A new class of Kadison–Singer algebras

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We show that the projection lattice generated by a maximal nest and a rank one projection in a separable infinite-dimensional Hilbert space is in general reflexive. Moreover we show that the corresponding reflexive algebra has a maximal triangular property, equivalently, it is a Kadison–Singer algebra. Similar results are also obtained for the lattice generated by a finite nest and a projection in a finite factor.

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1. Introduction

The development of the theory of non-self-adjoint operator algebras is parallel to that of the self-adjoint theory. The maximal triangular algebras introduced by Kadison and Singer \cite{11} and the reflexive algebras are two important classes of non-self-adjoint operator algebras. Many important results obtained in non-self-adjoint algebras depend on relations to compact operators which are almost absent in the self-adjoint theory. Therefore there is no fruitful interaction between these two theories in the past. In order to use the powerful tools in self-adjoint operator algebras, Liming Ge and Wei Yuan \cite{5} introduced a new class of non-self-adjoint algebras, Kadison–Singer algebras, which are reflexive and maximal with respect to their diagonals. The corresponding reflexive lattice is called Kadison–Singer lattice. Kadison–Singer algebras combine triangularity, reflexivity and von Neumann algebra properties together and makes the techniques of self-adjoint operator algebras more involved in the study of non-self-adjoint operator algebras. The results in \cite{5} and \cite{6} also establish surprising connections between classical geometry and non-self-adjoint operator algebras.

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Our aim in this paper is to give a new class of Kadison–Singer algebras. The corresponding Kadison–Singer lattices are generated by a nest and a projection. The main difficult is to show that the corresponding reflexive lattice is minimally generating for the von Neumann algebra it generates.

In the following we recall some basic definitions and results needed later. For basic theory on operator algebras, we refer to [10]. For the theory of non-self-adjoint operator algebras, we refer to [1–3,7,9,13] and [12]. For the basics of free probability theory, we refer to [14].

Suppose $\mathcal{H}$ is a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$. For a subset $\mathcal{F}$ of $\mathcal{B}(\mathcal{H})$, let $\mathcal{F}' = \{ T \in \mathcal{B}(\mathcal{H}) : TA = AT, \text{ for all } A \in \mathcal{F} \}$ and $\mathcal{F}'' = (\mathcal{F}')'$. Let $\mathcal{P}$ be a set of (orthogonal) projections in $\mathcal{B}(\mathcal{H})$. Define $\mathcal{Alg}(\mathcal{P}) = \{ T \in \mathcal{B}(\mathcal{H}) : TP = PT, \text{ for all } P \in \mathcal{P} \}$. Then $\mathcal{Alg}(\mathcal{P})$ is a weak-operator closed subalgebra of $\mathcal{B}(\mathcal{H})$. Similarly, for a subset $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, define $\mathcal{Lat}(\mathcal{A}) = \{ P \in \mathcal{B}(\mathcal{H}) : P \text{ is a projection}, TP = PT, \text{ for all } T \in \mathcal{A} \}$. Then $\mathcal{Lat}(\mathcal{A})$ is a strong-operator closed lattice of projections. A subalgebra $\mathcal{B}$ of $\mathcal{B}(\mathcal{H})$ is said to be a reflexive (operator) algebra if $\mathcal{B} = \mathcal{Alg}(\mathcal{Lat}(\mathcal{B}))$. Similarly, a lattice $\mathcal{L}$ of projections in $\mathcal{B}(\mathcal{H})$ is called a reflexive lattice (of projections) if $\mathcal{L} = \mathcal{Lat}(\mathcal{Alg}(\mathcal{L}))$. A nest is a totally ordered reflexive lattice. If $\mathcal{L}$ is a nest, then $\mathcal{Alg}(\mathcal{L})$ is called a nest algebra. Kadison and Singer [11] showed that nest algebras are the only maximal triangular reflexive algebras (with a commutative lattice of invariant projections).

The following definition was introduced in [5].

**Definition.** A subalgebra $\mathscr{A}$ of $\mathcal{B}(\mathcal{H})$ is called a Kadison–Singer (operator) algebra (or KS-algebra) if $\mathscr{A}$ is reflexive and maximal with respect to the diagonal subalgebra $\mathscr{A} \cap \mathcal{A}^*$ of $\mathscr{A}$, in the sense that if there is another reflexive subalgebra $\mathcal{B}$ of $\mathcal{B}(\mathcal{H})$ such that $\mathscr{A} \subseteq \mathcal{B}$ and $\mathcal{B} \cap \mathcal{A}^* = \mathscr{A} \cap \mathcal{A}^*$, then $\mathscr{A} = \mathcal{B}$. When the diagonal of a KS-algebra $\mathscr{A}$ is a factor, we say $\mathscr{A}$ is a Kadison–Singer factor (or KS-factor). A lattice $\mathcal{L}$ of projections in $\mathcal{B}(\mathcal{H})$ is called a Kadison–Singer lattice (or KS-lattice) if $\mathcal{L}$ is a minimal reflexive lattice that generates the von Neumann algebra $\mathcal{L}''$, or equivalently $\mathcal{L}$ is reflexive and $\mathcal{Alg}(\mathcal{L})$ is a Kadison–Singer algebra.

