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Axiomatisation and decidability of multi-dimensional Duration Calculus

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Abstract

The Shape Calculus is a spatio-temporal logic based on an *n*-dimensional Duration Calculus tailored for the specification and verification of mobile real-time systems. After showing non-axiomatisability, we give a complete embedding in *n*-dimensional interval temporal logic and present two different decidable subsets, which are important for tool support and practical use.

Keywords: Real-time systems; Mobile systems; Spatial logic; Temporal logic; Duration Calculus

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1. Introduction

1.1. Motivation

Mobile real-time systems are omnipresent today, e.g., in airplane and railroad control systems. Failures in these systems may have severe consequences which can even endanger lives. Formal specification and automatic verification are promising approaches to increase the safety of such systems. However, for these systems real-time aspects as well as spatial aspects are important.

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Thus, commonly used formalisms that concentrate on either timing or spatial behaviour fall short in these cases because they need to abstract from important issues.

A pivotal task in the UniForM [32] project in cooperation with the industrial partner Elpro was the development of a control for a single-tracked line segment (SLS) for tramways. The problem is to ensure the safety of trams if only one track is available and this track is passed in both directions and occupied by up to two trams simultaneously as long as they head into the same direction. A controller has been derived, simulated, and partially verified using techniques for real-time systems, namely PLC-Automata [19]. However, the main safety requirement, i.e., mutual exclusion of trams with opposite directions on the critical section, is a spatio-temporal property and cannot be expressed in purely time-dependent models like PLC Automata [18].

Similar problems arise in the specifications of mobile robots [42]. Each robot itself constitutes a real-time system, whereas the specification of the overall system behaviour has additional spatial requirements, for example collision avoidance.

1.2. Research contributions

The shortcomings described above led us to the idea to extend the Duration Calculus, a well-known formalism for real-time systems, with proven applicability [28], to be able to describe also spatial properties. The use of the formalism is similar to the use of the original Duration Calculus when no spatial reasoning is required. Thus, experienced users of temporal logics can easily adopt the new features.

We present the Shape Calculus[1] (SC), a spatio-temporal logic based on the Duration Calculus, extending the results in [42] and [43]. Shape Calculus is interval based and possesses an integral operator $\int$ for measuring time as well as space. We elaborate that this formalism is well suited for the application domain of mobile real-time systems. We present four major results for this formalism. First, we prove that the full logic is undecidable and non axiomatisable, even for discrete infinite models of time and space. To this end, we present a reduction of its validity problem to the emptiness problem of tiling languages. The full real-time logic Duration Calculus is known to be non-axiomatisable for continuous temporal domains, but still decidable for a subset in the discrete setting which shows to be undecidable in the multi-dimensional case. This relates to the undecidability of multi-dimensional products of decidable modal logics as discussed in [24]. Second, we present an axiomatisation of Shape Calculus relatively to an n-dimensional interval logic without the $\int$ operator, a result similar to the one for continuous time Duration Calculus. In practice, acceptance of formal methods is increased dramatically by tool support. Hence, we discuss decidable subsets of the Shape Calculus. Our third contribution is a decidable subset of Shape Calculus based on results for discrete Duration Calculus. There the subset assumes a discrete and infinite temporal domain but finite spatial domains. The decision procedure reduces validity to emptiness of regular languages. This subset has already led to a prototypical implementation of a model checker [38]. Forth, we elaborate a decidable subset of discrete Shape Calculus using ideas from logic combination [24] and relating a syntactical subset of Shape Calculus without chop alternation to fusions of Duration Calculus. This approach proceeds by reducing validity to iteratively checking emptiness of regular languages.

[1] The name Shape Calculus was proposed by A. Ravn during a presentation of early ideas.

Technical context. We review the technical context of our contribution.

1.3. Real-time systems

Concerning real-time aspects, the operational model of Timed Automata [2] is the most popular and widespread. Due to its decidable emptiness problem, it allows for the automatic verification of real-time properties. Thus, it enables the development of tools like Uppaal [4,5] and Kronos [8], which contributed much to the applicability of Times Automata as shown in several case studies [29,33].

The temporal logics Duration Calculus (DC) [12,28], TCTL [30], and TPTL [3] provide the possibility to specify and reason about real-time behaviour. The tool DCValid [36] is able to verify a restricted subset of Duration Calculus using the second order model checker MONA [31] as backend. A detailed discussion on decidability for subsets of Duration Calculus is given in [22]. However, neither the automata theoretic nor the logical formalism do provide support for specifying and verifying spatial properties.

Except for our approach, other extensions of Duration Calculus consider hybrid aspects [13] and superdense time [37].

1.4. Process calculi

For modelling mobility of concurrent processes, the π calculus was introduced by R. Milner [35]. However, it considers a notion of mobility different from the Shape Calculus. In the π calculus mobility stems from the change of links between processes. A spatial logic for the π calculus is proposed in [9]. This logic integrates support for reasoning about the behaviour and the structure of systems of concurrent π calculus systems. A model checker for a subset of this logic is implemented in [44].

Inspired by the π calculus, the Ambient Calculus [10] considers processes that are executed in hierarchically nested environments (called ambients) and that may be transfered from one ambient to another. For grasping the structure of the nested ambients and the process behaviour, in [34] a spatial logic for the Ambient Calculus based on TLA is proposed. Similarly to [9], the ambient logic based on modal logic is introduced in [11]. The model checking problem of the full logic against ambient calculus processes is undecidable and for finite processes (without replication) still PSPACE hard [16,15,14]. A compositional approach is proposed in [21]. It investigates the combination of logics via fusion and product. The notion of location and space is covered by a hybrid logic using nominals [7] whereas the temporal properties are expressed in a temporal logic. The combined logic is used to describe the overall system behaviour. However, all these approaches do not facilitate quantitative measuring, neither of time nor of space. Thus, it is impossible to express, for example, an upper bound on the reaction time or a minimal distance of two robots that needs to be kept.

1.5. Spatial and spatio-temporal logics

The Region Connection Calculus (RCC) [39] constitutes a spatial logic having regions as basic entities. Thereby, it permits *qualitative* reasoning about relations, e.g., the part-of-relation or tangentiality. Its main area of application is AI. As there is no notion of time in RCC, it has been extended in [25] to a spatio-temporal formalism for describing mobility *qualitatively*.

Spatio-temporal logics based on modal logics are proposed in [6,1,40] and different techniques of combining modal logics—namely fusion and product—is extensively investigated in [23,24]. The logical and mathematical background developed therein can be used to create various spatio-temporal logics. However, none of these logics proposed allows for quantitative spatial and temporal measures as needed for our intended application domain of physically mobile real-time systems. Yet, the second decidable subset we present in this paper is gained by treating our formalism as a fusion of instances of Duration Calculus.

1.6. Organisation of the paper

After giving a short introduction to SC in Section 2, we show in Section 3 that SC is not axiomatisable, but nevertheless it can be completely axiomatised *relatively* to the n-dimensional extension of interval temporal logic, which is presented in Section 4. In Section 5 we present two decidable subsets of discrete SC: one obtained by imposing restrictions on the class of models, another one by imposing restrictions on the class of formulae.

2. Shape Calculus

In this section, we introduce the Shape Calculus originally proposed in [42]. Here we make use of a simplified version.

In Duration Calculus [12], the behaviour of a system is modelled by a set of time-dependent variables (observables) whose values change over time. We adopt this approach and use Boolean observables that depend on space and time. We may choose to have discrete or continuous time and space depending on the current application. With the number of spatial and temporal dimensions, say n, being fixed a priori, the semantics of an observable X is given by a trajectory $\mathcal{I}$

$$\mathcal{I}[\![X]\!] : \mathbb{R}^n_{\geqslant 0} \to \{0,1\}$$

in the continuous case or as a function with domain $\mathbb{N}^n$ for the discrete case. In general, we denote the spatio-temporal domain by $\mathbb{T}$.

Example 1. To model a mobile robot moving on the floor, we need two spatial and one temporal dimension, so we fix $n = 3$. We employ two observables R and A. The observable R is true for a point in space and time if and only if the robot occupies this point in space at the given moment in time. Similarly, the bounded safe area is modelled by the observable A.

As we will measure time and space, we have to guarantee that an integral exists and therefore require Riemann-integrability of all functions.

The language of SC is built from state expressions, terms, and formulae. A state expression characterises properties of one point in time and space. They are denoted by π and built from Boolean combinations of observables. The semantics is given by a function $\mathcal{I}[\![\pi]\!] : \mathbb{R}^n_{\geqslant 0} \to \{0,1\}$ defined as a straightforward extension of trajectories of observables.

$$\begin{aligned}\mathcal{I}[\![\neg\pi]\!](\vec{z}) &\stackrel{df}{=} 1 - \mathcal{I}[\![\pi]\!](\vec{z})\\ \mathcal{I}[\![\pi \wedge \pi']\!](\vec{z}) &\stackrel{df}{=} \mathcal{I}[\![\pi]\!](\vec{z}) \cdot \mathcal{I}[\![\pi']\!](\vec{z})\end{aligned}$$

State expressions are formulae of propositional logic. Like in propositional logic, we therefore define two state expressions π and π' to be equivalent, denoted by $\pi \equiv \pi'$, if for all interpretations $\mathcal{I}$ the equality $\mathcal{I}[\![\pi]\!] = \mathcal{I}[\![\pi']\!]$ holds.

Example 2. The state expression $R \wedge \neg A$ describes exactly the points in space-time where the robot is outside its restricted area. The interpretation assigns 1 to all points satisfying the condition and 0 to all others.

