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Rational Tensor Product Bézier Volumes

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Abstract—Free form volumes in rational Bézier representation are derived via homogeneous coordinates. Some properties and constructions are presented and two applications of free form volumes are discussed: definition of solid primitives and curve and surface modelling by the way of volume deformation.

Keywords—Rational Bézier volumes, Solids, Free form deformation.

1. INTRODUCTION

While in the past, CAGD has been mostly concerned with curves and surfaces, more recently, there has been an increasing interest in higher dimensional, trivariate objects such as free form volumes which are suitable to describe inhomogeneous solids. The two most widely used methods of representing solids are the Constructive Solid Geometry Representation (CSG-Rep.)—a solid based method—and the Boundary Representation (B-Rep.)—a surface based method (see, e.g., [1]). However, free form character of both methods is not very substantial, and they also assume internal homogeneity. On the other hand, free form volumes, with which this paper is dealing, possess per definition a very high free form characteristic and describe every interior point as well as every point on the boundary surface of the volume uniquely. No assumption on internal homogeneity or structure is done. Besides solid modelling, there are some more applications of free form volumes, for instance, the description of spatial movement or deformation of a surface, the description of physical fields, such as temperature or pressure, etc., as functions of several variables, e.g., the positional coordinates, the modification of curves and surfaces through volume deformation, etc.

Free form volumes can be defined by the *tensor* (Cartesian) *product definition*,

$$\mathbf{V}(u,v,w) = \sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} \mathbf{V}_{i,j,k}\, u^i\, v^j\, w^k, \qquad u,v,w \in [0,1]. \tag{1}$$

The above given definition is based on monomials, in CAGD (*Computer Aided Geometric Design*) *Bernstein polynomials*, (see [2,3]),

$$B_k^n(w) = \binom{n}{k} w^k (1-w)^{n-k}, \qquad w \in [0,1], \tag{2}$$

of degree n in w, and analogously for v and u, are very popular. This is, because the expansion in terms of Bernstein polynomials yields, first, a numerically very stable behavior of all algorithms. Secondly, a geometric relationship between subject and coefficients of its defining equation. Note,

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that these properties are not available in the case of monomials and Lagrange or Hermite polynomials. Probably this is one of the main reasons why Bézier representations, which are based on the use of Bernstein polynomials, became the de facto industry standard in CAGD during the past years. Thus, a *tensor product Bézier volume*—briefly *TPB volume*—of degree (l, m, n) in (u, v, w) is defined by

$$\mathbf{V}(u,v,w) = \sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} \mathbf{V}_{i,j,k}\, B_i^l(u)\, B_j^m(v)\, B_k^n(w), \qquad u,v,w \in [0,1]. \tag{3}$$

The coefficients $\mathbf{V}_{i,j,k} \in \mathbb{R}^3$ are called *Bézier points*. They form, connected in the ordering given by their subscripts, a spatial net which is called *Bézier grid* [3].

Because of the polynomial character of (3), only polynomials can be represented exactly. Sphere (segments), for example, cannot be constructed by (3). Therefore, many (primary) elements of CAD systems cannot be converted exactly into this kind of free form representation. Rational Bézier representations overcome this disadvantage for the most part: Primary elements like conic sections and quadrics, as well as tori and cyclids, for example, can be constructed, thus allowing exact conversion from CSG-Rep. and B-Rep. based solid primitives into rational free form representation.

This paper is concerned with free form volumes in rational Bézier representation defined via the Cartesian product. Mathematical description using homogeneous coordinates, properties and constructions such as point and derivative evaluation, degree raising, subdivision, and continuity conditions are presented in Section 2. Section 3 and 4 discuss two examples of application of free form volumes: generation of solid primitives defined by rational TPB volumes and curve and surface modelling by free form deformation.

2. BÉZIER VOLUMES

Using homogeneous coordinates r^i, points $\mathbf{R} = (x, y, z)^\top$ of $\mathbb{R}^3$ can be represented by points $\mathsf{R} = \left(\mathsf{r}^1, \mathsf{r}^2, \mathsf{r}^3, \mathsf{r}^4\right)^\top$ of $\mathbb{R}^4$ via the projection of $\mathbb{R}^4$: into the hyperplane $\mathsf{r}^4 = 1$ according to (cf. Figure 1)

$$\mathcal{H}(\mathsf{R}) = \begin{cases} \dfrac{1}{\mathsf{r}^4}\left(\mathsf{r}^1, \mathsf{r}^2, \mathsf{r}^3\right)^\top, & \text{for } \mathsf{r}^4 \neq 0, \\ \left(\mathsf{r}^1, \mathsf{r}^2, \mathsf{r}^3\right)^\top, & \text{for } \mathsf{r}^4 = 0. \end{cases}$$

The center of projection is the origin of the 4D Cartesian coordinate system. A point $\mathbf{R} = (x, y, z)^\top$ is the projection of $\omega(x, y, z, 1)^\top$, where $\omega \neq 0$, $\omega \in \mathbb{R}$. The real number ω is called *weight* of the corresponding point. Note, $\mathsf{R} \in \mathbb{R}^4$ and $\varrho\mathsf{R} \in \mathbb{R}^4$, $\varrho \neq 0$, $\varrho \in \mathbb{R}$, describe the same point of $\mathbb{R}^3$. $\mathcal{H}(\mathsf{R})$ is unlimited for $\mathsf{r}^4 \to 0$, $\mathsf{r}^4 \neq 0$. In this way, infinite points of $\mathbb{R}^3$ can be described by finite points of $\mathbb{R}^4$ with $\mathsf{r}^4 = 0$. In $\mathbb{R}^3$ these points will be represented by direction vectors [4].

Now, a *rational tensor product Bézier volume*—briefly *rational TPB volume*—of degree (l, m, n) in (u, v, w) is defined by [5,6],

$$\mathbf{V}(\mathbf{u}) = \mathcal{H}(\mathsf{V}(\mathbf{u})), \tag{4}$$

where

$$\mathsf{V}(\mathbf{u}) = \sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} \mathsf{V}_{i,j,k}\, B_i^l(u)\, B_j^m(v)\, B_k^n(w), \qquad u,v,w \in [0,1],$$

and $\mathbf{u} = (u, v, w)$. $\mathsf{V}(\mathbf{u})$ is called homogeneous form of $\mathbf{V}(\mathbf{u})$. Thus, a rational TPB volume of degree (l, m, n) in $\mathbb{R}^3, \mathbf{V}(\mathbf{u})$, is defined by a nonrational, polynomial TPB volume of degree (l, m, n) in $\mathbb{R}^4, \mathsf{V}(\mathbf{u})$, i.e., is the projection of a polynomial TPB volume into the hyperplane $\mathsf{r}^4 = 1$.

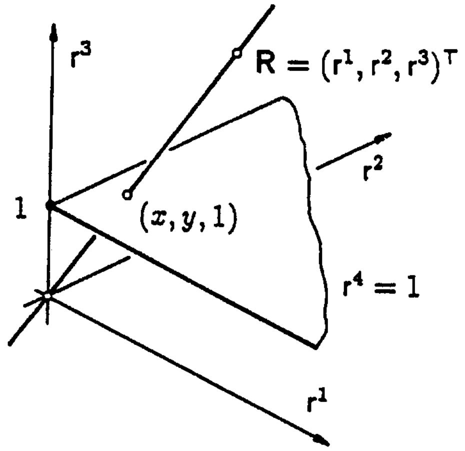

Figure 1. Introducing homogeneous coordinates via projection $\mathcal{H}(\cdot)$ of points of $\mathbb{R}^3$ into the plane $r^3 = 1$.