Kadison–Singer algebras with hyperfinite factors as diagonals were constructed in [5]. Let $G_n$ be the free product of $\mathbb{Z}_2$ with itself $n$ times, for $n \geq 2$ or $n = \infty$. Let $\mathcal{L}_{G_n}$ be the group von Neumann algebra associated to the group $G_n$ [10]. If $U_1, \ldots, U_n$ are canonical generators for $\mathcal{L}_{G_n}$ corresponding to the generators of $G_n$ with $U_j^2 = I$, then $\frac{I - U_j}{2}$ ($j = 1, \ldots, n$) are projections of trace $\frac{1}{2}$. Let $\mathcal{F}_n$ be the lattice consisting of these $n$ free projections with trace $\frac{1}{2}$ and 0 and $I$. Then $\mathcal{F}_n$ is a minimal lattice which generates $\mathcal{L}_{G_n}$ as a von Neumann algebra. It is shown in [6] that $\mathcal{Alg}(\mathcal{F}_n)$ ($n \leq 3$) is a Kadison–Singer algebra and $\mathcal{Lat}(\mathcal{Alg}(\mathcal{F}_3)) \setminus \{ 0, I \}$ is homeomorphic to the two-dimensional sphere $S^2$.

This paper contains three sections. In Section 2, we consider the lattices generated by a maximal nest (i.e. the nest generates a maximal abelian self-adjoint algebra) and a rank one projection on a separable infinite-dimensional Hilbert space. We show that these lattices are Kadison–Singer lattices in general and the corresponding algebras are Kadison–Singer algebras with diagonals equal to $\mathbb{C}I$. Three examples are given. Assume that $\mathcal{H}$ is a Hilbert space and $\mathcal{M}$ is a finite factor in $\mathcal{B}(\mathcal{H})$ with a normal faithful tracial state $\tau$. Suppose $\mathcal{L}_0 = \{ 0, P_1, P_2, \ldots, P_n = I \}$ ($n \geq 2$) is a nest of projections in $\mathcal{M}$ with $\tau(P_k) = \frac{k}{n}$ ($k = 1, 2, \ldots, n$) and $Q \in \mathcal{M}$ is a projection with $\tau(Q) = \frac{1}{2}$. Let $\mathcal{L}$ be the lattice generated by $\mathcal{L}_0$ and $Q$. In Section 3, we give a sufficient condition for $\mathcal{L}$ to be a Kadison–Singer lattice.

2. Lattice generated by a maximal nest and a rank one projection

In this section, we shall assume that $\mathcal{H}$ is a separable infinite-dimensional Hilbert space and $\mathcal{N}$ is a nest in $\mathcal{B}(\mathcal{H})$ such that $\mathcal{N}'$ is a maximal abelian self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. In this case $\mathcal{N}' = \mathcal{N}''$ (denote it by $\mathfrak{A}$) and $\mathcal{N}$ is a maximal nest. Suppose $\xi \in \mathcal{H}$ is a separating vector for $\mathfrak{A}$ and $P_\xi$ is the orthogonal projection from $\mathcal{H}$ onto the one dimensional subspace of $\mathcal{H}$ generated by $\xi$.

**Lemma 2.1.** With the above notations, we have that the von Neumann algebra generated by $\mathcal{N}$ and $P_\xi$ is $\mathcal{B}(\mathcal{H})$, i.e. $\{ \mathcal{N}, P_\xi \}' = \mathcal{B}(\mathcal{H})$.

**Proof.** Let $Q$ be a projection in $\{ \mathcal{N}, P_\xi \} \subset \mathcal{N}' (= \mathfrak{A})$. Then $QP_\xi = P_\xi Q$ and therefore $Q\xi = P_\xi Q\xi = (Q\xi, \xi)\xi$. Thus $(Q - (Q\xi, \xi)I)\xi = 0$. Since $\xi$ is separating for $\mathfrak{A}$, we have $Q = (Q\xi, \xi)I$. Since $Q$ is a projection, $Q = 0$ or $Q = I$. This shows that $\{ \mathcal{N}, P_\xi \}' = \mathbb{C}I$, or equivalently, $\{ \mathcal{N}, P_\xi \}'' = \mathcal{B}(\mathcal{H})$. \qed
Since $\xi$ is a separating vector for $\mathcal{A}$, it follows from [11] that the map $\Phi(Q) = \langle Q\xi, \xi \rangle$ is an order preserving homeomorphism of $(\mathcal{N}, \prec)$ onto a compact subset $\delta$ of $[0, 1]$. From now on, for any $t \in \delta$, we denote $Q_t = \Phi^{-1}(t)$ the inverse image of $t$ under $\Phi$. In other words, $Q_t \in \mathcal{N}$ and $\langle Q_t\xi, \xi \rangle = t$.

By the definition, we have $Q_0 = 0$, $Q_1 = I$ and $Q_{t_1} < Q_{t_2}$ for $t_1, t_2 \in \delta$ with $t_1 < t_2$. Also we have $Q_{t_1} = Q_{t_2}$ if and only if $t_1 = t_2$.