A *term* θ is either a measure $\int \pi$, where π is a state expression, a rigid variable x, i.e., a variable that does not change over time, the special symbol $\ell_{\vec{e}_i}$ denoting the diameter of the n-dimensional interval (hypercube) under consideration along the i-th unit vector $\vec{e}_i$ or the application of a function f. Commonly used functions are summation or multiplication.

$$\theta ::= \textstyle\int \pi \mid x \mid \ell_{\vec{e}_i} \mid f(\theta_1, \ldots, \theta_k)$$

The value of a rigid variable is a real number or a natural number, depending on the time domain. It is determined by a valuation $\mathcal{V}$ which is a function mapping the variables to the spatio-temporal domain. The set of valuations is denoted by *Val*. The semantics of terms assigns a real number to each n-dimensional interval from the set $\mathsf{Int}^n \stackrel{df}{=} \{[b_1, f_1] \times \cdots \times [b_n, f_n] | b_i, f_i \in \mathbb{R}\}$ of all n-dimensional intervals. Thus, it is a function $\mathcal{I}[\![\theta]\!] : \mathsf{Int}^n \times Val \to \mathbb{R}$ and defined in the expected way, i.e., let

$$\mathcal{M} = [b_1, f_1] \times \cdots \times [b_i, f_i] \times \cdots \times [b_n, f_n] \in \mathsf{Int}^n$$

and $\mathcal{V} \in Val$ then

$$\begin{aligned}\mathcal{I}[\![\textstyle\int \pi]\!](\mathcal{V}, \mathcal{M}) &\stackrel{df}{=} \int_{\mathcal{M}} \mathcal{I}[\![\pi]\!]\\ \mathcal{I}[\![\ell_{\vec{e}_i}]\!](\mathcal{V}, \mathcal{M}) &\stackrel{df}{=} f_i - b_i\\ \mathcal{I}[\![x]\!](\mathcal{V}, \mathcal{M}) &\stackrel{df}{=} \mathcal{V}(x)\\ \mathcal{I}[\![f(\theta_1, \ldots, \theta_k)]\!](\mathcal{V}, \mathcal{M}) &\stackrel{df}{=} f_{\mathcal{I}}(\mathcal{I}[\![\theta_1]\!](\mathcal{V}, \mathcal{M}), \ldots, \mathcal{I}[\![\theta_k]\!](\mathcal{V}, \mathcal{M}))\end{aligned}$$

Example 3. The term $\int(R \wedge \neg A)$ is the measure of all points violating the requirement.

Formulae are interpreted over n-dimensional intervals and incorporate a special "chop" operator ($\langle\rangle$) to partition the current interval into two parts. A formula $F \langle \vec{e}_x \rangle G$ is evaluated to true, if the interval can be split along the x-axis into two parts, the first satisfying F and the second satisfying

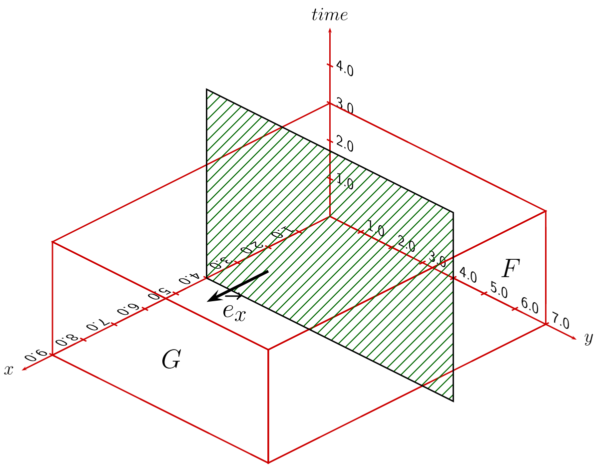

Fig. 1. Illustration for $F \langle \vec{e}_x \rangle G$.

G. This is sketched in Fig. 1. As we consider a many-dimensional logic, we allow chops along each cartesian axis. Formally, we define the set of *formulae* by

$$F ::= F_1 \langle \vec{e}_i \rangle F_2 \mid p(\theta_1, \ldots, \theta_k) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \exists x : F$$

where p is a predicate symbol like $=$ or $\leqslant$, x a rigid variable that does not change over time and $\vec{e}_i$ the ith unit vector. The other Boolean connectives can be defined as the usual abbreviations. We only give the definition of "chop" here as the other operators and the existential quantifier are defined according to in First-Order Logic.

We employ the following notation for describing the application of the chop operation on intervals.

Definition 4 (*Notation*)**.** Let $\mathcal{M} = [b_1, f_1] \times \cdots \times [b_m, f_m]$ denote an m-dimensional interval. We denote the lower bound b_i of the ith dimension by $\min_i \mathcal{M}$ and the upper bound $\vec{e}_i$ by $\max_i \mathcal{M}$, respectively. Furthermore, denote by $\mathcal{M} \prec_i r \stackrel{df}{=} [b_1, f_1] \times \cdots \times [b_i, r] \times \cdots \times [b_m, f_m]$ the first subinterval obtained by chopping the original interval along the ith axis at position r and the second part by $\mathcal{M} \succ_i r \stackrel{df}{=} [b_1, f_1] \times \cdots \times [r, f_i] \times \cdots \times [b_m, f_m]$. The "interior" $\mathcal{M}^-$ of $\mathcal{M}$ is defined by $\mathcal{M}^- = [b_1, f_1) \times \cdots \times [b_m, f_m)$.

Using this notation the semantics of the chop operator is defined as follows.

$$\mathcal{I}[\![F_1 \langle \vec{e}_i \rangle F_2]\!](\mathcal{V}, \mathcal{M}) = \text{true}$$

iff there is an $m \in [\min_i \mathcal{M}, \max_i \mathcal{M}]$ such that

$$\begin{aligned} \mathcal{I}[\![F_1]\!](\mathcal{V}, \mathcal{M} \prec_i m) &= \text{true and} \\ \mathcal{I}[\![F_2]\!](\mathcal{V}, \mathcal{M} \succ_i m) &= \text{true.} \end{aligned}$$

The *satisfaction relation* $\models$ is defined by

$$\mathcal{I}, \mathcal{V}, \mathcal{M} \models F \ \text{ iff } \mathcal{I}[\![F]\!](\mathcal{V}, \mathcal{M}) = \text{true}.$$

We define some abbreviations to make specifications more concise. The almost everywhere operator $\lceil \pi \rceil$ expresses that a state assertion π holds almost everywhere in the interval and the interval is non-empty. The empty interval is denoted by $\lceil\rceil$. The n-dimensional volume is measured by the term ℓ.

$$\ell \stackrel{df}{=} \textstyle\int 1 \qquad \lceil \pi \rceil \stackrel{df}{=} (\textstyle\int \pi = \ell \wedge \ell > 0) \qquad \lceil\rceil \stackrel{df}{=} \textstyle\int 1 = 0$$

The somewhere operator $\diamond_{\vec{e}_i} F$ chops the n-dimensional interval twice in the ith direction such that in the middle interval F holds, hence it expresses that F holds on some region along the ith axis.

$$\diamond_{\vec{e}_i} F \stackrel{df}{=} \text{true} \ \langle \vec{e}_i \rangle \ F \ \langle \vec{e}_i \rangle \ \text{true}$$

The dual globally operator is $\Box_{\vec{e}_i}$ defined by

$$\Box_{\vec{e}_i} \stackrel{df}{=} \neg \diamond_{\vec{e}_i} \neg F$$

and expresses that F holds in every region along the ith axis. Although chop is associative only for chopping in the same direction, $\diamond_{\vec{e}_1} \diamond_{\vec{e}_2} F$ still is equivalent to $\diamond_{\vec{e}_2} \diamond_{\vec{e}_1} F$. We will denote the unit vector corresponding to the time dimension by $\vec{e}_t$ and to spatial dimensions by $\vec{e}_x, \vec{e}_y$, etc.

Example 5. The initial requirement, that at most 10 cm^2 of the robot R is ever outside a restricted area defined by A can be expressed by

$$\Box_{\vec{e}_t} \left(\textstyle\int (R \wedge \neg A) \leqslant (10 \cdot \ell_{\vec{e}_t}) \right)$$

where the unit is omitted. The formula reads as follows: for every temporal interval the volume of all points of R outside of A is less than 10 multiplied by the temporal length. The scenario is sketched in Fig. 2a. The observable R modelling the robot is true for all points between the solid lines, the observable A is true for all points between the dashed lines. For simplicity we omitted the second spatial dimension in the drawing.

Example 6 (*Ensuring a minimal distance*)**.** Consider the scenario of two moving robots using a collision avoidance system as depicted in Fig. 2b. We require that the minimal distance is always at least than 1 cm. This is specified by

$$\begin{aligned} \Box_{\vec{e}_t} \Box_{\vec{e}_x} \big((\diamond_{\vec{e}_x} & \lceil R_1 \rceil \wedge \diamond_{\vec{e}_x} \lceil R_2 \rceil) \Rightarrow \\ \diamond_{\vec{e}_t} & \ ((\diamond_{\vec{e}_x} \lceil R_1 \rceil \wedge \neg \diamond_{\vec{e}_x} \lceil R_2 \rceil) \ \langle \vec{e}_x \rangle \\ & \ (\lceil \neg R_1 \wedge \neg R_2 \rceil \wedge \ell_{\vec{e}_x} \geqslant 1) \ \langle \vec{e}_x \rangle \\ & \ (\diamond_{\vec{e}_x} \lceil R_2 \rceil \wedge \neg \diamond_{\vec{e}_x} \lceil R_1 \rceil)) \big) \end{aligned}$$

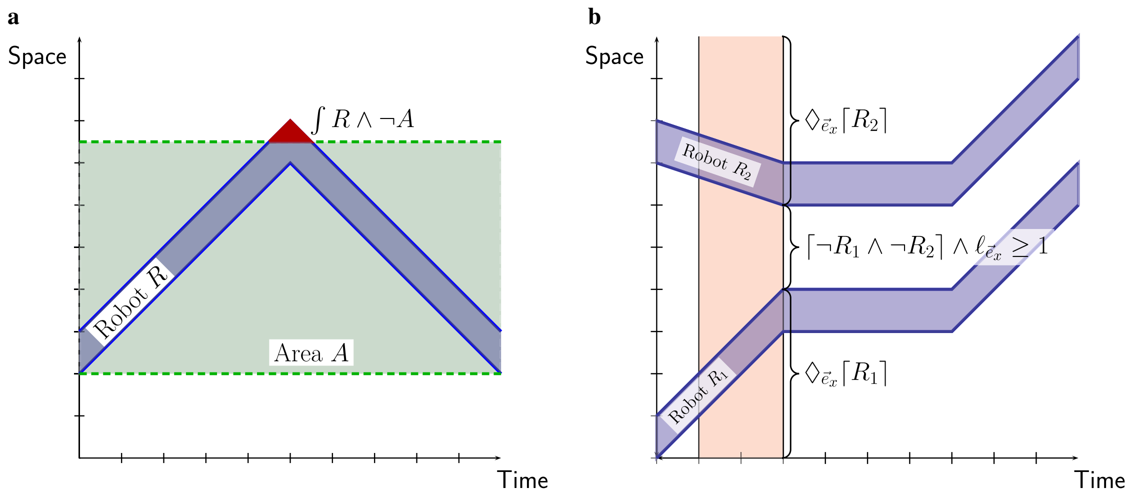

Fig. 2. (a) Moving robot scenario and (b) Minimal distance scenario.

This formula reads as follows. For all spatio-temporal subintervals such that Robot R_1 and Robot R_2 are contained somewhere in this interval, there is a temporal subinterval such that we can split space into three parts such that

(1) the lower part contains R_1
(2) the middle part neither contains Robot R_1 nor R_2 and it has length greater than or equal to 1
(3) and the upper part contains R_2.

As indicated in Fig. 2b arbitrarily large temporal intervals do not need to satisfy the Condition (2).