If we write $\mathbf{V}_{i,j,k} = \left(V^1_{i,j,k}, V^2_{i,j,k}, V^3_{i,j,k}, V^4_{i,j,k}\right)^\top = \left(\omega_{i,j,k}\, \mathbf{V}^\top_{i,j,k}, \omega_{i,j,k}\right)^\top \in \mathbb{R}^4$, where $\mathbf{V}_{i,j,k} = (x_{i,j,k}, y_{i,j,k}, z_{i,j,k})^\top \in \mathbb{R}^3$, $\omega_{i,j,k} \in \mathbb{R}$, and assume

$$\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} \omega_{i,j,k}\, B_i^l(u)\, B_j^m(v)\, B_k^n(w) \neq 0, \qquad u, v, w \in [0,1],$$

then,

$$\mathbf{V}(\mathbf{u}) = \frac{\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} \omega_{i,j,k}\, \mathbf{V}_{i,j,k}\, B_i^l(u)\, B_j^m(v)\, B_k^n(w)}{\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} \omega_{i,j,k}\, B_i^l(u)\, B_j^m(v)\, B_k^n(w)}, \tag{5}$$

and *3D Bézier points* $\mathbf{V}_{i,j,k}$ are projections, $\mathbf{V}_{i,j,k} = \mathcal{H}\left(\mathbf{V}_{i,j,k}\right)$, of *4D Bézier points* $\mathbf{V}_{i,j,k}$. Bézier points form in their natural ordering, given by their subscripts, the vertices of the *Bézier grid*. Scalars $\omega_{i,j,k} \in \mathbb{R}$ are called *weights*: If we increase one $\omega_{i,j,k}$ the volume will be pulled towards the corresponding $\mathbf{V}_{i,j,k}$.

The assumption of (5) will be fulfilled if weights $\omega_{i,j,k}$ are nonnegative and all 8 corner Bézier points of the grid are not infinite control points that will be required from now on. If a control point $\mathbf{V}_{i,j,k}$ is a *point at infinity*, i.e., $\omega_{i,j,k} = 0$, we replace, according to definition of $\mathcal{H}(\cdot), \omega_{i,j,k}\mathbf{V}_{i,j,k}$ by $\mathbf{V}_{i,j,k}$ in the numerator of (5). Note, infinite control points, also called *control vectors*, can be eliminated by degree elevation (cf., [2, p. 260]) as stated by (7).

One of the weights (e.g., $\omega_{0,0,0}$) can be normalized to be of value one and on boundary curves $u = 0, v = 0$ and $w = 0$, the parameterization can be chosen so that, for instance $\omega_{l,0,0,} = \omega_{0,m,0} = \omega_{0,0,n} = 1$. All other weights directly influence the shape of the volume. If all weights are equal, (5) yields the nonrational, polynomial TPB volume because Bernstein polynomials sum to one (see, e.g., [2, p. 42; 3, p. 116]).

Positive weights result in volumes which have all the properties and algorithms we do know from polynomial representations. Because of definition (4) via the projection $\mathcal{H}(\cdot)$, properties of rational TPB volumes can be deduced from properties of the nonrational TPB volume scheme. This means that relations between Bézier grid and rational TPB volume can be deduced from those of the underlying Bézier curve scheme, and that many constructions in different parameters commute, such as degree raising, de Casteljau construction and derived constructions, segmentation for example. We list a few properties (see [5,6]).

CONVEX HULL PROPERTY. The rational TPB volume lies completely within the convex hull of its Bézier points.

PARAMETRIC SURFACES. The parametric surfaces of constant u are rational TPB surfaces of degree (m, n) and analogously for parametric surfaces of constant v and w, respectively.

PARAMETRIC LINES. Lines u, v = constant, so-called w parameter lines, are rational Bézier curves of degree n and analogously for v and for u parameter lines.

BOUNDARY SURFACES. The boundary surfaces of a rational TPB volume are rational TPB surfaces whose Bézier points and weights are the corresponding boundary points and weights of the Bézier grid.

BOUNDARY CURVES. The boundary curves of a rational TPB volume are rational Bézier curves whose Bézier points and weights are the corresponding boundary points and weights of the Bézier grid.

VERTICES. The eight vertices of a rational TPB volume coincide with the eight corresponding vertices of the Bézier grid.

VISUAL APPROXIMATION. The Bézier grid gives a rough impression of the rational TPB volume.

DERIVATIVES. The partial derivatives of order (p, q, r) of a rational TPB volume of degree (l, m, n), $\mathbf{V}(\mathbf{u}) = \mathbf{N}(\mathbf{u})/D(\mathbf{u})$, are given by (cf., [2, p. 256])

$$\frac{\partial^\alpha}{\partial \mathbf{u}^\alpha}\mathbf{V}(\mathbf{u}) = \frac{1}{D(\mathbf{u})}\left(\frac{\partial^\alpha}{\partial \mathbf{u}^\alpha}\mathbf{N}(\mathbf{u}) - \sum_{\mathbf{i}\in\mathbf{I}}\binom{p}{i}\binom{q}{j}\binom{r}{k}\frac{\partial^\beta}{\partial \mathbf{u}^\beta}D(\mathbf{u})\,\frac{\partial^{\alpha-\beta}}{\partial \mathbf{u}^{\alpha-\beta}}\mathbf{N}(\mathbf{u})\right), \tag{6}$$

where $\mathbf{I} = \{\mathbf{i} = (i,j,k) : i = 0\,(1)\,p,\ j = 0\,(1)\,q,\ k = 0\,(1)\,r \text{ and } (i,j,k) \neq (0,0,0)\}$ and $\beta = |\mathbf{i}| = i + j + k$, using the notation

$$\frac{\partial^\alpha}{\partial \mathbf{u}^\alpha}\mathsf{F}(\mathbf{u}) = \frac{\partial^{|\mathbf{p}|}}{\partial \mathbf{u}^{\mathbf{p}}}\mathsf{F}(\mathbf{u}) = \frac{\partial^{p+q+r}}{\partial u^p\,\partial v^q\,\partial w^r}\mathsf{F}(\mathbf{u}),$$

where $\mathbf{p} = (p,q,r), \alpha = |\mathbf{p}| = p + q + r$, to indicate partial derivatives of a (vector valued) function $\mathsf{F}(\mathbf{u})$. Derivative calculation of $D(\mathbf{u})$ is according to

$$\frac{\partial^\alpha}{\partial \mathbf{u}^\alpha}D(\mathbf{u}) = \frac{l!}{(l-p)!}\,\frac{m!}{(m-q)!}\,\frac{n!}{(n-r)!}\sum_{i=0}^{l-p}\sum_{j=0}^{m-q}\sum_{k=0}^{n-r}\Delta^{p,q,r}\,\omega_{i,j,k}\,B_i^{l-p}(u)\,B_j^{m-q}(v)\,B_k^{n-r}(w),$$

with forward differences $\Delta^{p,q,r}\,\omega_{i,j,k}$, and similarly for $\mathbf{N}(\mathbf{u})$, employing forward differences $\Delta^{p,q,r}\,(\omega_{i,j,k}\mathbf{V}_{i,j,k})$.

DEGREE RAISING. If we write a rational TPB volume, (4), of degree (l, m, n) as one of degree $(l + \lambda, m, n)$, the new 4D Bézier points $\mathsf{V}^\lambda_{i,j,k}$ are given by

$$\mathsf{V}^\lambda_{i,j,k} = \sum_{I=0}^{\lambda}\lambda_I\,\frac{\binom{\lambda}{I}\binom{l}{i-I}}{\binom{l+\lambda}{i}}\,\mathsf{V}_{i-I,j,k}. \tag{7}$$

This can be seen as follows: $\lambda(u)\mathsf{V}(\mathbf{u})$, with an arbitrary real-valued function $\lambda(u)$ which does not vanish on $[0,1]$, and $\mathsf{V}(\mathbf{u})$ are associated with the same rational TPB volume $\mathbf{V}(\mathbf{u})$. If we choose $\lambda(u)$ to be of degree $\lambda, \lambda(u) = \sum_{I=0}^{\lambda}\lambda_I B_I^\lambda(u)$, then $\lambda(u)\mathsf{V}(\mathbf{u})$ is of degree $l + \lambda$ in u, and equation (7) results. As a special case, $\lambda = 1$, the degree $(l + 1, m, n)$ Bézier points are given by

$$\mathsf{V}^1_{i,j,k} = \lambda_0\left(1 - \frac{i}{l+1}\right)\mathsf{V}_{i,j,k} + \lambda_1\frac{i}{l+1}\mathsf{V}_{i-1,j,k}.$$

Equation (7) means, since $\lambda_I \in \mathbb{R}$, that there is an infinite number of ways to represent a degree (l, m, n) volume as one of degree $(l + \lambda, m, n)$. The same holds, with corresponding formulae, for degree raising in the v and w variables.