**Lemma 2.2.** Let $\mathcal{L} = \{Q, Q \lor P_\xi : t \in \delta\}$. Then $\mathcal{L}$ is a lattice.

**Proof.** It suffices to show $Q_{t_1} \land (Q_{t_2} \lor P_\xi) = Q_{\min(t_1, t_2)}$, for $t_1$ and $t_2$ in $\delta \setminus \{1\}$. It is clear that the equality holds when $t_1 \leq t_2$. It remains to consider the case when $t_1 > t_2$. Let $P = Q_{t_1} \land (Q_{t_2} \lor P_\xi)$. Note that $Q_{t_2} \leq P \leq Q_{t_1}$ if $P = Q_{t_2}$, our claim follows. Now we assume $P > Q_{t_2}$. Then $Q_{t_2} < P \leq Q_{t_1} \lor P_\xi$. Since the dimension of $(Q_{t_2} \lor P_\xi - Q_{t_1})(\mathcal{H})$ is at most 1, we have $P = Q_{t_2} \lor P_\xi$. It follows that $\xi \in Q_{t_1}(\mathcal{H})$ and $Q_{t_1}\xi = \xi$. Since $\xi$ is separating for $\mathcal{A}$, we have $Q_{t_1} = I$. This contradicts with the assumption that $t_1 \neq 1$ and ends the proof. \[\square\]

Next we will show that $\mathcal{L} = \{Q, Q \lor P_\xi : t \in \delta\}$ is a reflexive lattice, i.e., $\mathcal{L} = \mathcal{L}(\text{Alg}(\mathcal{L}))$.

**Lemma 2.3.** Suppose $E \in \mathcal{L}(\text{Alg}(\mathcal{L}))$. If there exist a unit vector $\zeta$ in $E(\mathcal{H})$ and a $t \in \delta \setminus \{1\}$ such that $(I - Q_t)\zeta \not\in \text{span}\{(I - Q_t)\xi\}$, then $Q_t \leq E$.

**Proof.** Let $\eta = (I - Q_1)\zeta - \left(\frac{(I - Q_t)\zeta}{\|I - Q_t\xi\|}\right)\left(\frac{(I - Q_2)\zeta}{\|I - Q_2\xi\|}\right)$. Then $\eta \neq 0$. For any $\beta \in Q_{t_1}(\mathcal{H})$, we consider the map $T_\beta$ from $\mathcal{H}$ into $\mathcal{H}$ defined by $T_\beta(\alpha) = \langle \alpha, \eta \rangle \beta$ for $\alpha \in \mathcal{H}$. Then $T_\beta$ belongs to $\text{Alg}(\mathcal{L})$. Indeed, for any $s \in \delta$, $s \leq t$, if $\alpha \in Q_s(\mathcal{H})$, then $T_\beta(\alpha) = \langle \alpha, \eta \rangle \beta = \langle (I - Q_s)\alpha, \eta \rangle \beta = 0$. This shows that $T_\beta$ leaves $Q_s\xi$ ($s \leq t$) invariant. And obviously $T_\beta$ leaves $P_\xi(\mathcal{H})$ and $Q_{t_1}(\mathcal{H})$ ($s > t$) invariant. Thus $T_\beta \zeta = \left(\|I - Q_1\zeta\|^2 - \|\zeta - \frac{(I - Q_2)\zeta}{\|I - Q_2\xi\|}\|^2\right) \beta \in E(\mathcal{H})$. It follows that $\beta \in E(\mathcal{H})$. Since $\beta$ is arbitrary in $Q_{t_1}(\mathcal{H})$, we have $Q_t \leq E$. \[\square\]

**Lemma 2.4.** Let $\zeta \in \mathcal{H}$. If there exist $t_1$ and $t_2$ in $\delta \setminus \{0, 1\}$ such that $(I - Q_{t_1})\zeta = a_1(I - Q_{t_2})\zeta$, $i = 1, 2$, then $a_{t_1} = a_{t_2}$.

**Proof.** Without loss of generality, we may assume that $a_{t_1} = 0$ (from replacing $\zeta$ by $\zeta - a_{t_1}\xi$). Then we have $(I - Q_{t_1})\zeta = 0$ and $(I - Q_{t_2})\zeta = a_{t_2}(I - Q_{t_2})\zeta$. This is equivalent to $Q_{t_1}\xi = \zeta$ and $Q_{t_2}(\zeta - a_{t_2}\xi) = \zeta - a_{t_2}\xi$. It follows from Lemma 2.2 that $\zeta \in (Q_{t_1} \land (Q_{t_2} \lor P_\xi))(\mathcal{H}) = Q_{\min(t_1, t_2)}(\mathcal{H})$. This implies that $a_{t_1}Q_{t_2}\xi = 0$. If $a_{t_2} \neq 0$, then $\zeta \in Q_{t_2}(\mathcal{H})$. Since $\zeta$ is separating for $\mathcal{A}$, we have $Q_{t_2} = I$. This contradicts with the fact that $t_2 \neq 1$. This completes the proof. \[\square\]

The following result is an immediate consequence of Lemma 2.4.

**Corollary 2.5.** Suppose $\zeta \in \mathcal{H}$ and $\{t_n\}_{n=1}^{\infty} \subset \delta \setminus \{0\}$ is a sequence of numbers such that $\lim_{t \to \infty} t_n = 0$. If, for any $t_n$, there is a complex number $a_{t_n}$ such that $(I - Q_{t_n})\zeta = a_{t_n}(I - Q_{t_n})\xi$, then there is a complex number a such that $\zeta = a\xi$.