Definition 7 (*Validity/satisfiability*)**.** A formula F is valid iff it evaluates to true for all interpretations, valuations and intervals. It is satisfiable iff there is an interpretation, a valuation and an interval such that F holds.

Note 1. Duration Calculus and one-dimensional Shape Calculus coincide. In this sense the extention of Duration Calculus is conservative.

3. Undecidability and non-axiomatisability

In this section, we show that validity for Shape Calculus is undecidable and even not recursively enumerable. Henceforth, Shape Calculus is not recursively axiomatisable by Craig's Theorem [17].

In [28] it is shown that Duration Calculus is decidable for the discrete time domain and for the formulae restricted to phase expressions $\lceil\cdot\rceil$, chop and Boolean operators. This result does not transfer to Shape Calculus when considering more than one dimension. Since one-dimensional Shape

Calculus and Duration Calculus coincide, for one-dimensional discrete Shape Calculus and this restricted subset, validity is still decidable.

Theorem 8. *For two dimensions and above, the set of valid SC formulae is not recursively enumerable, neither interpreted in the continuous nor in the discrete domain.*

By Craig's Theorem [17] a theory is recursively axiomatisable if and only if the set of valid formulas is recursively enumerable. Therefore from the above theorem, we obtain the following corollary.

Corollary 9. *There is not a sound and complete proof system for SC .*

Extending the undecidability proof in [42], we provide a reduction from a non recursively enumerable tiling problem.

For this proof we restrict ourselves to the class of formulae given by

$$F ::= \lceil \pi \rceil \mid F \wedge G \mid \neg F \mid F \langle \vec{e}_1 \rangle G \mid F \langle \vec{e}_2 \rangle G \mid \ell_{\vec{e}_i} = r$$

for some fixed r. The corresponding subset of Duration Calculus interpreted in discrete or continuous time domain is known to be decidable [28]. Without loss of generality, we choose $r = 1$.

3.1. Tiling systems

The theory of string languages can be extended to two-dimensional (also called picture) languages. We shortly review the main definitions and results. A detailed discussion can be found in [26].

We fix an alphabet Σ and a fresh boundary character #. A two-dimensional string (picture) over Σ is a two-dimensional rectangular matrix of elements of Σ such that the boundary is marked by the fresh symbol #.

A tile p is a 2×2 matrix with elements in $\Sigma \cup \#$ and a tiling system Θ is a finite set of tiles. The local language $L(\Theta)$ for a tiling system Θ is the set of all $n \times m$ matrices such that each 2×2 block is in Θ and the boundaries of the matrix consist only of # and # does not occur in the interior.

Giammarresi and Restivo show in [26] that the emptiness problem

Given a tiling system Θ, is $L(\Theta) = \varnothing$?

is undecidable. This problem can be reformulated as follows:

There is no $n \times m$ matrix for $n, m \in \mathbb{N}$ such that every 2×2 submatrix is contained in the set Θ and the boundaries of the matrix consist of # only and # does not occur in the interior.

They provide a reduction such that a Turing Machine $\mathcal{M}$ has no successful computation iff $L(\Theta)$ is empty. With this reduction to the termination problem, the emptiness problem for tiling systems is not recursively enumerable. Both problems are co-recursively enumerable.

3.2. Encoding tilings in Shape Calculus

We provide a reduction of the emptiness problem for tiling systems as described above to the validity problem of Shape Calculus. For a set of tiles $\Theta = \{p_1, \ldots, p_k\}$, we define a formula F_Θ in SC, such that $L(\Theta) \neq \varnothing$ iff F_Θ is satisfiable which is equivalent to $L(\Theta) = \varnothing$ iff $\neg F_\Theta$ is valid.

We present an encoding which does not rely on continuous or discrete time and space domain. Therefore, to avoid chopping at arbitrary positions, we impose a chess-board marking by a fresh observable $\bigstar$ as a region marker to clearly identify 2×2 blocks in the continuous case. We specify the grid by a formula F_{grid} as follows:

$$
\begin{aligned}
F_{\text{grid}} \overset{df}{=}\ & \ell_{\vec{e}_1} \geqslant 2 \wedge \ell_{\vec{e}_2} \geqslant 2 \wedge \\
& \lceil \bigstar \iff (\bigstar_1 \iff \bigstar_2) \rceil \wedge && (*) \\
& \Box_{\vec{e}_1} \big(((\lceil \bigstar_1 \rceil \langle \vec{e}_1 \rangle \ell_{\vec{e}_1} = 1 \Rightarrow \lceil \bigstar_1 \rceil \langle \vec{e}_1 \rangle \lceil \neg \bigstar_1 \rceil) \wedge \\
& \quad (\lceil \neg \bigstar_1 \rceil \langle \vec{e}_1 \rangle \ell_{\vec{e}_1} = 1 \Rightarrow \lceil \neg \bigstar_1 \rceil \langle \vec{e}_1 \rangle \lceil \bigstar_1 \rceil)\big) \wedge && (**) \\
& \quad \ell_{\vec{e}_1} \geqslant 1 \Rightarrow (\lceil \bigstar_1 \rceil \wedge \ell_{\vec{e}_1} = 1 \langle \vec{e}_1 \rangle \text{true}) \wedge && (***) \\
& \Box_{\vec{e}_2} \big(((\lceil \bigstar_2 \rceil \langle \vec{e}_2 \rangle \ell_{\vec{e}_2} = 1 \Rightarrow \lceil \bigstar_2 \rceil \langle \vec{e}_2 \rangle \lceil \neg \bigstar_2 \rceil) \wedge \\
& \quad (\lceil \neg \bigstar_2 \rceil \langle \vec{e}_2 \rangle \ell_{\vec{e}_2} = 1 \Rightarrow \lceil \neg \bigstar_2 \rceil \langle \vec{e}_2 \rangle \lceil \bigstar_2 \rceil)\big) \wedge && (**) \\
& \quad \ell_{\vec{e}_2} \geqslant 1 \Rightarrow (\lceil \bigstar_2 \rceil \wedge \ell_{\vec{e}_2} = 1 \langle \vec{e}_2 \rangle \text{true}) && (***)
\end{aligned}
$$

We use two auxiliary observables $\bigstar_1$ and $\bigstar_2$. The observable $\bigstar_1$ is true on intervals $[i, i+1] \times [a, b]$ and false on $[i+1, i+2] \times [a, b]$ when i is even and a, b are arbitrary. The same holds for $\bigstar_2$ and intervals $[a, b] \times [i, i+1]$ and $[a, b] \times [i+1, i+2]$, respectively. This fact can be easily proven by induction on i. The quantified subformulae $(**)$ specify that a $\lceil \bigstar_i \rceil$ slice is succeeded by a $\lceil \neg \bigstar_i \rceil$ slice and vice versa. The initial condition that the first slice has a size of 1 and satisfies $\lceil \bigstar_1 \rceil$, respectively $\lceil \bigstar_2 \rceil$, is specified separately by $(***)$. The chess-board marking by $\bigstar$ is obtained using the equivalence operation on $\bigstar_1$ and $\bigstar_2$ in $(*)$. This idea is formalised in the following lemma.

Lemma 10. *Let $\mathcal{I}$ be an interpretation and $k \in \mathbb{N}, a, b \in \mathbb{T}$. Then $\mathcal{I}, [0, k] \times [a, b] \models F_{\text{grid}}$ if and only if $k \geqslant 2$, $b - a \geqslant 2$, and for all $i \in \mathbb{N}$, $i \leqslant k$ and arbitrary $[a', b'] \subseteq [a, b]$ the following holds:*

$$
(\alpha) \quad \mathcal{I}, [i, i+1] \times [a', b'] \models \begin{cases} \lceil \bigstar_1 \rceil & \text{if } i \text{ is even,} \\ \lceil \neg \bigstar_1 \rceil & \text{otherwise} \end{cases}
$$

$$
(\beta) \quad \mathcal{I}, [a', b'] \times [i, i+1] \models \begin{cases} \lceil \bigstar_2 \rceil & \text{if } i \text{ is even,} \\ \lceil \neg \bigstar_2 \rceil & \text{otherwise} \end{cases}
$$

$$
(\gamma) \quad \mathcal{I}, [i, i+1] \times [j, j+1] \models \begin{cases} \lceil \bigstar \rceil & \text{if } i, j \text{ are both even or both odd} \\ \lceil \neg \bigstar \rceil & \text{otherwise.} \end{cases}
$$

To describe a 2×2 block in this grid satisfying the observables P_1, P_2, P_3, P_4 in its four cells starting with P_1 in the lower left corner, we use the pattern

$$
\begin{aligned}
F_{2\times 2}(P_1, P_2, P_3, P_4) \overset{df}{=}\ & ((\lceil \bigstar \wedge P_1 \rceil) \langle \vec{e}_1 \rangle (\lceil \neg \bigstar \wedge P_2 \rceil) \langle \vec{e}_2 \rangle \\
& (\lceil \neg \bigstar \wedge P_3 \rceil) \langle \vec{e}_1 \rangle (\lceil \bigstar \wedge P_4 \rceil)) \vee \\
& ((\lceil \neg \bigstar \wedge P_1 \rceil) \langle \vec{e}_1 \rangle (\lceil \bigstar \wedge P_2 \rceil) \langle \vec{e}_2 \rangle \\
& (\lceil \bigstar \wedge P_3 \rceil) \langle \vec{e}_1 \rangle (\lceil \neg \bigstar \wedge P_4 \rceil))
\end{aligned}
$$

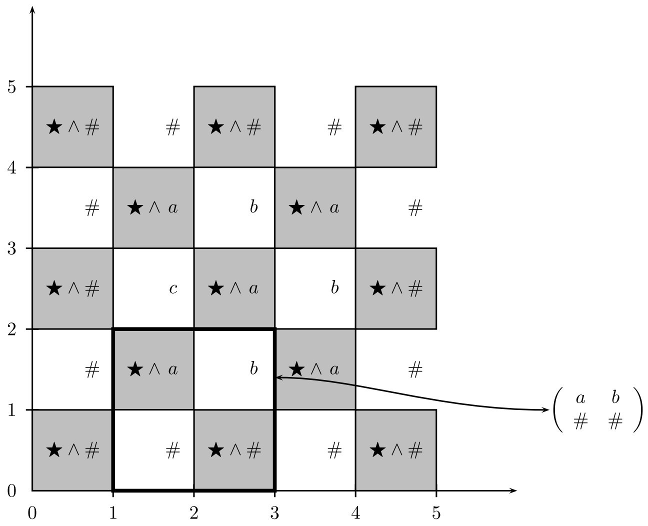

Fig. 3. Sample encoding of tilings in a grid structure.

and assign to every tile $p_i = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$ a formula $F_{p_i} \overset{df}{=} F_{2\times 2}(a, b, c, d)$. With these sub-formulae we define F_Θ to be

$$
\begin{aligned}
F_\Theta \overset{df}{=}\ & F_{\text{grid}} \\
& \wedge \Box_{\vec{e}_1} \Box_{\vec{e}_2} \left(F_{2\times 2}(\text{true}, \text{true}, \text{true}, \text{true}) \Rightarrow \bigvee_{i=1}^{k} F_{p_i} \right) && (*) \\
& \wedge \lceil \# \rceil \langle \vec{e}_1 \rangle (\lceil \# \rceil \langle \vec{e}_2 \rangle \lceil \neg \# \rceil \langle \vec{e}_2 \rangle \lceil \# \rceil) \langle \vec{e}_1 \rangle \lceil \# \rceil && (**) \\
& \wedge \lceil \bigwedge_{s, s' \in \Sigma, s \neq s'} s \Rightarrow \neg s' \rceil && (***)
\end{aligned}
$$

The second conjunct $(*)$ states that each 2×2 block in the grid must be in Θ, whereas the third conjunct $(**)$ states that the picture must be framed by # and # does not occur in the interior, as sketched in Fig. 3. The last conjunct ensures mutual exclusion of symbols. With this definition, F_Θ is satisfiable if and only if the local language $L(\Theta)$ is not empty, so $\neg F_\Theta$ is valid if and only if the local language $L(\Theta)$ is empty.

