in the 2-category $\mathsf{Mnd}(\mathsf{OpMon})$. Clearly, for a Cauchy complete monoidal category $\mathscr{M}$, the category $\mathsf{Sfbm}(\mathscr{M})$ is a subcategory in the opposite of the category underlying $\mathsf{Mnd}(\mathsf{OpMon})$. We may consider also the full subcategory of the underlying category of $\mathsf{Mnd}(\mathsf{OpMon})$, with objects the bimonads on bimodule categories over separable Frobenius monoids R in Cauchy complete monoidal categories. In this way (using the correspondence in Remark 2.4), we can define more general morphisms between weak bimonads than the arrows in $\mathsf{Wbm}(\mathscr{M})$ for a fixed $\mathscr{M}$. These more general morphisms do not need to preserve the underlying separable Frobenius monoid R.

3. An example: Weak bimonoids in braided monoidal categories

In this section we show that weak bimonoids in a Cauchy complete braided monoidal category $\mathscr{M}$ induce weak bimonads on $\mathscr{M}$.

Theorem 3.1. *For a monoid (B, μ, η) in a Cauchy complete braided monoidal category $(\mathscr{M}, \otimes, K, c)$, there is a bijection between*

(1) *weak bimonoids of the form $(B, \mu, \eta, \delta, \varepsilon)$ in $\mathscr{M}$;*
(2) *weak bimonads $(\bullet \otimes B, \bullet \otimes \mu, \bullet \otimes \eta, \tau, \tau_0)$ on $\mathscr{M}$ for which the diagram*

$$\begin{tikzcd} X\otimes Y\otimes B \arrow[r, "X\otimes Y\otimes\tau_{K,K}"] \arrow[dr, "\tau_{X,Y}"'] & X\otimes Y\otimes B\otimes B \arrow[d, "X\otimes c_{Y,B}\otimes B"] \\ & X\otimes B\otimes Y\otimes B \end{tikzcd} \tag{3.1}$$

commutes for all objects X, Y of $\mathscr{M}$.

Remark 3.2. Consider a monoid B in a Cauchy complete braided monoidal category $(\mathcal{M}, \otimes K, c)$ such that $\bullet \otimes B$ is a weak bimonad on $\mathcal{M}$. By naturality, for all morphisms $f : K \to X$, $g : K \to Y$, $h : K \to B$, the natural transformation $\tau_{X,Y} : X \otimes Y \otimes B \to X \otimes B \otimes Y \otimes B$ makes

$$\begin{tikzcd}
K \arrow[r, "f\otimes g\otimes h"] \arrow[dr, "f\otimes g\otimes h"] \arrow[d, "h"'] & X \otimes Y \otimes B \arrow[r, "X\otimes Y\otimes \tau_{K,K}"] & X \otimes Y \otimes B \otimes B \arrow[dd, "X\otimes c_{Y,B}\otimes B"] \\
B \arrow[d, "\tau_{K,K}"'] & X \otimes Y \otimes B \arrow[dr, "\tau_{X,Y}"] & \\
B \otimes B \arrow[rr, "f\otimes B\otimes g\otimes B"'] & & X \otimes B \otimes Y \otimes B
\end{tikzcd}$$

commute. Hence (3.1) holds provided that the monoidal unit K is a '*cubic generator*' in the following sense: If, for some morphisms $p, q : X \otimes Y \otimes Z \to W$ in $\mathcal{M}$, the equality $p \circ (f \otimes g \otimes h) = q \circ (f \otimes g \otimes h)$ holds, for *all* morphisms $f : K \to X$, $g : K \to Y$, $h : K \to Z$, then $p = q$.

The monoidal unit is a 'cubic generator', for example, in the symmetric monoidal category $\mathsf{Mod}(k)$ of modules over a commutative ring k. With this observation in mind, Theorem 3.1 includes Szlachányi's description in [21, Corollary 6.5] of *weak bialgebras* over k as *weak bimonads* on $\mathsf{Mod}(k)$.

Proof of Theorem 3.1. Suppose that $(B, \mu, \eta, \delta, \varepsilon)$ is a weak bimonoid in $\mathcal{M}$. By [16, Proposition 3.8], the category of B-modules is monoidal and there is a strong monoidal functor from it to a certain bimodule category ${}_R\mathcal{M}_R$. Furthermore, the resulting monoid R is a separable Frobenius monoid by [16, Proposition 1.4]. In view of Example 1.2, this proves that $\bullet \otimes B$ is a weak bimonad. Its opmonoidal structure comes out with $\tau_{X,Y}$ equal to the composite

$$X \otimes Y \otimes B \xrightarrow{X\otimes Y\otimes \delta} X \otimes Y \otimes B \otimes B \xrightarrow{X\otimes c_{Y,B}\otimes B} X \otimes B \otimes Y \otimes B$$

and $\tau_0 = \varepsilon$. Hence (3.1) is satisfied.