**Lemma 2.6.** If $E \in \mathcal{L}(\text{Alg}(\mathcal{L})) \setminus \{0\}$ and $E \not\in P_\xi$, then there exists a $t \in \delta \setminus \{0\}$ such that $Q_t \leq E$.

**Proof.** Since $E \not\in P_\xi$, there exists a $\zeta \in E(\mathcal{H})$ such that $\zeta \not\in \text{span}\{\xi\}$. If there is a $t \in \delta \setminus \{0, 1\}$ such that $(I - Q_t)\zeta \not\in \text{span}\{(I - Q_t)\xi\}$, then we have $Q_t \leq E$ by Lemma 2.3. Otherwise for any $t \in \delta \setminus \{0, 1\}$, we have $(I - Q_t)\zeta \in \text{span}\{(I - Q_t)\xi\}$. Since $\zeta \not\in \text{span}\{\xi\}$, there is an $\epsilon > 0$ such that $(0, \epsilon) \cap \delta \neq \emptyset$, $\epsilon > 0$. Since $E$ is a minimal projection. Suppose $e$ is the unit vector that spans $Q_{t_0}(\mathcal{H})$. Let $\beta = \zeta - \langle \zeta, e \rangle e$. Then the linear operator $T_e$ on $\mathcal{H}$ defined by $T_e(\alpha) = \langle \alpha, \beta \rangle e$ (\(\alpha \in \mathcal{H}\)) is in $\text{Alg}(\mathcal{L})$. Thus $e \in E(\mathcal{H})$ and $Q_{t_0} \leq E$. \[\square\]

**Theorem 2.7.** $\mathcal{L} = \{Q, Q \lor P_\xi : t \in \delta\}$ is a reflexive lattice.

**Proof.** Suppose $E \in \mathcal{L}(\text{Alg}(\mathcal{L})) \setminus \{0, 1\}$, then $E \not\in P_\xi$. Let $t_0 = \sup\{t \in \delta : Q_t \leq E\}$. It follows from Lemma 2.6 that $t_0 > 0$. If $E = Q_{t_0}$, then the result follows. Now assume $E \neq Q_{t_0}$. Given $\zeta \in E(\mathcal{H})$
such that \((I - Q_{t_0})\zeta \neq 0\). By Lemma 2.3, we have for any \(t \in \delta, t > t_0, (I - Q_t)\zeta \in \text{span}\{(I - Q_t)\xi\}.\) If \((I - Q_{t_0})\zeta \notin \text{span}\{(I - Q_t)\xi\},\) then we have \((t_0, 1) \cap \delta \neq \emptyset.\) Let \(t_1 = \inf\{t \in \delta | t > t_0\}.\) It is not hard to show that \(1 > t_1 > t_0, t_1 \in \delta\) and \((t_0, t_1) \cap \delta = \emptyset.\) Let \(e\) be the unit vector that spans \((Q_{t_1} - Q_{t_0})\mathcal{H}\) and \(\beta = (I - Q_{t_0})\zeta - \zeta, (I - Q_{t_0})\xi - (I - Q_{t_0})\xi\). Then the operator \(T_e\) on \(\mathcal{H}\) defined by \(T_e(\alpha) = (\alpha, \beta) e\) \((\alpha \in \mathcal{H}\) is in \(\text{Alg}(\mathcal{L})\) and \(e \in E(\mathcal{H}).\) This means that \(Q_{t_1} \leq E\) and we get a contradiction. Thus for any \(\zeta \in E(\mathcal{H}),\) there is a number \(a_{\zeta} \in \mathbb{C}\) such that \((I - Q_{t_0})\zeta = a_{\zeta}(I - Q_{t_0})\xi.\) Note that there is a vector \(\zeta\) such that \(a_{\zeta} \neq 0\) (otherwise \(E = Q_{t_0}\)). Hence \((I - Q_{t_0})\zeta \in E(\mathcal{H}).\) Since \(Q_{t_0}\xi \in E(\mathcal{H}),\) we have \(\xi \in E(\mathcal{H})\) and \(\zeta - a_{\zeta} \xi \in Q_{t_0}(\mathcal{H}).\) Therefore we have \(E = Q_{t_0} \vee P_{t_0}.\) Thus \(\mathcal{L} = \mathcal{L} \text{at}(\text{Alg}(\mathcal{L}))\) and \(\mathcal{L}\) is a reflexive lattice. \(\square\)

Next we will show that \(\mathcal{L} = \{Q_t, Q_t \vee P_{t_0} : t \in \delta\}\) is a Kadison–Singer lattice. Let \(\mathcal{L}_0\) be a reflexive sublattice of \(\mathcal{L}\).