Proof

"only if" Let $\mathcal{I}$ be a satisfying interpretation and $[0, k_1] \times [0, k_2]$ an interval such that $\mathcal{I}, [0, k_1] \times [0, k_2] \models F_\Theta$. Note that by definition of the grid and F_Θ a satisfying interval must have integer bounds. Let $(p_{i,j})_{i,j}$ be the matrix defined by

$$p_{i,j} = a \iff \mathcal{I}, [i, i+1] \times [j, j+1] \models \lceil a \rceil$$

for $a \in \Sigma \cup \{\#\}$. By $(***)$ there is at most one observable $a \in \Sigma \cup \{\#\}$ satisfied on $[i, i+1] \times [j, j+1]$ and by $(*)$ there is at least one observable satisfied. Therefore $(p_{i,j})_{i,j}$ is well-defined. By $(*)$ each interval of size 2×2 satisfies some F_{p_i}. Therefore by construction each 2×2 submatrix in $(p_{i,j})_{i,j}$ is in Θ. Furthermore, since the boundary satisfies $\lceil \# \rceil$ the matrix boundaries of $(p_{i,j})_{i,j}$ consists of #. So, $(p_{i,j})_{i,j} \in L(\Theta)$.

"if" Let $(p_{i,j})_{i,j} \in L(\Theta)$. Define an interpretation $\mathcal{I}$ for the observables $a \in \Sigma \cup \{\#\}$ by

$$\mathcal{I}[\![a]\!](x, y) = \begin{cases} 1 & \text{if } p_{i,j} = a \wedge x \in [i, i+1], y \in [j, j+1], \\ 0 & \text{otherwise} \end{cases}$$

and for the auxiliary observables by

$$\begin{aligned} \mathcal{I}[\![\bigstar_1]\!](x, y) &= \begin{cases} 1 & \text{if there is an even } i \text{ such that } x \in [i, i+1], \\ 0 & \text{otherwise} \end{cases} \\ \mathcal{I}[\![\bigstar_2]\!](x, y) &= \begin{cases} 1 & \text{if there is an even } i \text{ such that } y \in [i, i+1], \\ 0 & \text{otherwise} \end{cases} \\ \mathcal{I}[\![\bigstar]\!](x, y) &= \begin{cases} 1 & \text{if } \mathcal{I}[\![\bigstar_1]\!](x, y) \iff \mathcal{I}[\![\bigstar_2]\!](x, y), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

It is straightforward to see that $\mathcal{I}, [0, i+1] \times [0, j+1] \models F_\Theta$. $\square$

We haven proven so far that satisfiability corresponds to non-emptiness of local picture languages for tiling systems. Therefore validity corresponds to language emptiness, which is known to be undecidable and not recursively enumerable. By Craig's Theorem [17] this proves: SC is not recursively enumerable and not axiomatisable.

4. Relative completeness

In the previous section, we have demonstrated that Shape Calculus is not axiomatisable. Despite this negative result, it is still possible to give an axiomatisation relatively to an n-dimensional extension of Interval Temporal Logic (ITL^n). We assume an inference system for ITL^n, i.e, a set of inference rules such that every valid ITL^n formula can be derived by finitely many applications of the inference rules. The inference relation is denoted by $\vdash_{\mathsf{ITL}^n}$. Assuming the existence of this inference system for ITL^n, we derive a system for the Shape Calculus such that every valid Shape Calculus formula can be derived. Therefore, this system is called complete relatively to ITL^n.

Thereby, we extend the axiomatisation result for the Duration Calculus presented in [27] to the Shape Calculus. Duration Calculus itself allows an axiomatisation relatively to interval temporal logic (ITL). For this axiomatisation, we require a stronger finite variability assumption, namely every finite n-dimensional interval can be partitioned into finitely many sub-intervals such that $\mathcal{I}$ is constant on each sub-interval. Considering arbitrary integrable functions would require an axiomatisation of the integral calculus which is out of scope for this paper. The axiomatisation result for Duration Calculus presented in [27,28] relies on the same requirement, namely finite variability. Our proof follows the lines of [27,28] and considers only the two-dimensional case, but it can easily be generalised to more dimensions.

4.1. Interval temporal logic (ITL)

We shortly introduce the n-dimensional ITL. One-dimensional ITL is discussed in [20,28]. ITL^n does not use state assertions nor the integral operator but instead uses flexible variables v whose values depend on the interval. Furthermore, it incorporates rigid variables x and lengths $\ell_{\vec{e}_i}$ as terms.

$$\theta^{\mathsf{ITL}^n} ::= x \mid v \mid \ell_{\vec{e}_i} \mid f\left(\theta_1^{\mathsf{ITL}^n}, \ldots, \theta_k^{\mathsf{ITL}^n}\right)$$

The semantics of flexible variables is given by an interpretation $\mathcal{I}_{\mathsf{ITL}^n}$ that assigns a real number to each n-dimensional interval. This is extended to terms as follows:

$$\begin{aligned}
\mathcal{I}_{\mathsf{ITL}^n}[\![x]\!](\mathcal{V},\mathcal{M}) &\stackrel{df}{=} \mathcal{V}(x)\\
\mathcal{I}_{\mathsf{ITL}^n}[\![v]\!](\mathcal{V},\mathcal{M}) &\stackrel{df}{=} \mathcal{I}(v)(\mathcal{M})\\
\mathcal{I}_{\mathsf{ITL}^n}[\![\ell_{\vec{e}_i}]\!](\mathcal{V},\mathcal{M}) &\stackrel{df}{=} f_i - b_i\\
\mathcal{I}_{\mathsf{ITL}^n}[\![f\left(\theta_1^{\mathsf{ITL}^n}, \ldots, \theta_k^{\mathsf{ITL}^n}\right)]\!](\mathcal{V},\mathcal{M}) &\stackrel{df}{=} f_{\mathcal{I}_{\mathsf{ITL}^n}}(\mathcal{I}_{\mathsf{ITL}^n}[\theta_1^{\mathsf{ITL}^n}]\!](\mathcal{V},\mathcal{M}),\\
&\qquad \ldots,\\
&\qquad \mathcal{I}_{\mathsf{ITL}^n}[\theta_k^{\mathsf{ITL}^n}]\!](\mathcal{V},\mathcal{M}))
\end{aligned}$$

Like in Shape Calculus, $\mathcal{V}$ is a valuation of the rigid variables, i.e., variables that do not change over time and $\mathcal{M} = [b_1, f_1] \times \cdots \times [b_n, f_n]$ is an n-dimensional interval. Furthermore, we define the abbreviation $\ell \stackrel{df}{=} \ell_{\vec{e}_1} \cdot \ell_{\vec{e}_2}$ to measure the two-dimensional area. For formulae, ITL^n incorporates Boolean combinations, chop and quantification as in SC. Formally, it is given by the following BNF.

$$\begin{aligned}
F^{\mathsf{ITL}^n} ::= {} & F_1^{\mathsf{ITL}^n} \langle \vec{e}_i \rangle F_2^{\mathsf{ITL}^n} \mid p\left(\theta_1^{\mathsf{ITL}^n}, \ldots, \theta_k^{\mathsf{ITL}^n}\right) \mid \neg F_1^{\mathsf{ITL}^n} \mid F_1^{\mathsf{ITL}^n} \wedge F_2^{\mathsf{ITL}^n} \mid\\
& \exists x : F^{\mathsf{ITL}^n}
\end{aligned}$$

The semantics of the Boolean connectives and quantifiers is the same as in first order logic. The semantics of the chop operator is the same as in Shape Calculus.

4.2. Axiomatisation

We present the main theorem of this section and give a short proof sketch. To make the presentation more concise, we introduce negated unit vectors and define $F \langle -\vec{e}_i \rangle G \stackrel{df}{=} G \langle \vec{e}_i \rangle F$

Theorem 11. *Two-dimensional* SC *is axiomatised relatively to* ITL^2 *by the following axioms.*

$$\textstyle\int 0 = 0 \tag{SC1}$$
$$\textstyle\int 1 = \ell \tag{SC2}$$
$$\textstyle\int \pi \geqslant 0 \tag{SC3}$$
$$\textstyle\int \pi_1 + \int \pi_2 = \int(\pi_1 \vee \pi_2) + \int(\pi_1 \wedge \pi_2) \tag{SC4}$$
$$\textstyle\int \pi = x \,\langle \vec{e}_i \rangle\, \int \pi = y \Rightarrow \int \pi = x + y \tag{SC5}$$
$$\lceil\rceil \vee ((\lceil \pi \rceil \vee \lceil \neg\pi \rceil \,\langle \vec{e}_1 \rangle\, \mathit{true})\,\langle \vec{e}_2 \rangle\, \mathit{true}) \tag{FV1}$$
$$\lceil\rceil \vee ((\lceil \pi \rceil \vee \lceil \neg\pi \rceil \,\langle \vec{e}_1 \rangle\, \mathit{true})\,\langle -\vec{e}_2 \rangle\, \mathit{true}) \tag{FV2}$$
$$\lceil\rceil \vee ((\lceil \pi \rceil \vee \lceil \neg\pi \rceil \,\langle -\vec{e}_1 \rangle\, \mathit{true})\,\langle \vec{e}_2 \rangle\, \mathit{true}) \tag{FV3}$$
$$\lceil\rceil \vee ((\lceil \pi \rceil \vee \lceil \neg\pi \rceil \,\langle -\vec{e}_1 \rangle\, \mathit{true})\,\langle -\vec{e}_2 \rangle\, \mathit{true}) \tag{FV4}$$

The set of axioms can be separated into two groups. The first group (SC1) up to (SC5) specify properties of the integral calculus need for piecewise constant functions. The second group (FV1)–(FV4) specifies finite variability, by demanding that for every point we can find 4 rectangles to the lower left, lower right, upper left and upper right, respectively, such that the value of a state expression is constant. The proof of relative completeness proceeds as follows. For a valid SC formula F we have to construct a derivation using the set of axioms defined previously. To this end, we construct a valid ITL^n formula. As we consider relative completeness, we can assume an ITL^n deduction of this formula. This deduction is lifted to a Shape Calculus deduction of F.