Assume conversely that (2) holds. We claim that $(B, \mu, \eta, \delta := \tau_{K,K}, \varepsilon := \tau_0)$ is a weak bimonoid in $\mathcal{M}$. The functor $\bullet \otimes B$ is opmonoidal and so sends comonoids to comonoids; in particular, it sends the comonoid K in $\mathcal{M}$ to a comonoid, which turns out to be (B, δ, ε). Use (3.1) to write $\tau_{X,Y}$ as $(X \otimes c_{Y,B} \otimes B) \circ (X \otimes Y \otimes \delta)$, for any objects X, Y of $\mathcal{M}$. Substituting this expression in conditions (1.3)–(1.7), we obtain the following commutative diagrams.

$$\begin{tikzcd}
B^2 \arrow[r, "B\otimes\eta\otimes B"] \arrow[d, "B\otimes\delta"] & B^3 \arrow[r, "B\otimes\delta\otimes B"] & B^4 \arrow[r, "B\otimes c^{-1}_{B,B}\otimes B"] & B^4 \arrow[d, "\mu\otimes\mu"'] \\
B^3 \arrow[d, "B\otimes c^{-1}_{B,B}"] & & & B^2 \arrow[d, "\varepsilon\otimes B"'] \\
B^3 \arrow[r, "\mu\otimes B"'] & B^2 \arrow[rr, "\varepsilon\otimes B"'] & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
B^2 \arrow[r, "B\otimes\eta\otimes B"] \arrow[dd, "B\otimes\delta"] & B^3 \arrow[r, "B\otimes\delta\otimes B"] & B^4 \arrow[d, "\mu\otimes\mu"'] \\
& & B^2 \arrow[d, "\varepsilon\otimes B"'] \\
B^3 \arrow[r, "\mu\otimes B"'] & B^2 \arrow[r, "\varepsilon\otimes B"'] & B
\end{tikzcd}
\tag{3.2}$$

$$\begin{tikzcd}
K \arrow[r, "\eta"] \arrow[d, "\eta\otimes\eta"] & B \arrow[rr, "\delta"] & & B^2 \arrow[d, "\delta\otimes B"'] \\
B^2 \arrow[r, "\delta\otimes\delta"'] & B^4 \arrow[r, "B\otimes c^{-1}_{B,B}\otimes B"'] & B^4 \arrow[r, "B\otimes\mu\otimes B"'] & B^3
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[r, "\eta"] \arrow[d, "\eta\otimes\eta"] & B \arrow[r, "\delta"] & B^2 \arrow[d, "\delta\otimes B"'] \\
B^2 \arrow[r, "\delta\otimes\delta"'] & B^4 \arrow[r, "B\otimes\mu\otimes B"'] & B^3
\end{tikzcd}
\tag{3.3}$$

$$
\begin{tikzcd}
B^4 \arrow[rr, "B\otimes c_{B,B}\otimes B"] & & B^4 \arrow[d, "\mu\otimes\mu"'] \\
B^2 \arrow[u, "\delta\otimes\delta"] \arrow[r, "\mu"'] & B \arrow[r, "\delta"'] & B^2
\end{tikzcd}
\tag{3.4}
$$

Condition (3.4) is identical to axiom (b), and the identities in (3.3) are identical to axiom (w) in the definition of a weak bimonoid in [16, Definition 2.1]. Thus we only need to show that, whenever the diagrams in (3.2), (3.3), and (3.4) commute, then axiom (v) in [16, Definition 2.1] holds; that is, the following diagram commutes.

$$
\begin{tikzcd}
B^3 \arrow[r, "B\otimes\delta\otimes B"] \arrow[dr, "\mu^2"] \arrow[dd, "B\otimes\delta\otimes B"'] & B^4 \arrow[r, "B\otimes c^{-1}_{B,B}\otimes B"] & B^4 \arrow[d, "\mu\otimes\mu"] \\
 & B \arrow[dr, "\varepsilon"] & B^2 \arrow[d, "\varepsilon\otimes\varepsilon"] \\
B^4 \arrow[r, "\mu\otimes\mu"'] & B^2 \arrow[r, "\varepsilon\otimes\varepsilon"'] & K
\end{tikzcd}
\tag{3.5}
$$

Commutativity of the lower triangle in (3.5) follows by commutativity of

$$
\begin{tikzcd}
B^3 \arrow[r, equal] \arrow[dd, "B\otimes\delta\otimes B"'] & B^3 \arrow[r, "B\otimes\mu"] \arrow[d, "B\otimes\eta\otimes B^2"] & B^2 \arrow[r, equal] \arrow[d, "B\otimes\eta\otimes B"] & B^2 \arrow[r, "\mu"] \arrow[d, "B\otimes\delta"] & B \arrow[dddd, "\varepsilon"] \\
 & B^4 \arrow[d, "B\otimes\delta\otimes B^2"] & B^3 \arrow[d, "B\otimes\delta\otimes B"] & B^3 \arrow[dd, "\mu\otimes B"] & \\
B^4 \arrow[r, phantom, "(3.2)"] \arrow[dd, "\mu\otimes B\otimes B"'] & B^5 \arrow[d, "\mu\otimes\mu\otimes B"] & B^4 \arrow[r, phantom, "(3.2)"] \arrow[d, "\mu\otimes\mu"] & & \\
 & B^3 \arrow[d, "\varepsilon\otimes B\otimes B"] & B^2 \arrow[d, "\varepsilon\otimes B"] & B^2 \arrow[d, "\varepsilon\otimes B"] & \\
B^3 \arrow[r, "\varepsilon\otimes B\otimes B"'] & B^2 \arrow[r, "\mu"'] & B \arrow[r, equal] & B \arrow[r, "\varepsilon"'] & K.
\end{tikzcd}
$$

The undecorated regions commute by naturality, counitality of δ and by associativity of μ. Commutativity of the upper triangle in (3.5) is proved similarly, making use of the first identity in (3.2). □

4. The antipode

The first attempt to equip Moerdijk's bimonad with an antipode, i.e. to define a Hopf monad, was made by Bruguières and Virelizier in [6]. Here the authors studied bimonads on autonomous monoidal categories such that the (left/right) duals lift to the Eilenberg–Moore category. This generalizes finite dimensional Hopf algebras to the categorical setting.