**Lemma 2.8.** If \(\mathcal{L}_0'' = \mathcal{B}(\mathcal{H}),\) then \(P_{t_0} \in \mathcal{L}_0.\)

**Proof.** Suppose \(P_{t_0} \notin \mathcal{L}_0.\) Let \(t_0 = \inf\{t \in \delta | Q_t \vee P_{t_0} \in \mathcal{L}_0\}.\) Since \(P_{t_0} \notin \mathcal{L}_0, t_0 > 0.\) Also since \(\mathcal{L}_0'' = \mathcal{B}(\mathcal{H}),\) there exists a \(t \in \delta \cap \{1\}\) such that \(Q_t \vee P_{t_0} \in \mathcal{L}_0.\) Hence \(t_0 < 1.\) Since \(\delta\) is compact, we have \(t_0 \in \delta.\) Finally it is easy to see that \(Q_{t_0} \in \mathcal{L}_0'.\) This is a contradiction and the result follows. \(\square\)

If there is no non-trivial projection \(Q_0(\neq 0, 1)\) in \(\mathcal{L}_0,\) then \(\mathcal{L}_0''\) is abelian. Hence \(\delta_0 = \{t \in \delta | Q_t \in \mathcal{L}_0\} \neq \{0, 1\}.\) By the completeness of \(\mathcal{L}_0,\) we have that \(\delta_0\) is a closed (and hence compact) subset of \(\delta.\)

**Lemma 2.9.** Suppose \(\mathcal{L}_0'' = \mathcal{B}(\mathcal{H}).\) If \(t \in \delta \setminus \delta_0,\) then \((t, 1) \cap \delta_0 \neq \emptyset.\)

**Proof.** Assume that \((t, 1) \cap \delta_0 = \emptyset.\) Then \(\text{dim}(I - Q_{t_0})(\mathcal{H}) \geq 1.\) Let \(t_0 = \sup\{s \in \delta_0 | s < t\} < t.\) Then \(\text{dim}(I - Q_t)(\mathcal{H}) \geq 2\) and \((t_0, 1) \cap \delta_0 = \emptyset.\) Now it is easy to see that \(Q_{t_0} \vee P_{t_0} \in \mathcal{L}_0''\) and \(Q_{t_0} \vee P_{t_0} \notin I.\) This is a contradiction and the result follows. \(\square\)

**Theorem 2.10.** If \(\mathcal{L}_0\) is a reflexive sublattice of \(\mathcal{L}\) and \(\mathcal{L}_0'' = \mathcal{B}(\mathcal{H}),\) then \(\mathcal{L}_0 \subseteq \mathcal{L}.\)

**Proof.** Suppose there is a \(t \in \delta \setminus \delta_0.\) By the lemma above there is a \(t_0 \in (t, 1) \cap \delta_0.\) Let \(t_1 = \sup\{s \in \delta_0 | s < t\} < t\) and \(t_2 = \inf\{s \in \delta_0 | s > t\} \in (t, 1).\) Note that for any \(s \in (t_1, t_2) \cap \delta, Q_s \vee P_{t_0} \notin \mathcal{L}_0\) (otherwise, \(Q_s = Q_{t_0} \wedge (Q_s \vee P_{t_0}) \in \mathcal{L}_0\) which implies that \(s \in \delta_0,\) but this contradicts with the fact that \(t_1 < s < t_2.\)) Since \(\text{dim}(Q_{t_2} - Q_{t_1})(\mathcal{H}) \geq 2,\) we can choose a subprojection \(F\) of \((Q_{t_2} - Q_{t_1})\) such that \(F(Q_{t_2} - Q_{t_1})\xi = 0.\) Then \(F \in \mathcal{L}_0''\) and we get a contradiction. Thus \(\delta_0 = \delta.\) Since \(P_{t_0} \in \mathcal{L}_0,\) we have \(\mathcal{L}_0 = \mathcal{L}.\) \(\square\)

The following theorem is our main result which follows from the above results.

**Theorem 2.11.** The notations are as above, \(\mathcal{L} = \{Q_t, Q_t \vee P_{t_0} : t \in \delta\}\) is a Kadison–Singer lattice and \(\text{Alg}(\mathcal{L})\) is a Kadison–Singer algebra.

In the rest of this section, we shall give three examples of Kadison–Singer algebras with the corresponding Kadison–Singer lattices generated by a maximal nest and a rank one projection. The first example concerns the lattice generated by a \(\mathbb{N}\)-ordered maximal nest and a rank one projection.

**Example 2.12.** Suppose \(\mathcal{H}\) is a separable infinite-dimensional Hilbert space with an orthogonal bases \(\{e_i : i \in \mathbb{N}\}.\) For each \(i \in \mathbb{N},\) let \(P_i\) be the orthogonal projection of \(\mathcal{H}\) onto the linear subspace of \(\mathcal{H}\) generated by \(\{e_1, e_2, \ldots, e_i\}.\) Let \(\xi = \sum_{n=1}^{\infty} a_n e_n \in \mathcal{H}\) be a vector with all \(a_n\) nonzero. Without loss of generality, we can assume that \(a_1 \neq 1.\) Let \(P_0\) be the orthogonal projection of \(\mathcal{H}\) onto the one dimensional subspace generated by \(\zeta \xi.\) Let \(0 \text{ and } I\) be the zero operator and identity operator on \(\mathcal{H}\) respectively. Since \(P_0 \wedge P_0 = 0\) for any \(n \in \mathbb{N},\) \(\mathcal{L} = \{0, I, P_0, P_{t_0} \vee P_{t_0} : n \in \mathbb{N}\}\) is the lattice generated by \(\{P_n : n \in \mathbb{N}\}\) and \(P_{t_0}\). It follows from Theorem 2.11 that \(\mathcal{L}\) is a Kadison–Singer lattice and \(\text{Alg}(\mathcal{L})\) is a Kadison–Singer algebra with diagonal equal to \(CI.\)