POINT AND DERIVATIVE EVALUATION. A volume point $\mathbf{V}(\mathbf{u}_0)$, for any $\mathbf{u}_0 = (u_0, v_0, w_0)$, u_0, v_0, $w_0 \in [0,1]$, can be computed by repeated *de Casteljau* steps adapted on 4D Bézier points in u, v and w, for example, in the u direction by the recursion formula

$$\mathsf{V}^{\beta,\delta,\xi}_{\alpha,\gamma,\epsilon}(\mathbf{u}_0) = (1-u_0)\,\mathsf{V}^{\beta-1,\delta,\xi}_{\alpha,\gamma,\epsilon}(\mathbf{u}_0) + u_0 \mathsf{V}^{\beta,\delta,\xi}_{\alpha+1,\gamma,\epsilon}(\mathbf{u}_0) \tag{8}$$

and analogously for v and w, where

$$\begin{aligned}\mathsf{V}^{\alpha,\gamma,\epsilon}_{\alpha,\gamma,\epsilon} &= \mathsf{V}_{\alpha,\gamma,\epsilon},\\ \mathsf{V}(\mathbf{u}_0) &= \mathsf{V}^{l,m,n}_{0,0,0}(\mathbf{u}_0),\\ \mathbf{V}(\mathbf{u}_0) &= \mathcal{H}(\mathsf{V}(\mathbf{u}_0)).\end{aligned}$$

This result stems from the fact that in (4), the de Casteljau algorithm can be applied separately to each of the parameter directions. The de Casteljau steps in different directions commute, and the result is independent of the order. Similarly, the derivative of order (p,q,r) of a rational TPB volume of degree (l,m,n) can be found using the de Casteljau algorithm. Indeed, starting with equation (6) derivatives of $\mathbf{N}(\mathbf{u})$ are given by

$$\frac{\partial^{p+q+r}}{\partial u^p\,\partial v^q\,\partial w^r}\mathbf{N}(\mathbf{u}) = \frac{l!}{(l-p)!}\frac{m!}{(m-q)!}\frac{n!}{(n-r)!}\Delta^{p,q,r}\left(\omega^{l-p,m-q,n-r}_{0,0,0}\mathbf{V}^{l-p,m-q,n-r}_{0,0,0}\right),$$

with forward differences $\Delta^{p,q,r}\left(\omega^{\beta,\delta,\xi}_{\alpha,\gamma,\epsilon}\mathbf{V}^{\beta,\delta,\xi}_{\alpha,\gamma,\epsilon}\right)$ which are now operating on both the subscripts and superscripts, and similarly for derivatives of $D(\mathbf{u})$.

SUBDIVISION. A rational TPB volume of degree (l,m,n) can be subdivided into two rational TPB volumes of the same degree which join along the parametric surface corresponding to $u = u_0$ with l continuous derivatives in u direction. The Bézier points $\mathsf{V}^{i,j,k}_{0,j,k}$ and $\mathsf{V}^{l,j,k}_{i,j,k}$ of the two subsegments can be found by applying the de Casteljau algorithm for $u = u_0$ to all i polygons of the Bézier grid. Using the parameter transformations $\bar{u} = u/u_0$ for $u \in [0, u_0]$ and $\bar{u} = (u-u_0)/(1-u_0)$ for $u \in [u_0, 1]$, respectively, we can reparameterize the subsegments to again be defined on the unit cube.

CONTINUITY CONDITIONS. To construct GC^r or C^r continuous rational volumes, $\mathbf{V}(\mathbf{u})$ and $\overline{\mathbf{V}}(\bar{\mathbf{u}})$, defined via homogeneous coordinates, i.e., as volumes $\mathsf{V}(\mathbf{u})$ and $\overline{\mathsf{V}}(\bar{\mathbf{u}})$, we have to take into account that all $\eta(\mathbf{u})\mathsf{V}(\mathbf{u})$ and $\mathsf{V}(\mathbf{u})$ with an arbitrary real valued function $\eta(\mathbf{u})$ which does not vanish for any $\mathbf{u}$ of parameter domain, are associated with the same rational volume $\mathbf{V}(\mathbf{u})$ in $\mathbb{R}^3$, and similarly for $\overline{\mathbf{V}}(\bar{\mathbf{u}})$. In view of this, the two 3D volumes meet continuously along a common regular boundary surface $\mathbf{B}$, $\overline{\mathbf{V}}(\bar{\mathbf{u}})|_{\mathbf{B}} = \mathbf{V}(\mathbf{u})|_{\mathbf{B}}$, iff the $\mathbb{R}^4$ representations satisfy

$$\overline{\mathsf{V}}(\bar{\mathbf{u}})\big|_{\mathbf{B}} = \varrho(\mathbf{u})\,\mathsf{V}(\mathbf{u})\big|_{\mathbf{B}}. \tag{9}$$

Note, V and $\overline{\mathsf{V}}$ do not have to meet continuously, for $\mathbf{V}$ and $\overline{\mathbf{V}}$ joining continuously.

Suppose $\mathbf{B}$ is given by $v = 1$, $\bar{v} = 0$, $\overline{\mathbf{V}}(\bar{u}, 0, \overline{w}) = \mathbf{V}(u, 1, w)$ for all $\bar{u} = u$, $\overline{w} = w$, then continuity condition (9) results in

$$\overline{\mathsf{V}}_{i,0,k} = \varrho_0\,\mathsf{V}_{i,m,k},$$

where $\varrho_0 = \varrho(\bar{u}, 0, \overline{w}) = \text{const.} \neq 0$ or in terms of inhomogeneous coordinates of $\mathbb{R}^3$

$$\begin{aligned}\overline{\omega}_{i,0,k}\overline{\mathbf{V}}_{i,0,k} &= \varrho_0\,\omega_{i,m,k}\,\mathbf{V}_{i,m,k},\\ \overline{\omega}_{i,0,k} &= \varrho_0\,\omega_{i,m,k},\end{aligned}$$

and, therefore,

$$\overline{\mathbf{V}}_{i,0,k} = \mathbf{V}_{i,m,k}.$$

Iff $\varrho_0 = 1$, then $\overline{\omega}_{i,0,k} = \omega_{i,m,k}$ and polynomial denominators of $\mathbf{V}(\mathbf{u})$ and $\overline{\mathbf{V}}(\bar{\mathbf{u}})$ join continuously too.

$\mathbf{V}(\mathbf{u})$ and $\overline{\mathbf{V}}(\bar{\mathbf{u}})$ join GC^r continuously (with contact of order r) across $\mathbf{B}$ iff (cf. [3])

$$\frac{\partial^\alpha}{\partial \bar{v}^\alpha} \overline{\mathbf{V}}(\bar{\mathbf{u}})\Big|_{\mathbf{B}} = \frac{\partial^\alpha}{\partial \bar{v}^\alpha} \Big(\varrho(\bar{\mathbf{u}})\, \mathbf{V}(\mathbf{u}(\bar{\mathbf{u}}))\Big)\Big|_{\mathbf{B}}, \qquad \alpha = 0\,(1)\,r, \tag{10}$$

where $\mathbf{V}(\mathbf{u})$ has been reparameterized by $\mathbf{u} \to \mathbf{u}(\bar{\mathbf{u}})$. Applying the chain rule, we get the following additional condition for a GC^1 continuous connection in v direction

$$\overline{\mathbf{V}}_{\bar{v}} = \varrho_0 \left(u_1 \mathbf{V}_u + v_1 \mathbf{V}_v + w_1 \mathbf{V}_w\right) + \varrho_1 \mathbf{V}, \tag{11}$$

where $\varrho_0 > 0$, $b_1 > 0$, and for a GC^2 continuous join in v direction the additional condition

$$\begin{aligned}\overline{\mathbf{V}}_{\bar{v}\bar{v}} &= \varrho_0 \, (u_1\; v_1\; w_1) \begin{pmatrix} \mathbf{V}_{uu} & \mathbf{V}_{uv} & \mathbf{V}_{uw} \\ \mathbf{V}_{uv} & \mathbf{V}_{vv} & \mathbf{V}_{vw} \\ \mathbf{V}_{uw} & \mathbf{V}_{vw} & \mathbf{V}_{ww} \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} \\ &\quad + (\mathbf{V}_u\, \mathbf{V}_v\, \mathbf{V}_w) \begin{pmatrix} \varrho_0 u_2 + 2\varrho_1 u_1 \\ \varrho_0 v_2 + 2\varrho_1 v_1 \\ \varrho_0 w_2 + 2\varrho_1 w_1 \end{pmatrix} + \varrho_2 \mathbf{V},\end{aligned} \tag{12}$$

where we have used the abbreviations

$$u_\alpha = \frac{\partial^\alpha}{\partial \bar{v}^\alpha} u(\bar{\mathbf{u}})\Big|_{\mathbf{B}}, \quad v_\alpha = \frac{\partial^\alpha}{\partial \bar{v}^\alpha} v(\bar{\mathbf{u}})\Big|_{\mathbf{B}}, \quad w_\alpha = \frac{\partial^\alpha}{\partial \bar{v}^\alpha} w(\bar{\mathbf{u}})\Big|_{\mathbf{B}}, \quad \varrho_\alpha = \frac{\partial^\alpha}{\partial \bar{v}^\alpha} \varrho(\bar{\mathbf{u}})\Big|_{\mathbf{B}}.$$