A more general notion of Hopf monad was introduced in [5] (see also [7]). This is based on the observation of Lawvere [10] that a right adjoint preserves internal homs precisely when Frobenius reciprocity holds; this Frobenius reciprocity condition also appeared in [17, Theorem & Definition 3.5] in the context of Takeuchi bialgebroids. Based on this result, the following definition was proposed also for monoidal categories which are not necessarily closed. For any monad (T, m, u) on a monoidal category $(\mathscr{M}, \otimes, K)$ such that T admits an opmonoidal structure (τ, τ_0) (hence in particular for any (weak) bimonad T on $\mathscr{M}$), there is a canonical natural transformation, given for any objects X, Y of $\mathscr{M}$ by

$$\mathsf{can}_{X,Y} := \big(T(TX \otimes Y) \xrightarrow{\tau_{TX,Y}} T^2X \otimes TY \xrightarrow{m_X \otimes TY} TX \otimes TY\big). \tag{4.1}$$

By the terminology in [5], a bimonad is called a *right Hopf monad* whenever the associated natural transformation (4.1) is invertible. Similarly, a bimonad is a *left Hopf monad* when the analogous natural transformation $T(X \otimes TY) \to TX \otimes TY$ is invertible, and a *Hopf monad* when it is both left and right Hopf.

In this section we propose a definition of a *right weak Hopf monad* T on a monoidal category $\mathscr{M}$ – characterized by the property that, whenever $\mathscr{M}$ is also Cauchy complete, the associated bimonad $\widetilde{T}$ (on another monoidal category) in Remark 2.4, is a right Hopf monad, with analogous definitions for left weak Hopf monads and weak Hopf monads.

Suppose that T is a weak bimonad on a Cauchy complete monoidal category $\mathscr{M}$, and R the corresponding separable Frobenius monoid, with forgetful functor $V : {}_R\mathscr{M}_R \to \mathscr{M}$ and $G \dashv V$. To say that $\widetilde{T}$ is right Hopf is to say that for all $\widetilde{X}, \widetilde{Y} \in {}_R\mathscr{M}_R$, the canonical morphism

$$\widetilde{T}(\widetilde{T}\widetilde{X} \otimes_R \widetilde{Y}) \xrightarrow{\widetilde{\mathsf{can}}_{\widetilde{X},\widetilde{Y}}} \widetilde{T}\widetilde{X} \otimes_R \widetilde{T}\widetilde{Y}$$

is invertible. Since every $\widetilde{X} \in {}_R\mathscr{M}_R$ is (naturally) a retract of one of the form GX, this will be the case precisely when

$$\widetilde{T}(\widetilde{T}GX \otimes_R GY) \xrightarrow{\widetilde{\mathsf{can}}_{GX,GY}} \widetilde{T}GX \otimes_R \widetilde{T}GY$$

is invertible. Now

$$\widetilde{T}(\widetilde{T}GX \otimes_R GY) \cong \widetilde{T}\big(TX \otimes_R (R \otimes Y \otimes R)\big) \cong \widetilde{T}(TX \otimes Y \otimes R)$$

and $V\widetilde{T}(TX \otimes Y \otimes R)$ is a retract of $T(TX \otimes Y)$ by construction of $\widetilde{T}$, while

$$V(\widetilde{T}GX \otimes_R \widetilde{T}GY) \cong TX \,\square\, TY$$

which is a retract of $TX \otimes TY$. Thus we obtain a composite map

$$T(TX \otimes Y) \xrightarrow{q_{X,Y}} V\widetilde{T}(\widetilde{T}GX \otimes_R GY) \xrightarrow{\widetilde{\mathsf{can}}_{GX,GY}} V(\widetilde{T}GX \otimes_R \widetilde{T}GY) \xrightarrow{i_{TX,TY}} TX \otimes TY \tag{4.2}$$

which turns out to be the canonical map $\mathsf{can}_{X,Y}$ associated to T itself.

Now the inclusion $TX \,\square\, TY \to TX \otimes TY$ is the section for a splitting of the idempotent $E_{TX,TY}$ on $TX \otimes TY$ defined in (1.14).

On the other hand, the quotient $T(TX \otimes Y) \to V\widetilde{T}(\widetilde{T}GX \otimes_R GY)$ is the retraction of a splitting of an idempotent $F_{X,Y}$ on $T(TX \otimes Y)$ defined by

$$T(TX \otimes Y) \xrightarrow{T(\delta \circ \eta \otimes TX \otimes Y \otimes \eta)} T(R \otimes R \otimes TX \otimes Y \otimes R) \xrightarrow{T(R \otimes \beta_{TX} \otimes Y \otimes R)} T(R \otimes TX \otimes Y \otimes R)$$
$$\xrightarrow{\lambda_{TX \otimes Y}} T(TX \otimes Y), \tag{4.3}$$

where β_{TX} denotes the left R-action on TX and λ is the natural transformation in (2.4).

To say that $\widetilde{\mathsf{can}}$ is invertible, is to say that can induces an isomorphism between the splittings of the idempotents $F_{X,Y}$ and $E_{TX,TY}$. We then call T a weak right Hopf monad:

Definition 4.1. A weak bimonad T on a monoidal category $(\mathscr{M}, \otimes, K)$ is said to be a *weak right Hopf monad* provided that there is a natural transformation $\chi_{X,Y} : TX \otimes TY \to T(TX \otimes Y)$ such that, for the canonical natural transformation can of T in (4.1), for the idempotent morphisms $E_{TX,TY}$ and $F_{X,Y}$ (4.3), and for any objects X, Y of $\mathscr{M}$,

$$\chi_{X,Y} \circ E_{TX,TY} = \chi_{X,Y} = F_{X,Y} \circ \chi_{X,Y}, \qquad \chi_{X,Y} \circ \mathsf{can}_{X,Y} = F_{X,Y}, \qquad \mathsf{can}_{X,Y} \circ \chi_{X,Y} = E_{TX,TY}. \tag{4.4}$$

The definition just given makes sense for any monoidal category $\mathscr{M}$, but is motivated by the following theorem, which requires $\mathscr{M}$ to be Cauchy complete.