The second example concerns the lattice generated by a \(\mathbb{Z}\)-ordered maximal nest and a rank one projection.
Example 2.13. Suppose $\mathcal{H}$ is a separable infinite-dimensional Hilbert space with orthogonal basis $\{e_n : n \in \mathbb{Z}\}$. For each $n \in \mathbb{Z}$, let $P_n$ be the orthogonal projection of $\mathcal{H}$ onto the closed subspace spanned by $\{e_k : k \in \mathbb{Z}, k \leq n\}$. Note that $\lim_{n \to -\infty} P_n = 0$ and $\lim_{n \to \infty} P_n = I$ in the strong-operator topology. Given a vector $\xi = \sum_{k=\infty}^{n} a_k e_k \in \mathcal{H}$ with $a_k \neq 0$ for all $k \in \mathbb{Z}$. Let $P_\xi$ be the orthogonal projection of $\mathcal{H}$ onto the closed subspace of $\mathcal{H}$ spanned by $\xi$. Let $\mathcal{L}$ be the lattice generated by $P_n$ ($n \in \mathbb{Z}$) and $P_\xi$. Then since $P_n \wedge P_\xi = 0$ for all $n \in \mathbb{Z}$, $\mathcal{L} = \{P_n, P_\xi, P_n \vee P_\xi, 0, 1 : n \in \mathbb{Z}\}$. Then it follows from Theorem 2.11 that $\mathcal{L}$ is a Kadison–Singer lattice and Alg($\mathcal{L}$) is a Kadison–Singer algebra with diagonal equal to $\mathbb{C}I$.

The last example is about the lattice generated by a continuous nest which is of order $[0, 1]$ and a rank one projection.

Example 2.14. Suppose $\mathcal{H} = L^2[0, 1]$ is a Hilbert space with inner product given by $\langle f, g \rangle = \int_{[0,1]} \overline{g(x)}f(x)dx$ for all $f, g \in \mathcal{H}$. For every $f \in L^\infty[0, 1]$, define $M_f$ on $\mathcal{H}$ by $(M_f(g))(\xi) = f(\xi)g(\xi)$, $\forall g \in L^2[0, 1], \xi \in [0, 1]$. Then $M_f$ is a bounded linear operator on $\mathcal{H}$. Let $\mathcal{D} = \{M_f : f \in L^\infty[0, 1]\}$. Then $\mathcal{D}$ is a maximal abelian subalgebra of $B(\mathcal{H})$ (for the proof of these facts, see [10]).

For any $t \in [0, 1]$, let $\chi_{\{0,1\}}$ be the characteristic function of $[0, t]$ and $P_1 = M_{\chi_{\{0,1\}}}$ be the orthogonal projection on $L^2[0, 1]$ defined by $P_1(g) = \chi_{\{0,1\}}g$ for any $g \in \mathcal{H}$. Suppose $\xi \in \mathcal{H}$ is a measurable function that is nonzero almost everywhere (we may suppose that $\xi(t) \neq 0$ for all $t \in [0, 1]$). Let $P_\xi$ be the orthogonal projection of $\mathcal{H}$ onto the one dimensional subspace $\mathbb{C}\xi$ of $\mathcal{H}$. Let $\mathcal{L}$ be the lattice generated by $\{P_t : t \in [0, 1]\}$ and $P_\xi$. Then it follows from Theorem 2.11 that $\mathcal{L}$ is a Kadison–Singer lattice and Alg($\mathcal{L}$) is a Kadison–Singer algebra with diagonal equal to $\mathbb{C}I$.

3. Kadison–Singer lattices in finite factors

In this section we consider the lattices generated by a finite nest and a projection in finite factors. The following theorem is our main result in this section.

Theorem 3.1. Assume $\mathcal{H}$ is a Hilbert space and $\mathcal{M} \subset B(\mathcal{H})$ is a finite factor with a normal faithful tracial state $\tau$. Suppose $\mathcal{L}_0 = \{0, P_1, \ldots, P_{n-1}, I\}$ $(n \geq 2)$ is a nest of projections in $\mathcal{M}$ with $\tau(P_i) = \frac{1}{n}$ $(i = 1, 2, \ldots, n - 1)$ and $Q$ is a projection in $\mathcal{M}$ with $\tau(Q) = \frac{1}{n}$. Let $\mathcal{L}$ be the lattice generated by $\mathcal{L}_0$ and $Q$. If $Q \wedge P_i = 0$ ($i = 1, 2, \ldots, n - 1$) and $\mathcal{L}_0Q^\prime = \mathcal{M}$, then $\mathcal{L}$ is a Kadison–Singer lattice with $\mathcal{L}'' = \mathcal{M}$ and $\text{Alg}(\mathcal{L})$ is a Kadison–Singer algebra.