Notice, V and $\overline{\mathsf{V}}$ do not have to meet GC^r continuously, for $\mathbf{V}$ and $\overline{\mathbf{V}}$ joining GC^r continuously.

Suppose $\mathbf{V}(\mathbf{u})$, of degree (l, m, n), and $\overline{\mathbf{V}}(\bar{\mathbf{u}})$, of degree $(l, \overline{m}, n)$, join continuously along boundary surface $\mathbf{B}$ given as above, then, GC^1 continuity condition (11) results in

$$\begin{aligned}\overline{m}\Delta^{0,1,0}\overline{\mathbf{V}}_{i,0,k} &= \varrho_1 \mathbf{V}_{i,m,k} \\ &\quad + \varrho_0 \Big((l-i)a_0\, \Delta^{1,0,0}\, \mathbf{V}_{i,m,k} + i a_1\, \Delta^{1,0,0}\, \mathbf{V}_{i-1,m,k} + m b_0\, \Delta^{0,1,0}\, \mathbf{V}_{i,m-1,k} \\ &\qquad + (n-k)c_0\, \Delta^{0,0,1}\, \mathbf{V}_{i,m,k} + k c_1\, \Delta^{0,0,1}\, \mathbf{V}_{i,m,k-1}\Big),\end{aligned} \tag{*}$$

where, for the purpose of solving (11) by comparing coefficients and to keep calculations easy, we assumed functions ϱ_0, ϱ_1 and $b_0 \equiv v_1$ to be constants, such that $\varrho_0 b_0 > 0$, function u_1 to be linear in u, $u_1 = (1-u)a_0 + u a_1$, $a_0, a_1 \in \mathbb{R}$, and function w_1 to be linear in w, $w_1 = (1-w)c_0 + w c_1$, $c_0, c_1 \in \mathbb{R}$. $(*)$ represents four conditions: the fourth equation of $(*)$ is a condition on volume weights and looks exactly like $(*)$ but with weights $\omega_{i,j,k}$ and $\overline{\omega}_{i,j,k}$, the first three equations of $(*)$ are conditions on the products of weights and 3D Bézier points. If we solve the $\overline{\omega}_{i,j,k}$ equation for $\varrho_1 \omega_{i,m,k}$ and insert this expression in the condition for $\overline{\omega}_{i,1,k}\overline{\mathbf{V}}_{i,1,k}$ a formula for 3D Bézier points $\overline{\mathbf{V}}_{i,1,k}$, very similar to $(*)$ can be derived:

$$\begin{aligned}\overline{m}\, \Delta^{0,1,0}\, \overline{\mathbf{V}}_{i,0,k} = \frac{\varrho_0}{\overline{\omega}_{i,1,k}} \Big(& (l-i)a_0\, \omega_{i+1,m,k}\, \Delta^{1,0,0}\, \mathbf{V}_{i,m,k} + i a_1\, \omega_{i-1,m,k}\, \Delta^{1,0,0}\, \mathbf{V}_{i-1,m,k} \\ & + m b_0\, \omega_{i,m-1,k}\, \Delta^{0,1,0}\, \mathbf{V}_{i,m-1,k} \\ & + (n-k)\, c_0\, \omega_{i,m,k+1}\, \Delta^{0,0,1}\, \mathbf{V}_{i,m,k} + k c_1\, \omega_{i,m,k-1}\, \Delta^{0,0,1}\, \mathbf{V}_{i,m,k-1}\Big).\end{aligned}$$

To get, for GC^1 continuity, necessary and sufficient conditions for Bézier points and weights, more general factors ϱ_0, ϱ_1 and u_1, v_1, w_1 have to be allowed. However, calculations become very technical, extensive and difficult to view. We omit them, but want to point out that [7] gives a very detailed treatment of GC^r continuous Bézier surfaces that can be extended to volume representations.

3. SOLID PRIMITIVES

Compared with solid definitions such as CSG or B-Rep, TPB volumes possess a very high free form characteristic, describe every interior point as well as every point on the boundary surface of the volume uniquely, and no assumptions are done on internal homogeneity or structure. On the other side, they also allow the exact representation of solid primitives. Bézier volumes can be constructed in various ways. First, by specification of all control points (and weights), second, by interpolation or approximation of digitized points, and third, by performing sweep, spin, loft, etc., transformations on *profile surfaces*, [1,8]. Rational volumes are, in particular, very useful for giving exact descriptions of solid primitives such as sphere, cylinder, torus, etc. Following the idea of [9], we calculate Bézier points of volumes $\mathbf{V}(\mathbf{u})$ by performing a continuum of geometric transformations $\mathbf{M}(w)$ on a profile surface $\mathbf{F}(u,v)$. To be able to generate solid primitives as well as free form volumes, we base our considerations on rational representations and homogeneous coordinates. Therefore, $\mathbf{M}(w)$ might be given by the following 4×4 matrix

$$\mathsf{M}(w) = \begin{pmatrix} m^{1,1} & m^{1,2} & m^{1,3} & t^1 \\ m^{2,1} & m^{2,2} & m^{2,3} & t^2 \\ m^{3,1} & m^{3,2} & m^{3,3} & t^3 \\ p^1 & p^2 & p^3 & s \end{pmatrix},$$

where $m^{\alpha,\beta}$ indicates rotation, reflection, scaling and shear, t^β indicates translation, p^α perspective projection, and s an overall scaling. In the following, we demonstrate the construction of solid primitives applying this method. We start by looking at *Sweep volumes* which result by moving a given surface $\mathbf{F}(u,v)$ along a prescribed curve $\mathbf{K}(w)$ which is sometimes referred to as *directrix*. Translation volumes are special sweep volumes and we are going to discuss them first. *Translation volumes* are specified by Theorem 1.

THEOREM 1. *Let $\mathsf{K}(w)$ be the homogeneous form of a rational Bézier curve $\mathbf{K}(w)$ of degree n with 4D Bézier points $\mathsf{K}_k = (\mathsf{K}_k^1, \mathsf{K}_k^2, \mathsf{K}_k^3, K_k^4)^\top = (\beta_k \mathbf{K}_k^\top, \beta_k)^\top$, where $\mathbf{K}_k = (x_k, y_k, z_k)^\top$ and weights β_k (w.l.o.g. $\beta_0 = \beta_n = 1$).*

Let $\mathsf{F}(u,v)$ be the homogeneous form of a rational TPB surface $\mathbf{F}(u,v)$ of degree (l,m) with 4D Bézier points $\mathsf{F}_{i,j} = (\mathsf{F}_{i,j}^1, \mathsf{F}_{i,j}^2, \mathsf{F}_{i,j}^3, \mathsf{F}_{i,j}^4)^\top = (\beta_{i,j}\mathbf{F}_{i,j}^\top, \beta_{i,j})^\top$, where $\mathbf{F}_{i,j} = (x_{i,j}, y_{i,j}, z_{i,j})^\top$ and weights $\beta_{i,j}$ (w.l.o.g. $\beta_{0,0} = \beta_{l,0} = \beta_{0,m} = 1$).