Theorem 4.2. *For any weak bimonad T on a Cauchy complete monoidal category $(\mathscr{M}, \otimes, K)$, and the associated bimonad $\widetilde{T}$ in Remark* 2.4*, the following assertions are equivalent.*

(1) *The canonical natural transformation $\widetilde{\mathsf{can}}$ of $\widetilde{T}$ as in* (4.1)*, is an isomorphism*; *that is, $\widetilde{T}$ is a right Hopf monad.*
(2) *There is a natural transformation $\chi_{X,Y} : TX \otimes TY \to T(TX \otimes Y)$ obeying* (4.4); *that is, T is a weak right Hopf monad.*

Proof. The equations in (4.4) state exactly that the morphism induced by $\chi_{X,Y}$ between the splittings of $F_{X,Y}$ and $E_{TX,TY}$ is inverse to the morphism $\widetilde{\mathsf{can}}_{X,Y}$ induced by $\mathsf{can}_{X,Y}$ between the splittings of $E_{TX,TY}$ and $F_{X,Y}$. $\square$

Remark 4.3. Consider a weak right Hopf monad T on a Cauchy complete monoidal category $\mathscr{M}$ with corresponding separable Frobenius monoid R. By Theorem 4.2 and [5, Theorem 3.6] we conclude that whenever the category of R-bimodules is right closed, this closed structure lifts to the Eilenberg–Moore category $\mathscr{M}^T$. The category of R-bimodules is right closed whenever $\mathscr{M}$ is right closed (in which case the internal homs are defined by splitting an appropriate idempotent natural transformation).

Next we show that, as expected, a weak bimonoid B in a Cauchy complete braided monoidal category $\mathscr{M}$, induces a right weak Hopf monad $\bullet \otimes B$ on $\mathscr{M}$ if and only if it is a weak Hopf monoid in the sense of [1,16].

Lemma 4.4. *For an arbitrary category $\mathscr{C}$, consider a functor $T : \mathscr{C} \to \mathscr{C}$ which admits both a monad structure $\underline{T} = (T, m, u)$ and a comonad structure $\overline{T} = (T, d, e)$. Denote by $\underline{U} : \mathscr{C}^{\underline{T}} \to \mathscr{C}$ and by $\overline{U} : \mathscr{C}^{\overline{T}} \to \mathscr{C}$ the corresponding forgetful functors with respective left adjoint $\underline{F} : \mathscr{C} \to \mathscr{C}^{\underline{T}}$ and right adjoint $\overline{F} : \mathscr{C} \to \mathscr{C}^{\overline{T}}$. The following monoids* (*in* Set) *are isomorphic.*

(1) *The monoid of natural transformations $\overline{F}\underline{U} \to \overline{F}\underline{U}$, with multiplication given by the composition of natural transformations.*
(2) *The monoid of those natural transformations $\gamma : \overline{F}T \to \overline{F}T$ for which $\overline{F}m \circ \gamma T = \gamma \circ \overline{F}m$, with multiplication given by the composition of natural transformations.*
(3) *The monoid of natural transformations $T \to T$, with multiplication given by the 'convolution product' $\varphi * \varphi' := m \circ T\varphi' \circ \varphi T \circ d$.*

Proof. (1) $\cong$ (2) The stated isomorphism is given by the maps $\mathsf{Nat}(\overline{F}\underline{U}, \overline{F}\underline{U}) \ni \beta \mapsto \beta\underline{F}$, with the inverse $\gamma \mapsto \overline{F}\underline{U}\underline{\kappa} \circ \gamma\underline{U} \circ \overline{F}u\underline{U}$, where $\underline{\kappa}$ is the counit of the adjunction $\underline{F} \dashv \underline{U}$.

(1) $\cong$ (3) This is the adjunction isomorphism $\mathsf{Nat}(\overline{F}\underline{U}, \overline{F}\underline{U}) \cong \mathsf{Nat}(\overline{U}\,\overline{F}, \underline{U}\underline{F})$. $\square$

For a functor T as in Lemma 4.4, one may consider the so-called '*fusion operator*' in [20],

$$\gamma := \big(T^2 \xrightarrow{dT} T^3 \xrightarrow{Tm} T^2\big). \tag{4.5}$$

Clearly, it belongs to the monoid in Lemma 4.4 (2). The corresponding element of the isomorphic monoid in Lemma 4.4 (3) is the identity natural transformation $T \to T$. (Hence, incidentally, Lemma 4.4 provides an alternative proof of [13, Theorem 5.5].)