Proof. Since $P_{n-1} \wedge Q = 0$, we have $\tau(P_{n-1} \vee Q) = \tau(P_{n-1}) + \tau(Q) - \tau(P_{n-1} \wedge Q) = 1$ and hence $P_{n-1} \vee Q = I$. Given any $i, j \in \{1, 2, \ldots, n - 1\}$. Since $P \wedge Q = 0$ for any $P \in \mathcal{L}_0 \setminus \{I\}$, we have

$$
\tau((P_i \vee Q) \wedge P_j) = \tau(P_i) + \tau(P_j) + \tau(Q) - \tau(P_i \vee P_j \vee Q) = \tau(P_i) + \tau(P_j) + \tau(Q) - \tau(P_{\text{max}(i,j)}) - \tau(Q) = \tau(P_{\text{min}(i,j)}).
$$

Since $P_{\text{min}(i,j)} \leq (P_i \vee Q) \wedge P_j$ we have $(P_i \vee Q) \wedge P_j = P_{\text{min}(i,j)}$. It follows that $\mathcal{L} = \{0, P_1, Q, P_i \vee Q : i = 1, 2, \ldots, n - 1\}$. Now it is easy to check that $\mathcal{L}$ is distributive and therefore $\mathcal{L}$ is reflexive by [8].

The case $n = 2$ is easy to show. In the following we assume $n \geq 3$. We let $P_0 = 0$ and $P_n = I$.

Suppose $\mathcal{L}_1$ is a reflexive sublattice of $\mathcal{L}$ such that $\mathcal{L}_1'' = \mathcal{M}$. Since $\mathcal{L}_1 \cap \mathcal{M} = \mathbb{C}I$, we have $P_{n-1}, Q \in \mathcal{L}_1$ (otherwise $P_{n-2} \vee Q$ or $P_1$ will lie in $\mathcal{L}_1 \cap \mathcal{M} = \mathbb{C}I$ which is a contradiction). Suppose that there is a $P_i \in \{P_1, P_2, \ldots, P_{n-2}\}$ not in $\mathcal{L}_1$. Then $P_i \wedge Q$ is not in $\mathcal{L}_1$ (otherwise, $P \wedge Q \in \mathcal{L}_1$ implies $(P_i \vee Q) \wedge P_{n-1} = P_i \in \mathcal{L}_1$ and this leads to a contradiction). Let $k$ be the largest number $j$ in $\{0, \ldots, i - 1\}$ satisfying $P_j \in \mathcal{L}_1$. Let $s$ be the smallest number $j$ in $\{i + 1, \ldots, n - 1\}$ satisfying $P_j \in \mathcal{L}_1$. Then $P_k \in \mathcal{L}_1$ and $P_k \in \mathcal{L}_1$. Let $F = P_s - P_k$. Then $\tau(F) = \frac{k-s}{n} \geq \frac{2}{n}$. Let $E = F \wedge (I-Q)$. It is easy to see that $\tau(E) > 0$ and $E \neq 0$. Since $E$ commutes with $P_1, \ldots, P_k, P_s, \ldots, P_{n-1}$ and $Q$, we have $E \in \mathcal{L}_1'' \cap \mathcal{M} = \mathbb{C}I$ which is a contradiction. It follows from above that $\mathcal{L}_1 \not\subseteq \mathcal{L}$ and $\mathcal{L}$ is a Kadison–Singer lattice with $\mathcal{L}'' = \mathcal{M}$. □
The corresponding Hasse graph of the lattice $\mathcal{L}$ in Theorem 3.1 is given below.

Example 3.2. Suppose $n (n \geq 2)$ is a positive integer. Let $\mathcal{M} = M_n (C)$. We assume that $\mathcal{M}$ is acting on $\mathcal{H} = C^n$. Suppose $\{ e_{ij} : i, j = 1, \ldots, n \}$ is the usual system of matrix units of $M_n (C)$. Let $P_1 = e_{11}, P_2 = e_{11} + e_{22}, \ldots, P_{n-1} = e_{11} + e_{22} + \cdots + e_{n-1,n-1}, P_n = I$ is a maximal nest in $\mathcal{M}$. Let $\xi \in \mathcal{H}$ be a separating vector for $\mathcal{L}_0$ and $Q$ be the rank one projection corresponding the one dimensional subspace of $\mathcal{H}$ generated by $\xi$. Then it follows from Theorem 3.1 that the lattice $\mathcal{L}$ generated by $\mathcal{L}_0$ and $Q$ is a Kadison–Singer lattice with $\mathcal{L}'' = \mathcal{M}$ and $\text{Alg}(\mathcal{L})$ is a Kadison–Singer algebra.

Example 3.3. Suppose $n (n \geq 2)$ is a positive integer. Let $\mathcal{M}$ be a finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with a normal faithful tracial state $\tau$. Suppose $\mathcal{L}_0 = \{ 0, P_1, P_2, \ldots, P_n = I \}$ is a nest of projections in $\mathcal{M}$ with $\tau (P_k) = \frac{1}{k}$ for $k = 1, 2, \ldots, n$. We also denote $P_0 = 0$. Let $Q \in \mathcal{M}$ be a projection with $\tau (Q) = \frac{1}{n}$. Suppose $Q$ and $\{ P_1, P_2, \ldots, P_n \}$ are free in $\mathcal{M}$ with respect to $\tau$ [14].