4D Bézier points $\mathsf{V}_{i,j,k} = \left(\mathsf{V}_{i,j,k}^1, \mathsf{V}_{i,j,k}^2, \mathsf{V}_{i,j,k}^3, \mathsf{V}_{i,j,k}^4\right)^\top = \left(\omega_{i,j,k}\mathbf{V}_{i,j,k}^\top, \omega_{i,j,k}\right)^\top$, with weights $\omega_{i,j,k}$ and $\mathbf{V}_{i,j,k} = (x_{i,j,k}, y_{i,j,k}, z_{i,j,k})^\top$ of a rational Bézier solid, $\mathbf{V}(\mathbf{u})$, represented in homogeneous form $\mathsf{V}(\mathbf{u})$, and generated by the movement of $\mathbf{F}(u,v)$ along $\mathbf{K}(w)$, such that $\mathbf{V}(u,v,0) = \mathbf{F}(u,v)$, $\mathbf{V}(0,0,w) = \mathbf{K}(w)$, are given by

$$\mathsf{V}_{i,j,k} = \mathsf{M}_k\, \mathsf{F}_{i,j}, \tag{13}$$

where $\mathsf{M}_0 = \mathsf{I}_4$ and

$$\mathsf{M}_k = \frac{1}{(\mathsf{K}_0^4)^2} \begin{pmatrix} \mathsf{K}_0^4\mathsf{K}_k^4 & 0 & 0 & \mathsf{K}_0^4\mathsf{K}_k^1 - \mathsf{K}_0^1\mathsf{K}_k^4 \\ 0 & \mathsf{K}_0^4\mathsf{K}_k^4 & 0 & \mathsf{K}_0^4\mathsf{K}_k^2 - \mathsf{K}_0^2\mathsf{K}_k^4 \\ 0 & 0 & \mathsf{K}_0^4\mathsf{K}_k^4 & \mathsf{K}_0^4\mathsf{K}_k^3 - \mathsf{K}_0^3\mathsf{K}_k^4 \\ 0 & 0 & 0 & \mathsf{K}_0^4\mathsf{K}_k^4 \end{pmatrix}.$$

PROOF. Since all $\eta(w)\mathsf{V}(\mathbf{u})$, with nonvanishing real-valued function $\eta(w)$, are associated with the same sweep volume of coordinate space, $\mathsf{V}(\mathbf{u})$ has to be defined by

$$\mathsf{V}(\mathbf{u}) = \eta(w)\,\mathsf{M}(w)\,\mathsf{F}(u,v) \tag{14}$$

with

$$\mathbf{M}(w) = \begin{pmatrix} & \mathbf{I}_3 & & \mathbf{T} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\mathbf{I}_3$ is a 3×3 unit matrix and $\mathbf{T} = \mathbf{T}(w) = \left(t^1(w), t^2(w), t^3(w)\right)^\top$ describes moving of $\mathbf{F}(u,v)$ along $\mathbf{K}(w)$.

Since $\mathbf{V}(0,0,w) = \mathbf{K}(w)$ and $\mathbf{V}(u,v,0) = \mathbf{F}(u,v)$, (14) implies

$$\mathsf{K}(w) = \eta(w)\,\mathsf{M}(w)\,\mathsf{K}_0.$$

Solving this 4×4 system of linear equations for the components of $\eta(w)\,\mathsf{M}(w)$ yields

$$\eta(w)\,\mathsf{M}(w) = \frac{1}{(\mathsf{K}_0^4)^2} \begin{pmatrix} \mathsf{K}_0^4\,\mathsf{K}^4(w) & 0 & 0 & \mathsf{K}_0^4\,\mathsf{K}^1(w) - \mathsf{K}_0^1\,\mathsf{K}^4(w) \\ 0 & \mathsf{K}_0^4\,\mathsf{K}^4(w) & 0 & \mathsf{K}_0^4\,\mathsf{K}^2(w) - \mathsf{K}_0^2\,\mathsf{K}^4(w) \\ 0 & 0 & \mathsf{K}_0^4\,\mathsf{K}^4(w) & \mathsf{K}_0^4\,\mathsf{K}^3(w) - \mathsf{K}_0^3\,\mathsf{K}^4(w) \\ 0 & 0 & 0 & \mathsf{K}_0^4\,\mathsf{K}^4(w) \end{pmatrix}.$$

Now, $\mathsf{K}(w) = \left(\mathsf{K}^1(w), \mathsf{K}^2(w), \mathsf{K}^3(w), \mathsf{K}^4(w)\right)^\top$ is given in Bézier representation, therefore, we have

$$\eta(w)\,\mathsf{M}(w) = \sum_{k=0}^{n} \mathsf{M}_k\, B_k^n(w) \tag{*}$$

with M_k as given in Theorem 1. Substituting the Bézier representations of $\mathsf{F}(u,v)$ and of $\mathsf{V}(\mathbf{u})$ as well as $(*)$ into (14) yields by comparing coefficients (13). ∎

Note, while 3D Bézier points $\mathbf{V}_{i,j,k} \in \mathbb{R}^3$ result by moving the surface net defined by $\mathbf{F}_{i,j}$ according to the control points of directrix $\mathbf{K}(w)$, this is not true for volume weights $\omega_{i,j,k} \in \mathbb{R}^3$, according to Theorem 1 they are given by

$$\omega_{i,j,k} = \frac{\beta_{i,j}\,\beta_k}{\beta_{0,0}}.$$

Also notice, although $\mathsf{V}(\mathbf{u})$ was generated by moving a surface along a w line, parametric surfaces of constant u are congruent as well and the same holds for isoparametric surfaces of constant v.

For $n = 1$ the trivariate analogy of a cylindrical surface is created for which Bézier points in w direction define parallel lines,

$$\mathbf{V}_{i,j,0} = \mathbf{F}_{i,j}, \quad \mathbf{V}_{i,j,1} = \mathbf{F}_{i,j} + \mathbf{T},$$

where $\mathbf{T} = \mathbf{K}_1 - \mathbf{K}_0$ is translation vector and $\omega_{i,j,k} = \beta_{i,j}$, for all i,j and $k = 0,1$.

A generalization of the translation volume is the *blending volume* (*loft volume*, [8]): here two noncongruent surfaces $\mathsf{F}^0(u,v)$ and $\mathsf{F}^n(u,v)$ defined by Bézier points $\mathsf{F}_{i,j}^0$ and $\mathsf{F}_{i,j}^n$ are connected appropriately. For polynomial blending, n has to be specified and, in case of $n > 1$, Bézier points of *intermediate nets* $\mathsf{F}_{i,j}^k$, $k = 1\,(1)\,n-1$. Then,

$$\mathsf{V}_{i,j,k} = \mathsf{F}_{i,j}^k. \tag{15}$$

Bézier points $\mathsf{F}_{i,j}^k$ of intermediate nets can be provided in different ways. For example, via interpolation of $n-1$ intermediate surfaces (*skinning*, *lofting*), via continuity construction (see Section 2), if $\mathsf{F}_{i,j}^0$ and $\mathsf{F}_{i,j}^n$ are supposed to be boundary surfaces of connecting volume segments or, via interactive input procedures.

$n = 1$ gives a very special blending volume, the *linear blending volume.* It is the trivariate generalization of a ruled surface and thus is sometimes called *ruled volume* (see, e.g., [1]). Defined by taking a convex combination,

$$\mathbf{V}(\mathbf{u}) = (1-w)\,\mathbf{F}^0(u,v) + w\,\mathbf{F}^n(u,v), \qquad w \in [0,1],$$

of surfaces $\mathbf{F}^0(u,v)$ and $\mathbf{F}^n(u,v)$, $n=1$, Bézier points are given by (15).

Volumes of revolution also referred to as *spin volumes* are sweep volumes as well and can be created easily as stated by Theorem 2 (see [5,6]).