Lemma 4.5. *For a weak bimonoid $(B, \mu, \eta, \delta, \varepsilon)$ in a Cauchy complete braided monoidal category $(\mathscr{M}, \otimes, K, c)$, and its induced weak bimonad $T := \bullet \otimes B$, the following assertions hold, for any objects X, Y of $\mathscr{M}$.*

(i) *For the natural transformation* (4.1) *of $T = \bullet \otimes B$,*

$$\mathsf{can}_{X,Y} = (X \otimes c_{Y,B} \otimes B) \circ (X \otimes Y \otimes \gamma_K) \otimes \big(X \otimes c^{-1}_{Y,B} \otimes B\big),$$

where γ is the fusion operator (4.5) *for the monad and comonad $\bullet \otimes B$.*

(ii) *The idempotent natural transformation $E_{TX,TY}$ on $TX \otimes TY$* (1.14) *satisfies*

$$E_{TX,TY} = (X \otimes c_{Y,B} \otimes B) \circ (X \otimes Y \otimes E_{TK,TK}) \otimes \big(X \otimes c^{-1}_{Y,B} \otimes B\big). \tag{4.6}$$

Moreover, $\bullet \otimes E_{TK,TK}$ belongs to the monoid in Lemma 4.4 (2) *and the corresponding element of the isomorphic monoid in Lemma* 4.4 (3) *is $\bullet \otimes t$, where t is the composite*

$$B \xrightarrow{B\otimes\eta} B^2 \xrightarrow{B\otimes\delta} B^3 \xrightarrow{c_{B,B}\otimes B} B^3 \xrightarrow{B\otimes\mu} B^2 \xrightarrow{B\otimes\varepsilon} B.$$

(iii) *The idempotent natural transformation $F_{X,Y}$ on $T(TX \otimes Y)$* (4.3) *satisfies*

$$F_{X,Y} = (X \otimes c_{Y,B} \otimes B) \circ (X \otimes Y \otimes F_{K,K}) \circ \big(X \otimes c^{-1}_{Y,B} \otimes B\big). \tag{4.7}$$

Moreover, $\bullet \otimes F_{K,K}$ belongs to the monoid in Lemma 4.4 (2) *and the corresponding element of the isomorphic monoid in Lemma* 4.4 (3) *is $\bullet \otimes r$, where r is the composite*

$$B \xrightarrow{\eta\otimes B} B^2 \xrightarrow{\delta\otimes B} B^3 \xrightarrow{B\otimes c_{B,B}} B^3 \xrightarrow{\mu\otimes B} B^2 \xrightarrow{\varepsilon\otimes B} B.$$

(iv) *If in addition $T := \bullet \otimes B$ is a weak right Hopf monad*; *that is, there exists a natural transformation χ obeying* (4.4), *then*

$$\chi_{X,Y} = (X \otimes c_{Y,B} \otimes B) \circ (X \otimes Y \otimes \chi_{K,K}) \otimes \big(X \otimes c^{-1}_{Y,B} \otimes B\big) \tag{4.8}$$

and $\bullet \otimes \chi_{K,K}$ belongs to the monoid in Lemma 4.4 (2).

Proof. Assertion (i) is immediate by relation (3.1) between the opmonoidal structure $\tau_{X,Y}$ of $T = \bullet \otimes B$ and the comultiplication $\delta = \tau_{K,K}$ in $B = TK$.

(ii) Eq. (4.6) follows from the formula (1.14) for $E_{TX,TY}$ and $E_{TK,TK}$. Then the morphism

$$E_{TK,TK} = \big(B^2 \xrightarrow{B^2\otimes\eta} B^3 \xrightarrow{B^2\otimes\delta} B^4 \xrightarrow{B\otimes c_{B,B}\otimes B} B^4 \xrightarrow{\mu\otimes\mu} B^2\big)$$

renders commutative the first diagram in

$$\begin{tikzcd} B^3 \arrow[r, "B\otimes E_{TK,TK}"] \arrow[d, "\mu\otimes B"'] & B^3 \arrow[d, "\mu\otimes B"] \\ B^2 \arrow[r, "E_{TK,TK}"'] & B^2 \end{tikzcd} \qquad \begin{tikzcd} B^2 \arrow[r, "E_{TK,TK}"] \arrow[d, "B\otimes\delta"'] & B^2 \arrow[d, "B\otimes\delta"] \\ B^3 \arrow[r, "E_{TK,TK}\otimes B"'] & B^3 \end{tikzcd} \tag{4.9}$$

by associativity of μ. By self-duality of the axioms of a weak bimonoid, the dual of (3.2) holds; that is, the first diagram in

$$\begin{tikzcd} B \arrow[r, "\eta\otimes B"] \arrow[d, "\eta\otimes B"'] & B^2 \arrow[r, "\delta\otimes B"] & B^3 \arrow[r, "B\otimes c^{-1}_{B,B}"] & B^3 \arrow[dd, "B\otimes\mu"] \\ B^2 \arrow[d, "\delta\otimes\delta"'] & & & \\ B^4 \arrow[r, "B\otimes c^{-1}_{B,B}\otimes B"'] & B^4 \arrow[r, "B\otimes\mu\otimes B"'] & B^3 \arrow[r, "B\otimes\varepsilon\otimes B"'] & B^2 \end{tikzcd} \qquad \begin{tikzcd} B^3 \arrow[r, "E_{TK,TK}\otimes B"] & B^3 \arrow[d, "B\otimes\varepsilon\otimes B"] \\ B^2 \arrow[u, "B\otimes\delta"] \arrow[r, "E_{TK,TK}"'] & B^2 \end{tikzcd}$$

commutes. Tensoring on the left with B and then composing with $\mu \otimes B$ gives commutativity of the diagram on the right. It follows by coassociativity of δ that also the second diagram in (4.9) commutes. This proves that $\bullet \otimes E_{TK,TK}$ belongs to the monoid in Lemma 4.4 (2). The corresponding element of the isomorphic monoid in Lemma 4.4 (3) is $(B \otimes \varepsilon) \circ E_{TK,TK} \circ (\eta \otimes B) = t$ as stated, by unitality of μ.