4.3. From Shape Calculus to ITL^n

For a given valid Shape Calculus formula F, we first elaborate an encoding of all Shape Calculus axioms that are possibly needed for the proof of F into one ITL^n formula.

Encoding the axioms in ITL^n. Let F be an arbitrary valid SC formula and let $X_1, \ldots, X_l$ be the set of Boolean observables occurring in F and $\mathcal{S}$ the set of all state expressions built from these observables. Note that, since state expressions are formulae of propositional logic, only finitely many state expressions can be nonequivalent. Let

$$[\pi] \overset{df}{=} \{\pi' \mid \pi' \equiv \pi\}$$

denote such an equivalence class and $\mathcal{S}_{\equiv} = \{[\pi] | \pi \in \mathcal{S}\}$ denote the set of equivalence classes. For every equivalence class $[\pi]$ we introduce an ITL^n flexible variable $v_{[\pi]}$ with the intuition that $v_{[\pi]}$ models the the value of $\int \pi$. We encode the SC Axioms by the following finite sets of ITL^n formulae.

$$\mathcal{H}_1 \overset{df}{=} \{v_{[0]} = 0\}$$
$$\mathcal{H}_2 \overset{df}{=} \{v_{[1]} = \ell\}$$

$$\mathcal{H}_3 \stackrel{df}{=} \{v_{[\pi]} \geqslant 0 \mid [\pi] \in \mathcal{S}_{\equiv}\}$$
$$\mathcal{H}_4 \stackrel{df}{=} \{(v_{[\pi_1]} + v_{[\pi_2]}) = (v_{[\pi_1 \vee \pi_2]} + v_{[\pi_1 \wedge \pi_2]}) \mid [\pi_1], [\pi_2] \in \mathcal{S}_{\equiv}\}$$
$$\mathcal{H}_5 \stackrel{df}{=} \{(v_{[\pi]} = x \langle \vec{e}_i \rangle v_{[\pi]} = y) \Rightarrow (v_{[\pi]} = x + y) \mid [\pi] \in \mathcal{S}_{\equiv}\}$$
$$\mathcal{H}_6 \stackrel{df}{=} \{\lceil\rceil \vee ((\lceil v_{[\pi]} \rceil \vee \lceil v_{[\neg\pi]} \rceil \langle \vec{d}_1 \rangle \mathit{true}) \langle \vec{d}_2 \rangle \mathit{true}) \mid$$
$$[\pi] \in \mathcal{S}_{\equiv}, d_i \in \{\vec{e}_i, -\vec{e}_i\}\}$$

where $\lceil v_{[\pi]} \rceil \stackrel{df}{=} (v_{[\pi]} = \ell \wedge \ell > 0)$ and $\lceil\rceil \stackrel{df}{=} (\ell_1 = 0 \vee \ell_2 = 0)$. We define H_F^I to be the conjunction of all formulae in $\mathcal{H}_1$ to $\mathcal{H}_6$ and F^I to be the ITL^n formula obtained from F by replacing every occurrence of $\int \pi$ by $v_{[\pi]}$. Note, that this formula depends on F.

We will now show, that an ITL^n interpretation $\mathcal{I}_{\mathsf{ITL}^n}$ and a valuation $\mathcal{V}$ that satisfy the axioms encoded in H_F^I can already be used to derive a Shape Calculus interpretation. We will use such interpretations, valuations and intervals frequently in what follows, so we aggregate them in triples.

Definition 12 (*H-Triple*)**.** A triple $(\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [b_1, f_1] \times [b_2, f_2])$ is called an H-triple if

$$\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [b_1, f_1] \times [b_2, f_2] \models_{\mathsf{ITL}^n} \Box_{\vec{e}_1} \Box_{\vec{e}_2} H_F^I$$

i.e., H_F^I holds for every subrectangle of $[b_1, f_1] \times [b_2, f_2]$.

Using this definition and the axioms, we derive some properties of H-Triples that will be used in the completeness proof. A complete proof of this lemma can be found in [28].

Lemma 13. *Let* $(\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [b_1, f_1] \times [b_2, f_2])$ *be an H-Triple. Then the following holds*:

(1) $\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [b_1, f_1] \times [b_2, f_2] \models_{\mathsf{ITL}^n} v_{[\pi]} + v_{[\neg\pi]} = \ell$
(2) $\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [b_1, f_1] \times [b_2, f_2] \models_{\mathsf{ITL}^n} v_{[\pi]} \leqslant \ell$
(3) $\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [b_1, f_1] \times [b_2, f_2] \models_{\mathsf{ITL}^n} v_{[\pi_1]} \leqslant v_{[\pi_1 \vee \pi_2]}$
(4) $\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [b_1, f_1] \times [b_2, f_2] \models_{\mathsf{ITL}^n} \lceil v_{[\pi]} \rceil \mathit{implies} \mathcal{I}, \mathcal{V}, [c_1, d_1] \times [c_2, d_2] \models \lceil v_{[\pi]} \rceil \mathit{for} [c_k, d_k] \subseteq [b_k, f_k]$, $k = 1, 2$.

Deriving the piecewise constant property. As we require Shape Calculus interpretations to be piecewise constant, i.e., there is a partition of time and space into intervals such that the interpretation of observables is constant on each interval, we show that this property can be derived in ITL^n from the encoded axioms. We need the instances in $\mathcal{H}_6$ and the Theorem of Heine Borel [41].

Lemma 14. *Given an arbitrary H-triple* $(\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [b_1, f_1] \times [b_2, f_2])$ *such that* $b_1 < f_1$ *and* $b_2 < f_2$*, i.e., the interval is non-empty, then for every* $\pi \in \mathcal{S}$ *there is a finite partition in sub-rectangles* $[b_1^1, f_1^1] \times [b_2^1, f_2^1], \ldots, [b_1^n, f_1^n] \times [b_2^n, f_2^n]$ *such that for every rectangle* $[b_1^i, f_1^i] \times [b_2^i, f_2^i]$ *holds either*

$$\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [b_1^i, f_1^i] \times [b_2^i, f_2^i] \models_{\mathsf{ITL}^n} \lceil v_{[\pi]} \rceil \text{ or } \mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [b_1^i, f_1^i] \times [b_2^i, f_2^i] \models_{\mathsf{ITL}^n} \lceil v_{[\neg\pi]} \rceil$$

Proof. Let $(x, y) \in [b_1, f_1] \times [b_2, f_2]$. Then by $\mathcal{H}_6$ there exists $x_1 \leqslant x \leqslant x_2$ and $y_1 \leqslant y \leqslant y_2$ such that

$$\begin{aligned}
&\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [x_1, x] \times [y_1, y] \models_{\mathsf{ITL}^n} \lceil v_{[\pi]} \rceil \vee \lceil v_{[\neg\pi]} \rceil \text{ and}\\
&\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [x_1, x] \times [y, y_2] \models_{\mathsf{ITL}^n} \lceil v_{[\pi]} \rceil \vee \lceil v_{[\neg\pi]} \rceil \text{ and}\\
&\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [x, x_2] \times [y_1, y] \models_{\mathsf{ITL}^n} \lceil v_{[\pi]} \rceil \vee \lceil v_{[\neg\pi]} \rceil \text{ and}\\
&\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [x, x_2] \times [y, y_2] \models_{\mathsf{ITL}^n} \lceil v_{[\pi]} \rceil \vee \lceil v_{[\neg\pi]} \rceil
\end{aligned}$$

Now $(x_1, x_2) \times (y_1, y_2)$ is an open interval covering the point (x, y) and the closed interval $[x_1, x_2] \times [y_1, y_2]$ has the desired property. Then by Heine-Borels Theorem there is a finite subset of this infinite partition covering $[b_1, f_1] \times [b_2, f_2]$. The cases where (x, y) is on the border are handled similarly. This yields the finite partition as required. □

From ITL^n interpretations to SC interpretations. We have to show that for every valid SC formula there is a valid ITL^n formula such that we can lift the derivation of the ITL^n formula to a derivation of the SC formula. We will show the contrapositive, i.e., that an ITL^n interpretation satisfying the axioms, corresponds to an SC interpretation. Let $(\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [b_1, f_1] \times [b_2, f_2])$, be an H-triple. We construct an SC-interpretation $\mathcal{I}_{\mathsf{SC}}$ by defining for every observable X the interpretation $\mathcal{I}_{\mathsf{SC}}(X)$ to be

$$\mathcal{I}_{\mathsf{SC}}(X)((x, y)) \stackrel{df}{=} \begin{cases} 1 \text{ if there are } x_1, x_2, y_1, y_2 \\ \quad x_1 \leqslant x < x_2, y_1 \leqslant y < y_2 \text{ such that} \\ \quad \mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [x_1, x_2] \times [y_1, y_2] \models_{\mathsf{ITL}^n} \lceil v_{[X]} \rceil \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

This interpretation has the required finite variability property: each interval can be partitioned into finitely many subintervals such that the value of each observable X is constant on each subinterval. It is to be shown that the SC interpretation given by this definition satisfies $\lceil \pi \rceil$ if and only if the ITL^n interpretation satisfies $\lceil v_{[\pi]} \rceil$. This result is established by the following lemma.