THEOREM 2. *Let $\mathsf{F}(u,v)$ be the homogeneous form of a rational TPB surface $\mathbf{F}(u,v)$ of degree (l,m) with 4D Bézier points $\mathsf{F}_{i,j} = \left(\mathsf{F}^1_{i,j}, \mathsf{F}^2_{i,j}, \mathsf{F}^3_{i,j}, \mathsf{F}^4_{i,j}\right)^\top = \left(\beta_{i,j}\mathbf{F}^\top_{i,j}, \beta_{i,j}\right)^\top$, where $\mathbf{F}_{i,j} = (x_{i,j}, y_{i,j}, z_{i,j})^\top$ and weights $\beta_{i,j}$ (w.l.o.g. $\beta_{0,0} = \beta_{l,0} = \beta_{0,m} = 1$).*

The rational TPB volume $\mathbf{V}(\mathbf{u})$ of degree (l,m,n), $n=2$, with 4D Bézier points

$$\mathsf{V}_{i,j,k} = \mathsf{M}_k\,\mathsf{F}_{i,j}, \tag{16}$$

where $\mathsf{M}_0 = \mathsf{I}_4$,

$$\mathsf{M}_1 = \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} & 0 & 0 \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} & 0 & 0 \\ 0 & 0 & \cos\frac{\alpha}{2} & 0 \\ 0 & 0 & 0 & \cos\frac{\alpha}{2} \end{pmatrix}, \quad \mathsf{M}_2 = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

represents a rotation of $\mathbf{F}(u,v)$ through the arc α, with $|\alpha| \le 180^0$, round the z-axis such that $\mathbf{V}(u,v,0) = \mathbf{F}(u,v)$.

PROOF. For a rotation of $\mathbf{F}(u,v)$ about the z-axis by the angle α, M of (14) now has to be given by

$$\mathsf{M} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

With the same arguments as in proof of Theorem 1, we result in equation (*) but with M_k now given by

$$\mathsf{M}_k = \begin{pmatrix} \dfrac{\mathsf{K}^1_k\mathsf{K}^1_0 + \mathsf{K}^2_k\mathsf{K}^2_0}{(\mathsf{K}^1_0)^2 + (\mathsf{K}^2_0)^2} & -\dfrac{\mathsf{K}^2_k\mathsf{K}^1_0 - \mathsf{K}^1_k\mathsf{K}^2_0}{(\mathsf{K}^1_0)^2 + (\mathsf{K}^2_0)^2} & 0 & 0 \\ \dfrac{\mathsf{K}^2_k\mathsf{K}^1_0 - \mathsf{K}^1_k\mathsf{K}^2_0}{(\mathsf{K}^1_0)^2 + (\mathsf{K}^2_0)^2} & \dfrac{\mathsf{K}^1_k\mathsf{K}^1_0 + \mathsf{K}^2_k\mathsf{K}^2_0}{(\mathsf{K}^1_0)^2 + (\mathsf{K}^2_0)^2} & 0 & 0 \\ 0 & 0 & \dfrac{\mathsf{K}^3_k}{\mathsf{K}^3_0} & 0 \\ 0 & 0 & 0 & \dfrac{\mathsf{K}^4_k}{\mathsf{K}^4_0} \end{pmatrix}. \tag{*}$$

Note, $k=0$ yields $\mathsf{M}_0 = \mathsf{I}_4$. Since M_k is supposed to represent a rotation, entries of M_k, $k = 1,2$ can be specified further:

Rotations (about the z-axis by the angle α) can be described exactly by rational quadratics defined by Bézier points $\mathsf{K}_k = \left(\mathsf{K}^1_k, \mathsf{K}^2_k, \mathsf{K}^3_k, \mathsf{K}^4_k\right)^\top = \left(\beta_k\mathbf{K}^\top_k, \beta_k\right)^\top$, $k = 0,1,2$, where $\beta_0 = 1$, $\beta_1 = \cos\frac{\alpha}{2}$, $\beta_2 = 1$, and rotation of $\mathbf{K}_0$ by α, $\mathbf{R}(\alpha)$, yields $\mathbf{K}_2$ while rotation of $\mathbf{K}_0$ by $\frac{\alpha}{2}$, $\mathbf{R}\left(\frac{\alpha}{2}\right)$, followed by a radial scaling with $1/\cos\frac{\alpha}{2}$, $\mathbf{S}\left(1/\cos\frac{\alpha}{2}\right)$, gives $\mathbf{K}_1$, see [3, p. 152].

Now, solving of $\mathsf{K}_2 = \eta\mathsf{R}(\alpha)\mathsf{K}_0$ and of $\mathsf{K}_1 = \eta\mathsf{S}\left(1/\cos\frac{\alpha}{2}\right)\mathsf{R}\left(\frac{\alpha}{2}\right)\mathsf{K}_0$ for the components of $\eta\mathsf{R}(\alpha)$ and of $\eta\mathsf{S}\left(1/\cos\frac{\alpha}{2}\right)\mathsf{R}\left(\frac{\alpha}{2}\right)$ proves that $\eta\mathsf{R}(\alpha)$ is equal to M_2 and $\eta\mathsf{S}\left(1/\cos\frac{\alpha}{2}\right)\mathsf{R}\left(\frac{\alpha}{2}\right)$ is equal to M_1 in (*) and that components are given as stated in Theorem 2. ∎

Note, Bézier points $\mathbf{V}_{i,j,1} = \mathbf{M}_1 \mathbf{F}_{i,j}$ become infinite control points for $|\alpha| = 180^0$. Figures 2 and 3 show two examples of volumes of revolution, in both cases solid definition involves several infinite control points.

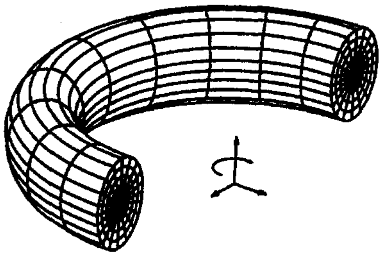

Figure 2. Solid half-torus described by a rational TPB volume of degree (1,2,2).

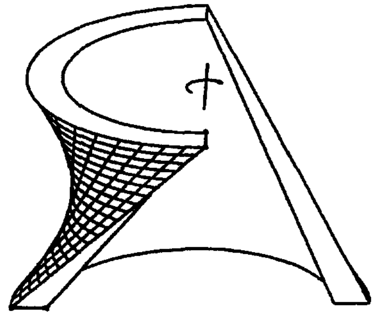

Figure 3. Part of a solid hyperboloid of revolution of one sheet described by a rational TPB volume of degree (1,1,2).

Rotations about the x-axis and the y-axis can be described analogously.

Since matrices $\mathbf{M}_k$ (and *start point* $\mathbf{K}_0$) define directrix $\mathbf{K}(w)$, different curves can be described by changing these matrices and all w parameter lines of $\mathbf{V}(\mathbf{u})$ will adapt the geometry introduced by the $\mathbf{M}_k$. For example, if we choose parameter s_2 in

$$\mathbf{K}(w) = \eta(w)\, \mathbf{M}(w)\, \mathbf{K}_0 = \sum_{k=0}^{2} \hat{\mathbf{M}}_k \mathbf{K}_0\, B_k^2(w) \tag{17}$$

with $\hat{\mathbf{M}}_k = s_k \mathbf{M}_k$, $\mathbf{M}_k$ according to (16), $s_0 = s_2 = 1$, appropriately, we obtain an arc of an ellipse, parabola or hyperbola, [5,6]: A *parabola* results for $s_2 = s^* \equiv 1/\cos\frac{\alpha}{2}$, because in that case $\mathbf{K}(w)$ is a nonrational, polynomial quadratic Bézier curve (all weights are equal!), for $s_2 > s^*$ a *hyperbola* results, for $s_2 < s^*$ an *ellipse* and, for $s_2 = 1$, we have Theorem 2, i.e., $\mathbf{K}(w)$ defines a circle, see Figure 4.

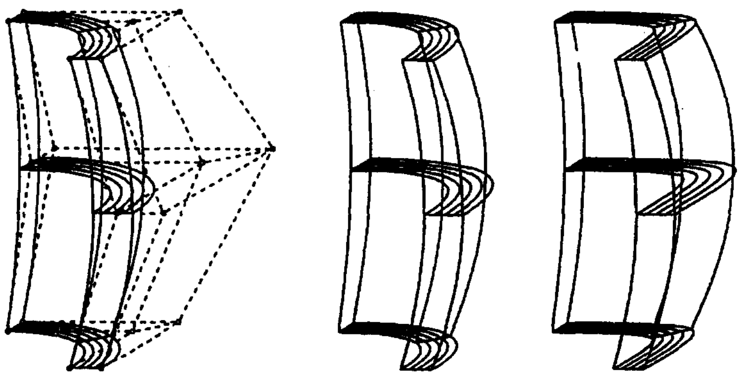

Figure 4. Three rational TPB volumes with arcs of circles ($s_2 = 1$), parabolas ($s_2 = s^*$, middle) and hyperbolas ($s_2 = 3s^*$), as w parameter lines.