(iii) Similarly to part (ii), one easily checks that

$$F_{X,Y} = (X \otimes c_{Y,B} \otimes B) \circ \big(X \otimes Y \otimes (\mu \otimes \varepsilon \circ \mu \otimes B) \circ \big(B \otimes c^{-1}_{B,B} \circ \delta \circ \eta \otimes \delta\big)\big) \circ \big(X \otimes c^{-1}_{Y,B} \otimes B\big),$$

which proves (4.8). By associativity of μ and by coassociativity of δ,

$$F_{K,K} = \big(B^2 \xrightarrow{B\otimes\eta\otimes B} B^3 \xrightarrow{B\otimes\delta\otimes\delta} B^5 \xrightarrow{B\otimes c^{-1}_{B,B}\otimes B^2} B^5 \xrightarrow{\mu\otimes\mu\otimes B} B^3 \xrightarrow{B\otimes\varepsilon\otimes B} B^2\big)$$

makes commute both diagrams in

$$\begin{tikzcd} B^3 \arrow[r, "B\otimes F_{K,K}"] \arrow[d, "\mu\otimes B"'] & B^3 \arrow[d, "\mu\otimes B"] \\ B^2 \arrow[r, "F_{K,K}"'] & B^2 \end{tikzcd} \qquad \begin{tikzcd} B^2 \arrow[r, "F_{K,K}"] \arrow[d, "B\otimes\delta"'] & B^2 \arrow[d, "B\otimes\delta"] \\ B^3 \arrow[r, "F_{K,K}\otimes B"'] & B^3. \end{tikzcd} \tag{4.10}$$

Thus $\bullet \otimes F_{K,K}$ is an element of the monoid in Lemma 4.4 (2). By unitality of μ and counitality of δ, the corresponding element $(B \otimes \varepsilon) \circ F_{K,K} \circ (\eta \otimes B)$ of the isomorphic monoid in Lemma 4.4 (3) is the stated morphism r.

(iv) In (4.2) we have seen a relationship between the canonical morphism can of T and the canonical morphism $\widetilde{\mathsf{can}}$ of the weakly lifted bimonad $\widetilde{T}$. Using this along with part (i), we deduce that $\widetilde{\mathsf{can}}_{R\otimes X\otimes R, R\otimes Y\otimes R}$ is equal to

$$p_{TX,TY} \circ (X \otimes c_{Y,B} \otimes B) \circ (X \otimes Y \otimes i_{TK,TK} \circ \widetilde{\mathsf{can}}_{R\otimes R, R\otimes R} \circ q_{K,K}) \circ \big(X \otimes c^{-1}_{Y,B} \otimes B\big) \circ j_{X,Y}$$

where $p_{TX,TY}$ is the epi part of the splitting of $E_{TX,TY}$, and $j_{X,Y}$ is the mono part of the splitting of $F_{X,Y}$. Hence in view of (4.6) and (4.7), $\widetilde{\mathsf{can}}^{-1}_{R\otimes X\otimes R,R\otimes Y\otimes R}$ is equal to

$$q_{X,Y}\circ(X\otimes c_{Y,B}\otimes B)\circ\big(X\otimes Y\otimes j_{K,K}\circ\widetilde{\mathsf{can}}^{-1}_{R\otimes R,R\otimes R}\circ p_{TK,TK}\big)\circ\big(X\otimes c^{-1}_{Y,B}\otimes B\big)\circ i_{TX,TY}.$$

Thus for $\chi_{X,Y}=j_{X,Y}\circ\widetilde{\mathsf{can}}^{-1}_{R\otimes X\otimes R,R\otimes Y\otimes R}\circ p_{TX,TY}$, the required condition (4.8) holds.

We need to show that $\chi_{K,K}$ induces a natural transformation as in Lemma 4.4 (2). By part (i), $\mathsf{can}_{K,K}=\gamma_K$ induces such a natural transformation. Hence in view of (4.2), since $i_{TX,TY}$ is a morphism of left B-modules and of right B-comodules, and by (4.10),

$$(\mu\otimes B)\circ(B\otimes j_{K,K})\circ\big(B\otimes\widetilde{\mathsf{can}}^{-1}_{R\otimes R,R\otimes R}\big)=j_{K,K}\circ\widetilde{\mathsf{can}}^{-1}_{R\otimes R,R\otimes R}\circ(\mu\otimes_R B)\quad\text{and}$$
$$(B\otimes\delta)\circ j_{K,K}\circ\widetilde{\mathsf{can}}^{-1}_{R\otimes R,R\otimes R}=(j_{K,K}\otimes B)\circ\big(\widetilde{\mathsf{can}}^{-1}_{R\otimes R,R\otimes R}\otimes B\big)\circ(B\otimes_R\delta).$$

Since $p_{TK,TK}$ is a morphism of left B-modules and of right B-comodules, this implies that $\chi_{K,K}=j_{K,K}\circ\widetilde{\mathsf{can}}^{-1}_{R\otimes R,R\otimes R}\circ p_{TK,TK}$ belongs to the monoid in Lemma 4.4 (2). □

Theorem 4.6. *For a weak bimonoid B in a Cauchy complete braided monoidal category $(\mathscr{M},\otimes,K,c)$, the induced functor $\bullet\otimes B$ is a weak right Hopf monad if and only if B is a weak Hopf monoid.*

Proof. By Lemma 4.5, $\bullet\otimes B$ is a weak right Hopf monad if and only if (using the same notation in the lemma) there is an element $X\otimes Y\otimes\chi_{K,K}:X\otimes Y\otimes B\otimes B\to X\otimes Y\otimes B\otimes B$ of the monoid in Lemma 4.4 (2), such that

$$\chi_{K,K}\circ E_{TK,TK}=\chi_{K,K}=F_{K,K}\circ\chi_{K,K},\qquad \chi_{K,K}\circ\mathsf{can}_{K,K}=F_{K,K},\qquad \mathsf{can}_{K,K}\circ\chi_{K,K}=E_{TK,TK}.$$