Let $\mathcal{L}$ be the von Neumann algebra generated by $\{ P_1, P_2, \ldots, P_n \}$ and $Q$. It follows from Theorem 2.3 in [4] that $\mathcal{M}$ is a type II$_1$ factor and $P_i \wedge Q = 0$ for $i = 1, 2, \ldots, n - 1$. Let $\mathcal{L}$ be the lattice generated by $\mathcal{L}_0$ and $Q$. Then it follows from Theorem 3.1 that $\mathcal{L} = \{ 0, P_i, Q, P_i \vee Q : i = 1, 2, \ldots, n - 1 \}$ is a Kadison–Singer lattice with $\mathcal{L}'' = \mathcal{M}$ and $\text{Alg}(\mathcal{L})$ is a Kadison–Singer algebra.

Let $m, n \in \mathbb{N}$ and $n \geq 2$. In the following we give some conditions for the lattice $\mathcal{L}$ generated by a nest $\mathcal{L}_0$ (not necessarily maximal) and a projection $Q$ in $\mathcal{L} = M_n (C) \otimes M_m (C)$ to generate $\mathcal{M}$. Assume $\mathcal{M} = M_n (C) \otimes M_m (C)$ acts on $\mathcal{H} = C^{nm}$. Let $\{ e_{ij} : i, j = 1, \ldots, n \}$ and $\{ f_{kl} : k, l = 1, 2, \ldots, m \}$ be the usual system of matrix units of $M_n (C)$ and $M_m (C)$ respectively. Suppose $\tau$ is the normal faithful tracial state of $\mathcal{M}$, then $\mathcal{L}_0 = \{ 0, E_1, \ldots, E_{n-1}, E_n = e_{11} + e_{22} + \cdots + e_{m1} \}$ is a maximal nest of $M_n (C)$ and $Q$ is a projection in $\mathcal{M}$ with rank $m$. We may assume $E_1 = e_{11}, E_2 = e_{11} + e_{22}, \ldots, E_{n-1} = e_{11} + e_{22} + \cdots + e_{n-1,n-1}$. Let $\{ \xi_i = (a_{i1}, a_{i2}, \ldots, a_{im}) : i = 1, 2, \ldots, m \}$ be the basis of $Q(\mathcal{H})$. Assume $A_j = \sum_{k=1}^m a_{j(k)m} E_k$ for $i = 1, 2, \ldots, n$. Denote $F_m = f_{11} + f_{22} + \cdots + f_{mm}$ and $P_i = E_i \otimes F_m (i = 2, \ldots, n)$. Then $\tau (P_i) = \frac{i}{n}$ and $\tau (Q) = \frac{1}{n}$. Let $\mathcal{L}_0 = \{ 0, P_1, P_2, \ldots, P_{n-1}, P_n \}$ be a nest in $\mathcal{M}$.

Proposition 3.4. The notations are as above. Let $\mathcal{L}$ be the lattice generated by $\mathcal{L}_0$ and $Q$. If $\{ X_1^{-1}(X_2^{-1})^*, X_3^{-1}(X_4^{-1})^*, \ldots, X_n^{-1}(X_{n-1}^{-1})^* \} = M_n (C)$, then $\mathcal{L}$ is a Kadison–Singer lattice with $\mathcal{L}'' = \mathcal{M}$ and $\text{Alg}(\mathcal{L})$ is a Kadison–Singer algebra.

Proof. To show $\mathcal{L}'' = \{ \mathcal{L}_0, Q \}'' = \mathcal{M}$, it is necessary and sufficient to show that if $T$ commutes with $\mathcal{L}_0$ and $Q$, then $T \in \mathcal{C}$. We may assume that $T$ is a self-adjoint operator. Thus we have $T = e_{11} \otimes T_1 + e_{22} \otimes T_2 + \cdots + e_{mm} \otimes T_n$ where $T_1, \ldots, T_n$ are $n$ self-adjoint $m \times m$ matrices. Let $A = (A_1, A_2^\top, \ldots, A_n^\top)^\top$ where $T$ denotes the transpose of a matrix. Since $T$ commutes with $Q$, $T \xi_i \in Q(\mathcal{H})$ for $i = 1, 2, \ldots, m$. Thus there exist numbers $b_{ij}$ such that $TA = A(\sum_{k=1}^m b_{ij} E_k)$. Let $W = \sum_{k=1}^m b_{ik} E_k$. Then $T_A = A W$ and $T_A^\top T_A = W (i = 1, 2, \ldots, n)$. Hence we have $T_1 = A_1 A_1^{-1} T_1 (A_1 A_1^{-1})^{-1}$ for $i = 2, 3, \ldots, n$. Since $X_i = A_i A_i^{-1}$ and $T$ is self-adjoint, we have $X_i T_1 X_1^{-1} = (X_1^{-1})^\top T_1 X_1^{-1} X_i$ for $i = 2, 3, \ldots, n$. Therefore $X_i^{-1}(X_1^{-1})^* T_1 = X_i T_1 X_i^{-1}(X_1^{-1})^*$ for $i = 2, 3, \ldots, n$. Since $\{ X_1^{-1}(X_2^{-1})^*, X_3^{-1}(X_4^{-1})^*, \ldots, X_n^{-1}(X_{n-1}^{-1})^* \} = M_n (C)$, we have $T_1 \in \mathcal{C}$ and $\{ \mathcal{L}_0, Q \}' = \mathcal{M}$. Since each $A_i$ is invertible, we have $P_j \wedge Q = 0$ for $j \in \{ 2, 3, \ldots, n - 1 \}$. Now the result follows from Theorem 3.1. □
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