4. CURVE AND SURFACE MODELLING

Free form volumes can be interpreted as deformation of the associated parameter domain, i.e., the unit cube, under the mapping induced by their defining equation. In general, the position of every point in the interior as well as those on the boundary surfaces is altered. It is of particular interest for designers to be able to embed curves, surfaces, and volumes in the parameter domain of a free form volume, since this allows concentrating on well-defined subsets of the parameter space. In simple cases, these subsets are in fact isoparametric lines and planes, which under the deformation are mapped to isoparametric curves and surfaces of the free form volume. The more interesting subsets, however, are nonisoparametric subsets, which might be given by any kind of representation. In context of these more general nonisoparametric subsets, the term FFD (*Free Form Deformation*) is commonly used [10–13].

In order to carry out an FFD modelling process, for every point of the object, we must first find the associated parameter value in the parameter space of the volume. For this, we define a box-shaped local coordinate system, the deformation domain, which includes all parts of the object to be deformed. It might be subdivided into sub-boxes corresponding to a segmented TPB volume. By comparing coordinates, we find the sub-box $\mathbf{Q}_0$, defined by $\mathbf{P}_0$, $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W}$ as in Figure 5 which contains the given point $\mathbf{P}$ of the object. We have

$$\mathbf{P} = \mathbf{P}_0 + u\,\mathbf{U} + v\mathbf{V} + w\,\mathbf{W}, \tag{18}$$

where local coordinates u, v, w are calculated by

$$u = \frac{\mathbf{V} \times \mathbf{W} \cdot (\mathbf{P} - \mathbf{P}_0)}{\mathbf{V} \times \mathbf{W} \cdot \mathbf{U}}, \quad v = \frac{\mathbf{U} \times \mathbf{W} \cdot (\mathbf{P} - \mathbf{P}_0)}{\mathbf{U} \times \mathbf{W} \cdot \mathbf{V}}, \quad w = \frac{\mathbf{U} \times \mathbf{V} \cdot (\mathbf{P} - \mathbf{P}_0)}{\mathbf{U} \times \mathbf{V} \cdot \mathbf{W}}.$$

Now we have to prescribe the polynomial degree, corresponding weights and a control point grid which covers the deformation domain. Figure 6 illustrates this process in case of a volume segment of degree $(3, 2, 2)$. The control points of the grid are given by

$$\mathbf{V}_{i,j,k} = \mathbf{P}_0 + \frac{i}{l}\mathbf{U} + \frac{j}{m}\mathbf{V} + \frac{k}{n}\mathbf{W}, \tag{19}$$

and initially weights $\omega_{i,j,k}$ are chosen to be equal to one.

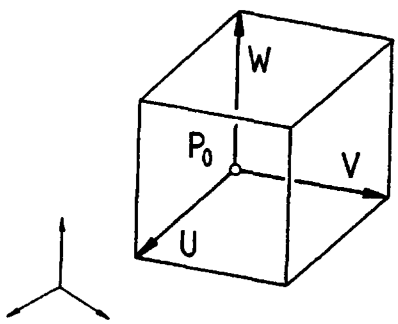

Figure 5. Local coordinate system.

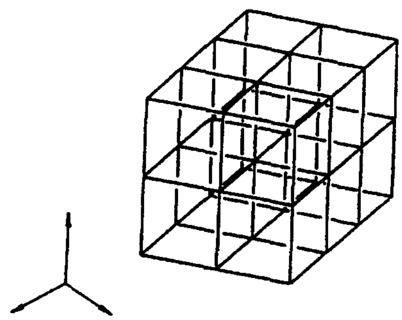

Figure 6. Control point grid.

The actual deformation of the object involves translation of points, $\mathbf{V}_{i,j,k} \mapsto \overline{\mathbf{V}}_{i,j,k}$, changing of weights, $\omega_{i,j,k} \mapsto \overline{\omega}_{i,j,k}$, and evaluation of the FFD defining equation with coefficients $\overline{\mathbf{V}}_{i,j,k}$, weights $\overline{\omega}_{i,j,k}$, and parameter values u, v, w of the object point $\mathbf{P}$ calculated above.

From a mathematical point of view, the FFD construction involves composing two mappings, the mappings of the object and of the volume definition (cf. Figure 7) and has been investigated in [14–16]. For the case of modelling a rational Bézier curve using a rational TPB volume, the mathematical foundation for exactly and explicitly describing the FFD is given by Theorem 3.

THEOREM 3. *Let* $\mathbf{K}(t) = (u(t), v(t), w(t))^\top$ *be a rational Bézier curve of degree* N *with Bézier points* $\mathbf{K}_I = (u_I, v_I, w_I)^\top$ *and weights* β_I.

Let $\mathbf{V}(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))^\top$ *be a rational TPB volume of degree* (l, m, n) *with Bézier points* $\mathbf{V}_{i,j,k} = (x_{i,j,k}, y_{i,j,k}, z_{i,j,k})^\top$ *and weights* $\omega_{i,j,k}$.

$\mathbf{V}(t) = \mathbf{V}(\mathbf{K}(t)) = \mathbf{V}(u(t), v(t), w(t))$ is rational and can be represented as rational Bézier curve of degree $rN, r = l + m + n$. We have, in terms of homogeneous coordinates,

$$\mathbf{V}(t) = \mathbf{V}(\mathbf{K}(t)) = \sum_{R=0}^{rN} \mathbf{V}_R B_R^{rN}(t), \tag{20}$$

where 4D Bézier points $\mathbf{V}_R = \left(\mathsf{V}_R^1, \mathsf{V}_R^2, \mathsf{V}_R^3, \mathsf{V}_R^4\right)^\top = \left(\Omega_R \mathbf{V}_R^\top, \Omega_R\right)^\top$ are given by[1]

$$\mathbf{V}_R = \sum_{|\mathbf{I}|=R} B^{l,m,n}(\mathbf{I})\, C_R^{l,m,n}(N, \mathbf{I})\, \mathbf{V}_{0,0,0}^{l,m,n}\left(u^l_{\mathbf{I}^u}, v^m_{\mathbf{I}^v}, w^n_{\mathbf{I}^w}\right), \tag{21}$$

with combinatorial constants

$$C_R^{l,m,n}(N, \mathbf{I}) = \frac{\prod_{Q^u=1}^{l} \binom{N}{I^u_{Q^u}} \prod_{Q^v=1}^{m} \binom{N}{I^v_{Q^v}} \prod_{Q^w=1}^{n} \binom{N}{I^w_{Q^w}}}{\binom{rN}{R}}, \tag{22}$$

and,

$$B^{l,m,n}(\mathbf{I}) = \prod_{Q^u=1}^{l} \beta_{I^u_{Q^u}} \prod_{Q^v=1}^{m} \beta_{I^v_{Q^v}} \prod_{Q^w=1}^{n} \beta_{I^w_{Q^w}}. \tag{23}$$

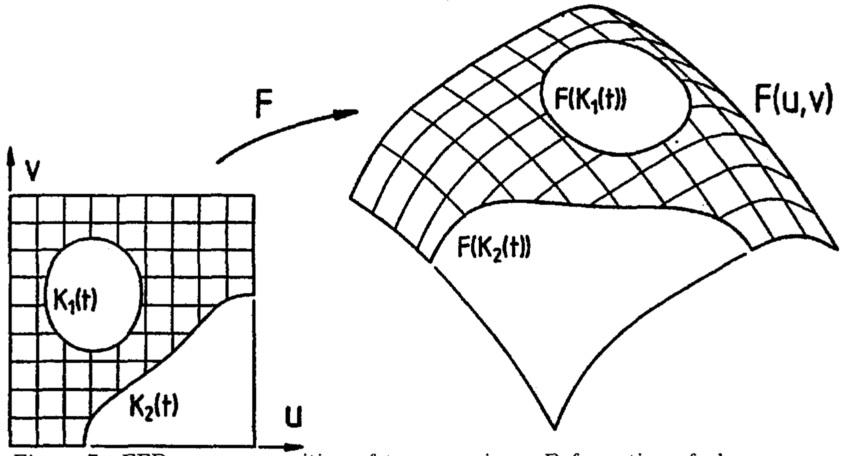

Figure 7. FFD as a composition of two mappings: Deformation of planar curves $\mathbf{K}_1(t)$ and $\mathbf{K}_2(t)$ by a surface mapping $\mathbf{F}(u, v) : \mathbb{R}^2 \mapsto \mathbb{R}^3$.