Lemma 15. *For an H-triple* $(\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [b_1, f_1] \times [b_2, f_2])$*, a state assertion* π*, and an SC-Interpretation as defined in Equation 1, there is a finite partition* $[m_1^1, m_1^2] \times [m_2^1, m_2^2], \ldots, [m_1^{n-1}, m_1^n] \times [m_2^{o-1}, m_2^o]$ *of the two-dimensional interval* $[b_1, f_1] \times [b_2, f_2]$ *such that for every point* $(x, y) \in [m_1^i, m_1^{i+1}) \times [m_2^j, m_2^{j+1})$ *holds*

$$\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [m_1^i, m_1^{i+1}) \times [m_2^j, m_2^{j+1}) \models_{\mathsf{ITL}^n} \lceil v_{[\pi]} \rceil \vee \lceil v_{[\neg\pi]} \rceil \tag{2}$$

and

$$\mathcal{I}_{\mathsf{SC}}[\![\pi]\!]\big((x,y)\big) = \begin{cases} 1 & \text{if } \mathcal{I}, \mathcal{V}, [m_1^i, m_1^{i+1}) \times [m_2^j, m_2^{j+1}) \models_{\mathsf{ITL}^n} \lceil v_{[\pi]} \rceil \\ 0 & \text{if } \mathcal{I}, \mathcal{V}, [m_1^i, m_1^{i+1}) \times [m_2^j, m_2^{j+1}) \models_{\mathsf{ITL}^n} \lceil v_{[\neg\pi]} \rceil . \end{cases} \tag{3}$$

This lemma can be proven by induction on the structure of π. As the integral $\int \pi$ is derived by summation over the piecewise constant parts we obtain the following corollary.

Corollary 16. *For the interpretation $\mathcal{I}_{SC}$ and every state assertion π and interval* $[b_1, f_1] \times [b_2, f_2]$

$$\mathcal{I}_{SC}[\![\textstyle\int \pi]\!]([b_1, f_1] \times [b_2, f_2]) = \mathcal{I}_{ITL^n}[\![v_{[\pi]}]\!]([b_1, f_1] \times [b_2, f_2])$$

4.4. Proving relative completeness

Starting with a valid Shape Calculus formula F, we have shown how to construct an Shape Calculus interpretation for every ITL^n interpretation that satisfies certain instances of the Shape Calculus axioms in the ITL^n formula $\Box_{\vec{e}_1}\Box_{\vec{e}_2}H_F^I$.

Corresponding to the Shape Calculus formula F, we define the ITL^n formula F^I by replacing the measure $\int \pi$ with a variable $v_{[\pi]}$. Using the above result, we can construct for every ITL^n interpretation $\mathcal{I}_{\mathsf{ITL}^n}$ which violates $\Box_{\vec{e}_1}\Box_{\vec{e}_2}H_F^I \Rightarrow F^I$, i.e, the interpretation satisfies $\Box_{\vec{e}_1}\Box_{\vec{e}_2}H_F^I$ but violates F^I, an SC interpretation $\mathcal{I}_{\mathsf{SC}}$ violating F.

This proves the following lemma.

Lemma 17. $\models_{SC} F$ *implies* $\models_{ITL^n} \Box_{\vec{e}_1}\Box_{\vec{e}_2}H_F^I \Rightarrow F^I$.

To show the converse implication, let $\mathcal{I}_{\mathsf{SC}}$ be an SC interpretation violating F. Define the violating ITL^n interpretation $\mathcal{I}_{\mathsf{ITL}^n}$ by

$$\mathcal{I}_{\mathsf{ITL}^n}(v_{[\pi]})([b_1, f_1] \times [b_2, f_2]) \stackrel{df}{=} \mathcal{I}_{\mathsf{SC}}[\![\textstyle\int \pi]\!]([b_1, f_1] \times [b_2, f_2]).$$

Using this interpretation and the soundness of our axiomatisation, we obtain

Lemma 18. $\models_{ITL^n} \Box_{\vec{e}_1}\Box_{\vec{e}_2}H_F^I \Rightarrow F^I$ *implies* $\models_{SC} F$.

To prove the relative completeness, suppose $\models_{\mathsf{SC}} F$. Then by Lemma 17 holds $\models_{\mathsf{ITL}^n} \Box_{\vec{e}_1}\Box_{\vec{e}_2}H_F^I \Rightarrow F^I$. Take the ITL^n derivation of $\models_{\mathsf{ITL}^n} \Box_{\vec{e}_1}\Box_{\vec{e}_2}H_F^I \Rightarrow F^I$ and replace every occurrence of $v_{[\pi]}$ by $\int \pi$ to obtain an SC derivation. As H_F is a boxed conjunction of instances of SC axioms, it can be easily deduced in SC and therefore we obtain a derivation of F by modus ponens.

5. Decidable subsets

For acceptance of formal methods in practice, tool support is essential. Decidable subsets therefore play an import role as they facilitate the implementation of model-checkers. Restricted discrete Duration Calculus is known to be decidable [28] and a model-checking method is implemented in DCVALID [36]. In this section, we present two different types of decidable subsets of SC. The first is obtained by imposing restrictions on the class of models and the second by imposing restrictions on the class of formulae. Duration Calculus is already undecidable in the continuous case except for very restricted subsets which, e.g., miss the possibility of taking quantitative measures. Henceforth we assume the time-space-domain to be discrete throughout this section. As First-Order logic is undecidable, we also omit first order variables. We assume all interpretations to change their values only at points in $\mathbb{N}$ and all variables to have a range in $\mathbb{N}$. A detailed discussion of decidable subsets and undecidability results for Duration Calculus can be found in [28] and [22].

5.1. Finite space with infinite time

This first subset $\mathsf{SC}_{\mathsf{fin}}$ imposes a restriction on the class of models: we allow one infinite temporal dimension and require the other spatial dimensions to be finite and the size to known beforehand. As we have shown in the previous section, Shape Calculus is already undecidable for two and more discrete infinite dimensions.

We give a decision procedure for the set of formulae given by the following BNF:

$$F ::= \lceil P \rceil \mid F \wedge G \mid \neg F \mid F \langle \vec{e}_i \rangle G$$

where $\vec{e}_i$ denotes a spatial or temporal unit vector. The more general integral operator is replaced by the more specialised everywhere operation $\lceil \cdot \rceil$ omitting the use of SC terms. We will see later that other operators can be defined as abbreviations in the discrete setting. The approach for the decision procedure is sketched in Fig. 4. As there are only finitely many observables, a configuration for a point in space-time can be represented by the finite set of observables evaluating to true. In Fig. 4b this set is represented as a bitvector.

We generalise this idea to an arbitrary number of spatial dimensions. All spatial dimensions are finite, so the spatial configuration for a moment in time can be represented by a finite function mapping the finite space to the powerset of the observables. Due to the finiteness of the domain, there are only finitely many spatial configurations and therefore only finitely many functions representing a temporal snapshot. For the decision procedure, we use these functions as an alphabet and represent interpretations over space and time by words over this alphabet. It proceeds by inductively constructing a regular language representing all satisfying interpretations. Conjunction is constructed by intersection, negation by complement, temporal chop by concatenation and spatial chop using inverse homomorphisms for "gluing" two words letterwise.

Definition 19 (*Alphabet Σ_F*)**.** Let F be an n-dimensional $\mathsf{SC}_{\mathsf{fin}}$ formula, *Obs* the set of observables occurring in F and D a finite rectangular subset of $\mathbb{N}^{n-1}$. The alphabet $\Sigma_F(D)$ of F for space D if defined by

$$\Sigma_F(D) = \{c \mid c : D \to \mathcal{P}(Obs)\}.$$

Let space be the subset of $\mathbb{N}^{n-1}$ denoting the finite space, we define

$$\Sigma_F = \bigcup_{D \subseteq \mathsf{space},\, D \text{ rectangular}} \Sigma_F(D).$$

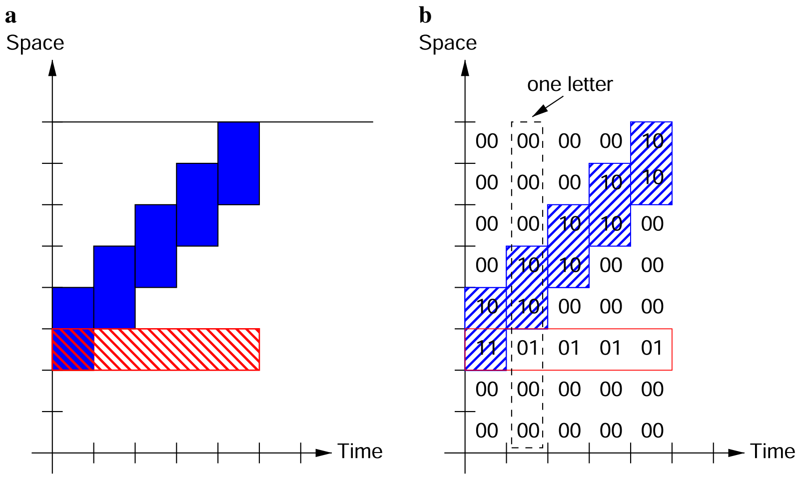

Fig. 4. (a) Two objects in finite space. (b) Their representation using a finite alphabet.

Each function in Σ_F represents a temporal snapshot of a spatial configuration for a point in time.

The intuition is that $c((x_1, \ldots, x_{n-1}))$ collects all observables that are true on the interval $[x_1, x_1 + 1) \times \cdots \times [x_{n-1}, x_{n-1} + 1)$. As we use discrete time and space, all interpretations are constant on these intervals. Therefore we can define the satisfaction relation $\Vdash$ for a state assertions π, a function $c \in \Sigma_F(D)$ and a spatial point $\vec{x} \in D$ inductively by

$$\begin{array}{ll} c(\vec{x}) \Vdash X & \text{iff } X \in c(\vec{x}) \\ c(\vec{x}) \Vdash \pi_1 \wedge \pi_2 & \text{iff } c(\vec{x}) \Vdash \pi_1 \text{ and } c(\vec{x}) \Vdash \pi_2 \\ c(\vec{x}) \Vdash \neg\pi & \text{iff } c(\vec{x}) \nVdash \pi \end{array}$$

Note that Σ_F is finite since space and the set of observables are both finite.

For handling the chop-operation, we need to restrict the domain of functions in Σ_F. This is done via a functional $h_{D \to D'}$ restricting the domain from D to a subset D'. This functional is extended to a homomorphism on words in Σ_F^* and using the inverse of this homomorphism, we can generate all possible spatial extentions to space D of the word describing space D'.

Definition 20 (*Homomorphism $h_{D \to D'}$*). Let $\Sigma_F(D)$ and $\Sigma_F(D')$ be alphabets as given by Definition 19 and $D' \subseteq D$. The function $h_{D \to D'}$ defined by

$$h_{D \to D'} : \begin{cases} \Sigma_F(D) \to \Sigma_F(D') \\ c \mapsto c|_{D'} \end{cases}$$

can be extended to a language homomorphism $\widehat{h_{D \to D'}} : \Sigma_F(D)^* \to \Sigma_F(D')^*$ in the usual way, i.e., $\widehat{h_{D \to D'}}(\varepsilon) \overset{df}{=} \varepsilon$ and $\widehat{h_{D \to D'}}(aw) = h_{D \to D'}(a) \circ \widehat{h_{D \to D'}}(w)$. We omit the distinction between $h_{D \to D'}$ and $\widehat{h_{D \to D'}}$ in the following.