By Lemma 4.4, this is equivalent to the existence of a morphism $\nu:B\to B$, such that

$$\nu*r=\nu=t*\nu,\qquad \nu*B=t,\qquad B*\nu=r,$$

where the morphisms $t,r:B\to B$ are introduced in Lemma 4.5 and $*$ denotes the convolution product $f*g=\mu\circ(f\otimes g)\circ\delta$, for any morphisms $f,g:B\to B$ in $\mathscr{M}$. □

Finally we turn to connections between right weak Hopf monads and left weak Hopf monads. Conditions (1.3)–(1.7) are invariant under replacing the monoidal product $\otimes$ with the opposite product $\overline{\otimes}$. That is, if (T,m,u,τ_0,τ) is a weak bimonad on a monoidal category $(\mathscr{M},\otimes,K)$, then $(T,m,u,\overline{\tau}_0,\overline{\tau})$ is a weak bimonad on $(\mathscr{M},\overline{\otimes},K)$, where $\overline{\tau}_0=\tau_0:TK\to K$ and $\overline{\tau}_{X,Y}=\tau_{Y,X}:T(X\,\overline{\otimes}\,Y)=T(Y\otimes X)\to TY\otimes TX=TX\,\overline{\otimes}\,TY$. We say that a weak bimonad (T,m,u,τ_0,τ) is a *left weak Hopf monad* on a monoidal category $(\mathscr{M},\otimes,K)$ provided that $(T,m,u,\overline{\tau}_0,\overline{\tau})$ is a right weak Hopf monad on $(\mathscr{M},\overline{\otimes},K)$. Clearly, this means that the *left canonical map*

$$T(X\otimes TY)\xrightarrow{\tau_{X,TY}}TX\otimes T^2Y\xrightarrow{TX\otimes m_Y}TX\otimes TY$$

induces an isomorphism between the retracts of $T(X\otimes TY)$ and $TX\otimes TY$ defined as above.

Some known facts about right weak Hopf monads immediately translate to left weak Hopf monads: obviously, for a weak Hopf monoid $(B,\mu,\eta,\delta,\varepsilon,\nu)$ in a braided monoidal category $(\mathscr{M},\otimes,K,c)$, the same data $(B,\mu,\eta,\delta,\varepsilon,\nu)$ describe a weak Hopf monoid in $(\mathscr{M},\overline{\otimes},K,\bar{c})$, where the braiding is given by $\bar{c}_{X,Y}=c_{Y,X}:X\,\overline{\otimes}\,Y=Y\otimes X\to X\otimes Y=Y\,\overline{\otimes}\,X$. From Theorem 4.6 we deduce

Proposition 4.7. *For a weak bimonoid B in a Cauchy complete braided monoidal category $(\mathscr{M},\otimes,K,c)$, the following assertions are equivalent*:

(1) *the weak bimonad* $\bullet \otimes B$ *on* $(\mathcal{M}, \otimes, K)$ *is a right weak Hopf monad*;
(2) *the weak bimonad* $\bullet \otimes B = B \overline{\otimes} \bullet$ *on* $(\mathcal{M}, \overline{\otimes}, K)$ *is a left weak Hopf monad*;
(3) *B is a weak Hopf monoid in* $(\mathcal{M}, \otimes, K, c)$;
(4) *B is a weak Hopf monoid in* $(\mathcal{M}, \overline{\otimes}, K, \bar{c})$.

In particular, the equivalence of the second and fourth assertions says that a weak bimonoid in a Cauchy complete braided monoidal category is a weak Hopf monoid if and only if it induces a left weak Hopf monad by tensoring on the left.

Our next aim is to describe those weak bimonoids $(B, \mu, \eta, \delta, \varepsilon)$ in a Cauchy complete braided monoidal category $(\mathcal{M}, \otimes, K, c)$ for which $\bullet \otimes B$ is both a right and a left weak Hopf monad.

Consider the weak bimonoid B^{op} in $(\mathcal{M}, \otimes, K, c^{-1})$ with the same comonoid structure (δ, ε) of B, multiplication $\mu^{op} := \mu \circ c^{-1}_{B,B}$ and unit η. Observe that via $c_{\bullet,B} : \bullet \otimes B \to B \otimes \bullet$, the weak bimonads $\bullet \otimes B$ and $B^{op} \otimes \bullet$ are isomorphic. Hence the following assertions on B are equivalent:

(1) the weak bimonad $\bullet \otimes B$ on $(\mathcal{M}, \otimes, K)$ is a left weak Hopf monad;
(2) the weak bimonad $B^{op} \otimes \bullet$ on $(\mathcal{M}, \otimes, K)$ is a left weak Hopf monad;
(3) B^{op} is a weak Hopf monoid in $(\mathcal{M}, \otimes, K, c^{-1})$;
(4) there is a morphism $\nu^{op} : B \to B$ (the *antipode* for B^{op}) such that the following diagrams commute.