The proof of Theorem 3 is analogous to the proofs presented in [16] (cf. [15]).

$\sum_{|\mathbf{I}|=R}$ has the meaning of summation over all $\mathbf{I} = (\mathbf{I}^u, \mathbf{I}^v, \mathbf{I}^w)$, where $\mathbf{I}^u = (I_1^u, \dots, I_l^u)$, $\mathbf{I}^v = (I_1^v, \dots, I_m^v)$, $\mathbf{I}^w = (I_1^w, \dots, I_n^w)$, and where $0 \le I_1^u, \dots, I_l^u \le N$, $0 \le I_1^v, \dots, I_m^v \le N$, $0 \le I_1^w, \dots, I_n^w \le N$, $|\mathbf{I}| = |\mathbf{I}^u| + |\mathbf{I}^v| + |\mathbf{I}^w| = I_1^u + \dots + I_l^u + I_1^v + \dots + I_m^v + I_1^w + \dots + I_n^w = R$.

$\mathbf{V}_{0,0,0}^{l,m,n}$ is defined recursively by de Casteljau's construction. The argument $\left(u^l_{\mathbf{I}^u}, v^m_{\mathbf{I}^v}, w^n_{\mathbf{I}^w}\right)$ indicates that $\mathbf{V}_{l,m,n}^{0,0,0}$ has to be calculated by performing l de Casteljau constructions in u direction for the u parameter values given by the indices $\mathbf{I}^u = (I_1^u, \dots, I_l^u)$, i.e., for the parameter values $u_{I_1^u}, \dots, u_{I_l^u}$, m de Casteljau constructions in v direction for the v parameter values given by the indices $\mathbf{I}^v = (I_1^v, \dots, I_m^v)$, i.e., for the parameter values $v_{I_1^v}, \dots, v_{I_m^v}$ and n de Casteljau constructions in w direction for the w parameter values given by the indices $\mathbf{I}^w = (I_1^w, \dots, I_n^w)$, i.e., for the parameter values $w_{I_1^w}, \dots, w_{I_n^w}$. Calculations for different parameter values commute, and the order in which we carry out calculations has no influence on the final result.

Note, isoparametric lines of (u, v, w) domain space should not be handled by Theorem 3 which, in that case, would yield degree raised isoparametric Bézier curves of $\mathbf{V}(u, v, w)$.

[1]Please note, notation varies from the one used in [16].

Spline curves are processed using Theorem 3 one curve segment at a time. Modelling by spline volumes requires the determination of intersections of $\mathbf{K}(t)$ and boundary planes of volume segments in domain space and splitting of $\mathbf{K}(t)$ at these intersection points by adding new knots applying de Casteljau's curve subdivision algorithm. Then, for all segments of $\mathbf{V}(\mathbf{u})$ Theorem 3 can be adapted. Smoothness of $\mathbf{V}(\mathbf{K}(t))$ results from application of the chain rule: if $\mathbf{K}(t)$ is C^a-continuous in $t = t^*$, and $\mathbf{V}(\mathbf{u})$ is C^b-continuous in the corresponding point $\mathbf{K}(t) = \mathbf{K}(t^*)$ of (u, v, w) domain space, $\mathbf{V}(\mathbf{K}(t))$ is C^e-continuous in $\mathbf{V}(\mathbf{K}(t^*)$, with $e = \min\{a, b\}$.

Generalizing Theorem 3, the mathematical foundation for the FFD approach to rational surface design via rational volume modelling is provided by Theorem 4 (see [15,16]).

THEOREM 4. *Let* $\mathbf{F}(\mu,\nu) = \left(u(\mu,\nu), v(\mu,\nu), w(\mu,\nu)\right)^{\top}$ *be a rational TPB surface of degree* (L, M) *with Bézier points* $\mathbf{F}_{I,J} = (u_{I,J}, v_{I,J}, w_{I,J})^{\top}$ *and weights* $\beta_{I,J}$.

Let $\mathbf{V}(u,v,w) = \left(x(u,v,w), y(u,v,w), z(u,v,w)\right)^{\top}$ *be a rational TPB volume of degree* (l, m, n) *with Bézier points* $\mathbf{V}_{i,j,k} = (x_{i,j,k}, y_{i,j,k}, z_{i,j,k})^{\top}$ *and weights* $\omega_{i,j,k}$.

$\mathbf{V}(\mu,\nu) = \mathbf{V}(\mathbf{F}(\mu,\nu)) = \mathbf{V}\left(u(\mu,\nu), v(\mu,\nu), w(\mu,\nu)\right)$ *is rational and can be represented as rational TPB surface of degree* $(rL, rM), r = l + m + n$. *We have, in terms of homogeneous coordinates,*

$$\mathsf{V}(\mu,\nu) = \mathsf{V}(\mathsf{F}(\mu,\nu)) = \sum_{R=0}^{rL} \sum_{S=0}^{rM} \mathsf{V}_{R,S}\, B_R^{rL}(\mu)\, B_S^{rM}(\nu), \tag{24}$$

where 4D Bézier points $\mathsf{V}_{R,S} = \left(\mathsf{V}^1_{R,S}, \mathsf{V}^2_{R,S}, \mathsf{V}^3_{R,S}, \mathsf{V}^4_{R,S}\right)^{\top} = \left(\Omega_{R,S}\mathbf{V}^{\top}_{R,S}, \Omega_{R,S}\right)^{\top}$ *are given by*[2]

$$\mathsf{V}_{R,S} = \sum_{|\mathbf{I}|=R} \sum_{|\mathbf{J}|=S} B^{l,m,n}(\mathbf{I},\mathbf{J})\, C_R^{l,m,n}(L,\mathbf{I})\, C_S^{l,m,n}(M,\mathbf{J})\, \mathsf{V}^{l,m,n}_{0,0,0}\left(u^l_{\mathbf{I}^u\mathbf{J}^u}, v^m_{\mathbf{I}^v\mathbf{J}^v}, w^n_{\mathbf{I}^w\mathbf{J}^w}\right). \tag{25}$$

$\sum_{|\mathbf{I}|=R}$ *with* $\mathbf{I} = (\mathbf{I}^u, \mathbf{I}^v, \mathbf{I}^w)$ *has the same meaning as in Theorem 3,* $\sum_{|\mathbf{J}|=S}$ *with* $\mathbf{J} = (\mathbf{J}^u, \mathbf{J}^v, \mathbf{J}^w)$ *similarly. Constants* $C_R^{l,m,n}(L,\mathbf{I})$ *and* $C_S^{l,m,n}(M,\mathbf{J})$ *are calculated according to equation (22) and* $B^{l,m,n}(\mathbf{I},\mathbf{J})$ *is defined as in (23) but now using surface weights* $\beta_{I,J}$. $\mathsf{V}^{l,m,n}_{0,0,0}$ *is specified by de Casteljau's construction.*

As with curves, remarks concerning isoparametric planes of domain space of $\mathbf{V}(\mathbf{u})$ as well as treatment of spline surfaces prove to be right again.

We observe that because FFDs involve functional composition, the degree of FFDs can grow very quickly. High degrees might cause problems in successive interrogation actions such as intersection calculations and also might not be supported by some CAD systems. Therefore, the exact description of FFDs will not always be the best approach. Suitable approximation techniques taking advantage of the knowledge of the exact and explicit FFD representation to describe $\mathbf{V}(\mathbf{K}(t))$ and $\mathbf{V}(\mathbf{F}(u,v))$ by lower polynomial degrees are under investigation.

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[2] Please note, calculation of combinatorial constants is wrong in [16, Section IV.2].

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