Definition 21 (*Language $\mathcal{L}(F)$*)**.** Let F be an n-dimensional $\mathsf{SC}_{\mathsf{fin}}$ formula, Σ_F the alphabet and $D \subseteq \mathsf{space}$. Using the notation from Definition 4 the regular language $\mathcal{L}_D(F)$ over the alphabet $\Sigma_F(D)$ is defined inductively by

$$\mathcal{L}_D(\lceil\pi\rceil) \stackrel{df}{=} \begin{cases} \varnothing & \text{if } \exists\, i : \min_i(D) \geqslant \max_i(D) \\ \{c \in \Sigma_F | \forall x \in D^- : c(x) \Vdash \pi\}^+ & \text{otherwise} \end{cases}$$
$$\mathcal{L}_D(\neg F) \stackrel{df}{=} \Sigma_F(D)^* \setminus \mathcal{L}_D(F)$$
$$\mathcal{L}_D(F_1 \wedge F_2) \stackrel{df}{=} \mathcal{L}_D(F_1) \cap \mathcal{L}_D(F_2)$$
$$\mathcal{L}_D(F_1 \langle \vec{e}_t \rangle F_2) \stackrel{df}{=} \mathcal{L}_D(F_1) \circ \mathcal{L}_D(F_2)$$

The definition for the spatial chop in direction $\vec{e}_i$ is more technical.

$$\mathcal{L}_D(F_1 \langle \vec{e}_i \rangle F_2) \stackrel{df}{=} \bigcup_{r \in [\min_i D, \max_i D]} \left(h^{-1}_{D \to (D \prec_i r)}(\mathcal{L}_{D \prec_i r}(F_1)) \cap h^{-1}_{D \to (D \succ_i r)}(\mathcal{L}_{D \succ_i r}(F_2)) \right)$$

The $r \in [\min_i D, \max_i D]$ is the point to chop the interval. We construct the language for the formula F_1 and the lower part of interval and use the inverse homomorphism, i.e. the preimage, to generate all possible extensions. The same approach is done for the upper part and formula F_2. The intersection assures that F_1 holds on the lower and F_2 on the upper part of the interval. n)

By definition, a formula is satisfiable if there is an interval satisfying it. Therefore, all finitely many spatial subintervals have to be considered for the language of F

$$\mathcal{L}(F) \stackrel{df}{=} \bigcup_{D \subseteq \mathsf{space}} \mathcal{L}_D(F)$$

As regular languages are closed under all operations used in this definition, $\mathcal{L}(F)$ is regular. The correspondence between words and interpretations is established by the following lemma which is proven by structural induction.

Lemma 22. *Let F be an n-dimensional $\mathsf{SC}_{\mathsf{fin}}$ formula, $[0, t]$ a temporal interval and $D = [b_1, f_1] \times \cdots \times [b_{n-1}, f_{n-1}]$ a spatial interval. Then the following holds*:

(α) *Let $\mathcal{I}$ be an interpretation satisfying F on the interval $D \times [0, t]$, then the word $\sigma \stackrel{df}{=} c_0 \ldots c_{t-1} \in \Sigma_F(D)^*$ representing $\mathcal{I}$ with*

$$c_k(\vec{x}) \stackrel{df}{=} \{X \in Obs \mid \mathcal{I}(X)(\vec{x}, k) = 1\}$$

for $k \in \{0, \ldots, t-1\}$ is in $\mathcal{L}_D(F)$.

(β) *Let* $\sigma = c_0 \ldots c_{t-1} \in \mathcal{L}_D(F)$, *then the corresponding interpretation* $\mathcal{I}$ *defined by*

$$\mathcal{I}(X)(\vec{x}, t) \stackrel{df}{=} \begin{cases} 1 & \textit{if } X \in c_t(\vec{x}) \\ 0 & \textit{otherwise} \end{cases}$$

satisfies $\mathcal{I}, D \times [0, t] \models F$.

The following corollary establishes the correctness of the construction.

Corollary 23. *An n-dimensional* $\mathsf{SC_{fin}}$ *Formula F is satisfiable on an interval* $D \times [0, t]$ *where D is a* $(n-1)$*-dimensional spatial and* $[0, t]$ *is a temporal interval iff* $\mathcal{L}_D(F)$ *is non-empty*.

As the emptiness problem for regular languages is decidable, this proves the decidability.

Theorem 24. *The validity and satisfiability problems for* $\mathsf{SC_{fin}}$ *are decidable*.

The complexity of the decision procedure is non-elementary as each negation may cause an exponential blow-up due to the complementation of finite automata. The complexity is polynomial in the assumed finite spatial cardinality, the degree of the polynomial is determined by the number of nested chops. The complexity of deciding validity for Duration Calculus is also already non-elementary [28]. Although non-elementary seems to be a serious concern, the application of the procedure for Duration Calculus on several case studies[36] demonstrates the feasibility of this approach.

Example 25. We illustrate the language construction for the decision procedure by constructing the language $\mathcal{L}(\lceil P \rceil \langle \vec{e}_y \rangle \lceil \neg P \rceil)$. We assume $Obs = \{P\}$, a two-dimensional space, and a maximal spatial cardinality of 2 in each dimension. For convenience the functions c are represented by matrices.

$$\mathcal{L}_{[0,2]\times[0,0]}(\lceil P \rceil) = \varnothing \qquad \mathcal{L}_{[0,2]\times[2,2]}(\lceil \neg P \rceil) = \varnothing$$

$$\mathcal{L}_{[0,2]\times[0,1]}(\lceil P \rceil) = \{(\{P\}\ \{P\})\}^+ \qquad \mathcal{L}_{[0,2]\times[1,2]}(\lceil \neg P \rceil) = \{(\{\}\ \{\})\}^+$$

$$\mathcal{L}_{[0,2]\times[0,2]}(\lceil P \rceil) = \left\{ \begin{pmatrix} \{P\} & \{P\} \\ \{P\} & \{P\} \end{pmatrix} \right\}^+ \qquad \mathcal{L}_{[0,2]\times[0,2]}(\lceil \neg P \rceil) = \left\{ \begin{pmatrix} \{\} & \{\} \\ \{\} & \{\} \end{pmatrix} \right\}^+$$

$$\begin{aligned}
\mathcal{L}_{[0,2]\times[0,2]}(\lceil P \rceil \langle \vec{e}_y \rangle \lceil \neg P \rceil) = {} & h^{-1}_{[0,2]\times[0,2]\to[0,2]\times[0,0]}(\mathcal{L}_{[0,2]\times[0,0]}(\lceil P \rceil)) \cap \\
& h^{-1}_{[0,2]\times[0,2]\to[0,2]\times[0,2]}(\mathcal{L}_{[0,2]\times[0,2]}(\lceil \neg P \rceil)) \\
& \cup \\
& h^{-1}_{[0,2]\times[0,2]\to[0,2]\times[0,1]}(\mathcal{L}_{[0,2]\times[0,1]}(\lceil P \rceil)) \cap \\
& h^{-1}_{[0,2]\times[0,2]\to[0,2]\times[1,2]}(\mathcal{L}_{[0,2]\times[1,2]}(\lceil \neg P \rceil)) \\
& \cup \\
& h^{-1}_{[0,2]\times[0,2]\to[0,2]\times[0,2]}(\mathcal{L}_{[0,2]\times[0,2]}(\lceil P \rceil)) \cap \\
& h^{-1}_{[0,2]\times[0,2]\to[0,2]\times[2,2]}(\mathcal{L}_{[0,2]\times[2,2]}(\lceil \neg P \rceil)) \\
= {} & \left\{ \begin{pmatrix} \{P\} & \{P\} \\ \{\} & \{\} \end{pmatrix} \right\}^+
\end{aligned}$$

Expressivity. Although the subset $\mathsf{SC}_{\mathsf{fin}}$ seems to be rather limited, there are several expressions of the original language that can be obtained as abbreviations using the restricted set and the fact that the temporal and spatial domain are discrete. The terms $\ell_{\vec{e}_i}$ are obtained because it is impossible in a discrete domain to chop an interval of length 1 into two parts of positive length. So, for $k \in \mathbb{N}^+$, we obtain measures as follows.

$$\begin{aligned}
\lceil\rceil &\overset{df}{\iff} \neg\lceil 1\rceil \\
\ell_{\vec{e}_i} = 1 \wedge \neg\lceil\rceil &\overset{df}{\iff} \lceil 1\rceil \wedge \neg(\lceil 1\rceil \langle \vec{e}_i\rangle \lceil 1\rceil) \\
\ell_{\vec{e}_i} = k + 1 \wedge \neg\lceil\rceil &\overset{df}{\iff} (\ell_{\vec{e}_i} = k \wedge \neg\lceil\rceil) \langle \vec{e}_i\rangle (\ell_{\vec{e}_i} = 1 \wedge \neg\lceil\rceil) \\
\ell_{\vec{e}_i} > k \wedge \neg\lceil\rceil &\overset{df}{\iff} (\ell_{\vec{e}_i} = k \wedge \neg\lceil\rceil) \langle \vec{e}_i\rangle \lceil 1\rceil;
\end{aligned}$$

The definition of the other operators $\leqslant, \geqslant, <$ is analogue. As interpretations may only change their value at discrete points, the measure $\int P$ can be expressed as follows:

$$\begin{aligned}
\textstyle\int P = 0 &\overset{df}{\iff} \lceil\neg P\rceil \vee \lceil\rceil \\
\textstyle\int P = 1 &\overset{df}{\iff} \textstyle\int P = 0 \\
&\qquad \langle \vec{e}_x\rangle \left(\textstyle\int P = 0 \langle \vec{e}_t\rangle (\lceil P\rceil \wedge \ell_x = 1 \wedge \ell_t = 1) \langle \vec{e}_t\rangle \textstyle\int P = 0\right) \\
&\qquad \langle \vec{e}_x\rangle \textstyle\int P = 0 \\
\textstyle\int P = k &\overset{df}{\iff} \bigvee_{\substack{k_1, k_2 > 0 \\ k_1 + k_2 = k}} \left(\left(\textstyle\int P = k_1\right) \langle \vec{e}_x\rangle \left(\textstyle\int P = k_2\right)\right) \vee \\
&\qquad \bigvee_{\substack{k_1, k_2 > 0 \\ k_1 + k_2 = k}} \left(\left(\textstyle\int P = k_1\right) \langle \vec{e}_t\rangle \left(\textstyle\int P = k_2\right)\right)
\end{aligned}$$

5.2. Non-alternating chop.

Another possibility of deriving a decidable subset is to use the fibrings and dovetailing ideas presented by Gabbay et al [23,24] to combine modal logics. In order to create a structure for the combined logic, they start with a structure for the first one, associate to each world a structure for the second logic and so on. The idea is depicted in Fig. 5a. Using this approach a lot of important properties like axiomatisability and decidability are inherited by the combination.