$$\begin{tikzcd}
B \arrow[r, "\delta"] \arrow[d, "\eta\otimes B"'] & B^2 \arrow[r, "\nu^{op}\otimes B"] & B^2 \arrow[d, "c^{-1}_{B,B}"] \\
B^2 \arrow[d, "\delta\otimes B"'] & & B^2 \arrow[d, "\mu"] \\
B^3 \arrow[r, "B\otimes\mu"'] & B^2 \arrow[r, "B\otimes\varepsilon"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\delta"] \arrow[d, "B\otimes\eta"'] & B^2 \arrow[r, "B\otimes\nu^{op}"] & B^2 \arrow[d, "c^{-1}_{B,B}"] \\
B^2 \arrow[d, "B\otimes\delta"'] & & B^2 \arrow[d, "\mu"] \\
B^3 \arrow[r, "\mu\otimes B"'] & B^2 \arrow[r, "\varepsilon\otimes B"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\delta^2"] \arrow[dd, "\nu^{op}"'] & B^3 \arrow[d, "\nu^{op}\otimes B\otimes\nu^{op}"] \\
 & B^3 \arrow[d, "(\mu\circ c^{-1}_{B,B})^2"] \\
B \arrow[r, equal] & B
\end{tikzcd}$$

We shall use the notations s, r, t in weak Hopf monoids, as in [16]; the forms of t and r are recalled in Lemma 4.5 above. The left-bottom path in the first diagram in (4) above (playing the role of t^{op}) is equal to s. The left-bottom path in the second diagram is conveniently denoted by r^{op}. The four morphisms s, t, r, r^{op} obey the following four equations

$$\nu \circ s = r, \qquad \nu \circ r^{op} = t, \qquad s \circ \nu = t, \qquad r^{op} \circ \nu = r. \tag{4.11}$$

The first one is (15) in Appendix B of [16], and the others are proved by similar steps.

Finally we are ready to provide the desired characterization:

Theorem 4.8. *For a weak bimonoid B in a Cauchy complete braided monoidal category $(\mathcal{M}, \otimes, K, c)$, the following conditions are equivalent*:

(1) *the weak bimonad* $\bullet \otimes B$ *on* $(\mathcal{M}, \otimes, K)$ *is both a right and a left weak Hopf monad*;
(2) *B is a weak Hopf monoid in* $(\mathcal{M}, \otimes, K, c)$ *and* B^{op} *is a weak Hopf monoid in* $(\mathcal{M}, \otimes, K, c^{-1})$;
(3) *B is a weak Hopf monoid in* $(\mathcal{M}, \otimes, K, c)$ *with an invertible antipode* ν;
(4) B^{op} *is a weak Hopf monoid in* $(\mathcal{M}, \otimes, K, c^{-1})$ *with an invertible antipode* ν^{op}.

In case (3), ν^{-1} *will be an antipode for* B^{op}; *in case* (4), $(\nu^{op})^{-1}$ *will be an antipode for* B.

Proof. We have already seen that (1) and (2) are equivalent. We show that (3) is equivalent to (2); the equivalence of (4) and (2) is similar.

Assume first that the (3) holds: B is a weak Hopf monoid in $(\mathcal{M}, \otimes, K, c)$ with an invertible antipode ν. In order to see that ν^{-1} provides an antipode for the weak Hopf monoid B^{op}, compose with ν^{-1} on the left the antipode axioms for B. The first two antipode axioms for B^{op} follow from the respective axiom for B, using the anti-multiplicativity of ν [16, (17)] and the identities $\nu \circ s = r$ and $\nu \circ r^{op} = t$, respectively. The third antipode axiom for B^{op} follows from the corresponding axiom for B by [16, (17), (6)]. Thus (2) holds.

Conversely, assume that (2) holds: B admits an antipode ν and B^{op} admits an antipode ν^{op}. In order to see that ν^{op} is a left inverse of ν, use associativity of the multiplication, and coassociativity of the comultiplication in B to compute the convolution product $(\nu^{op} \circ \nu) * \nu * B = \mu^2 \circ ((\nu^{op} \circ \nu) \otimes \nu \otimes B) \circ \delta^2$ in two different ways. On one hand,

$$\big(\big(\nu^{op} \circ \nu\big) * \nu\big) * B = \big(\mu \circ c^{-1}_{B,B} \circ \big(B \otimes \nu^{op}\big) \circ \delta \circ \nu\big) * B = \big(r^{op} \circ \nu\big) * B = r * B = B.$$

The first equality follows by anti-comultiplicativity of ν, cf. [16, (16)]. The second equality is a consequence of one of the antipode axioms for B^{op}. The third equality follows by the identity $r^{op} \circ \nu = r$ (4.11) and the last equality is easily derived from the form of r and axiom (b) in [16]. On the other hand,

$$\big(\nu^{op} \circ \nu\big) * (\nu * B) = \big(\nu^{op} \circ \nu\big) * t = \big(\nu^{op} \circ \nu\big) * (s \circ \nu) = \mu \circ c^{-1}_{B,B} \circ \big(s \otimes \nu^{op}\big) \circ \delta \circ \nu = \nu^{op} \circ \nu.$$

The first equality follows by one of the antipode axioms for B. The second equality follows by the identity $s \circ \nu = t$ (4.11) and the third one follows by anti-comultiplicativity of ν, cf. [16, (16)]. The last equality follows by the weak Hopf monoid identity $t * \nu = \nu$ applied to B^{op}. A symmetrical reasoning shows that ν^{op} is also a right inverse of ν: By [16, (17)], one of the antipode axioms for B^{op}, the identities $\nu \circ s = r$ (4.11) and $r * B = B$,

$$\big(\big(\nu \circ \nu^{op}\big) * \nu\big) * B = B.$$

On the other hand, by one of the antipode axioms for B, the identity $\nu \circ r^{op} = t$ (4.11), by [16, (17)] and the weak Hopf monoid identity $\nu * r = \nu$ applied to B^{op},

$$\big(\nu \circ \nu^{op}\big) * (\nu * B) = \nu \circ \nu^{op}.$$

Thus (3) holds. □

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