As we are interested in models isomorphic to the grid, we need to rule out models like those sketched in 5b where going up and right is not equivalent to going right first and up afterwards because our main goal is to reason about objects in $\mathbb{N}^n$. To this end, we do not allow chop-alternation. On the innermost nesting level we only allow formulae using $\langle \vec{e}_2\rangle$ nested in formulae using $\langle \vec{e}_1\rangle$ and so on. To preserve decidability, we restrict the interaction of formulae by adding a constraint on the length.

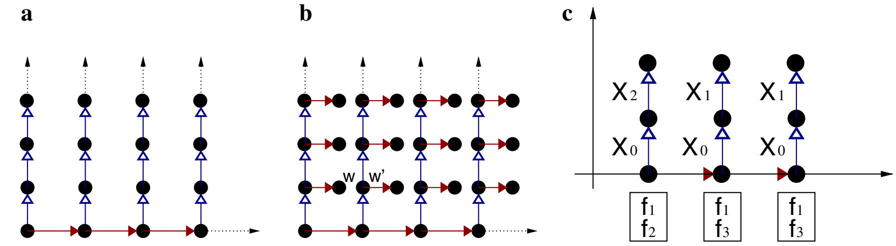

Fig. 5. (a) Dovetailing linear modal structures. (b) Points w and w' need not to be equal. (c) Dovetailing SC.

The language of this n-dimensional subset $\mathsf{SC}_{\mathsf{nAlt}}$ is the set of formulae F^n generated by the following BNF:

$$\begin{aligned} F^1 &::= \lceil P \rceil \mid F^1_1 \langle \vec{e}_1 \rangle F^1_2 \mid F^1_1 \wedge F^1_2 \mid \neg F^1_1 \\ F^{n+1} &::= F^{n+1}_1 \langle \vec{e}_{n+1} \rangle F^{n+1}_2 \mid F^{n+1}_1 \wedge F^{n+1}_2 \mid \neg F^{n+1}_1 \mid \left(F^n \wedge \ell_{\vec{e}_{n+1}} = 1\right) \end{aligned}$$

Let $\delta(F)$ denote the index of chop used on the topmost level, i.e, the minimal i such that F can be generated from F^i.

Note, that without this restriction, it is already possible to encode the tiling problem and the resulting subset is undecidable.

Remark 26. An $\mathsf{SC}_{\mathsf{nAlt}}$ formula of type $\delta(F) = n$ can not only be interpreted by n-dimensional interpretations $\mathcal{I}^n : \left(Obs \times \mathsf{Int}^n\right) \to \{0, 1\}$ but also by k-dimensional functions $\mathcal{I}^k \left(Obs \times \mathsf{Int}^k\right) \to \{0, 1\}$ for any $k \geqslant n$.

A simple consequence is obtained immediately.

Lemma 27. *Let F be an $\mathsf{SC}_{\mathsf{nAlt}}$ formula of type $\delta(F) = n$.*

(α) *Assume $\mathcal{I}^k$ for $k \geqslant n$ to be a k-dimensional interpretation and D a k-dimensional interval such that $\mathcal{I}^k, D \models F$. Define $\mathcal{I}^{k \to k+1}$ as a $k+1$-dimensional interpretation by $\mathcal{I}^{k \to k+1}(X)(\vec{x}, y) \overset{df}{=} \mathcal{I}^k(\vec{x})$. Then any interval $[b, f]$ satisfies $\mathcal{I}^{k \to k+1}, D \times [b, f] \models F$.*
(β) *Conversely, assume $\mathcal{I}^{k+1}, D \times [b, b+1] \models F$ for $b \in \mathbb{N}$, then for an interpretation $\mathcal{I}^{k+1 \to k}(X)(\vec{x}) \overset{df}{=} \mathcal{I}^{k+1}(\vec{x}, b)$ we obtain $\mathcal{I}^{k+1 \to k}, D \models F$.*
(γ) *Let $\mathcal{I}, \mathcal{I}'$ be two n-dimensional interpretations, $[0, f]$ a one-dimensional interval and D, D' two $(n-1)$-dimensional intervals such that*

$$\mathcal{I}, D \times [j, j+1] \models F^{n-1} \wedge \ell_{\vec{e}_n} = 1 \;\; \mathit{iff}\; \mathcal{I}', D' \times [j, j+1] \models F^{n-1} \wedge \ell_{\vec{e}_n} = 1$$

for all subformulae F^{n-1} of type $(n-1)$ occurring in F and all $j \in [0, f)$. Then $\mathcal{I}, D \times [0, f] \models F$ iff $\mathcal{I}', D' \times [0, f] \models F$.

The last proposition states that if validity for two interpretations coincides on every slice and every subformula then validity coincides for the whole formula. Like that for finite spatial domains, the decision procedure constructs regular languages associated to fulfilling interpretations. For a model of an n-dimensional formula, we encode each $(n-1)$-dimensional slice of this model by one letter. As this spatial slice is still infinite, we use the set of all $(n-1)$-dimensional subformulae that are true in this slice as a representative of the slice. Conversely, having no chop alternation, it is possible to obtain an n-dimensional model by joining $(n-1)$-dimensional slices.

This idea gives rise to the following definition of the alphabets where the dimension n is indicated by a superscript n if necessary for clarity and omitted if it is clear from the context

Definition 28 (*Alphabet* $\Sigma^n_{nA}(F)$)**.** Let F be an n-dimensional $\mathsf{SC_{nAlt}}$ formula.

Case $\delta(F) = n = 1$. If F is a pure DC formula (i.e. $n = 1$) a letter characterises which observables are true at the current position. Henceforth, we define the alphabet to be the powerset of the observables Obs as $\Sigma^1_{nA}(F) = \mathcal{P}(Obs)$.

Case $\delta(F) = n > 1$. In this case, the subformulae of type $(n-1)$ play the role of the observables. Let $Sub^{n-1}(F) = \{F_1, \ldots, F_y\}$ be the set of subformulae of F with type $\delta(F_i) = n-1$. A subset of $Sub^{n-1}(F)$ is used to characterise the set of all true *formulae* that hold for an interval of length one in direction n. Therefore the alphabet is defined by $\Sigma^n_{nA}(F) = \mathcal{P}(Sub^{n-1}(F))$.

The construction of the language $\mathcal{L}^n_{nA}(F)$ proceeds inductively on the structure of the formula.

Case $\delta(F) = 1$. In this case F is a pure DC formula and we construct the language in the same way as for discrete DC. Let $Obs = \{X_0, \ldots, X_z\}$ be the Boolean observables occurring in F. Then a subset a of Obs represents a valuation of these observables for an interval of unit length. Define $\mathcal{L}^1_{nA}(F)$ inductively by

$$\mathcal{L}^1_{nA}(\lceil\pi\rceil) \stackrel{df}{=} \{a \mid a \Vdash \pi\}^+, \qquad \mathcal{L}^1_{nA}(F_1 \wedge F_2) \stackrel{df}{=} \mathcal{L}^1_{nA}(F_1) \cap \mathcal{L}^1_{nA}(F_2),$$
$$\mathcal{L}^1_{nA}(F_1 \langle\vec{e}_1\rangle F_2) \stackrel{df}{=} \mathcal{L}^1_{nA}(F_1) \circ \mathcal{L}^1_{nA}(F_2), \quad \mathcal{L}^1_{nA}(\neg F_1) \stackrel{df}{=} \overline{\mathcal{L}^1_{nA}(F_1)}.$$

Case $\delta(F) = n+1$. In this case the subformulae of type n play the role of the observables. Let $Sub^n(F) = \{F_1, \ldots, F_y\}$ be the set of subformulae of F with $\delta(F_i) = n$. Then a set $a \subseteq Sub^n(F)$ can be used to describe which *formulae* are required to hold for an interval of length one. At first we construct an auxiliary regular language $\mathcal{L}'(F)$ for the formula F in the same way as in the above case such that for every word holds: if for every letter, i.e. a set of subformulae of F, there is a model satisfying all the formulae in this set and the models can be joined to obtain a model for the whole formula F.

$$\begin{aligned}
\mathcal{L}'(F^n \wedge \ell_{\vec{e}_{n+1}} = 1) &\stackrel{df}{=} \{a \mid F \in a\} \\
\mathcal{L}'(\neg F^{n+1}) &\stackrel{df}{=} \overline{\mathcal{L}'(F^{n+1})} \\
\mathcal{L}'(F_1^{n+1} \langle\vec{e}_{n+1}\rangle F_2^{n+1}) &\stackrel{df}{=} \mathcal{L}'(F_1^{n+1}) \circ \mathcal{L}'(F_2^{n+1}) \\
\mathcal{L}'(F_1^{n+1} \wedge F_2^{n+1}) &\stackrel{df}{=} \mathcal{L}'(F_1^{n+1}) \cap \mathcal{L}'(F_2^{n+1})
\end{aligned}$$

Different from the simple case, the language $\mathcal{L}'$ does not represent the set of satisfying interpretations, since a word in $\mathcal{L}'$ does not guarantee that there is a satisfying interpretation. A requirement that two subformulae F_1^n and F_2^n of type n hold jointly for the very same interval may not be satisfiable. Additionally, we have to ensure that

- for each letter occurring in a word, i.e., subset $a \subset Sub^n(F)$, there is an interpretation which satisfies exactly those formulae F where $F \in a$ and
- there is a common length k such that for all letters occurring in words, there is a satisfying interpretation of this length such that joining these models yields a *rectangular* model.

To capture these requirements, we introduce the notion of consistency, i.e. the existence of a common model, first for letters and then for alphabets. A letter $a \in \Sigma_{nA}^{n+1}(F)$ denotes a set of formulae of type n. It is consistent if there is an n-dimensional interpretation satisfying all formulae in a. For one formula $F \in a$, there is such an interpretation if the language $\mathcal{L}_{nA}^n(F)$ is not empty. A set a of formulae is consistent if there is a word in the conjunction of all such languages and only built from consistent letters of the lower dimensional type $n-1$. A set of letters $\Pi \subseteq \Sigma_{nA}^{n+1}(F)$ is consistent if there are satisfying n-dimensional interpretations and intervals for each letter having all the same size such that they can be concatenated to form an $n+1$-dimensional interpretation. Therefore, for every word over a consistent alphabet $\Pi \subseteq \Sigma_{nA}^{n+1}(F)$, there is an interpretation such that each letter corresponds to a slice of length one in direction $(n+1)$ satisfying all the n-dimensional subformulae occurring in the letter.

For regarding only the length of a word, we employ the following homomorphism, which "obscures" the actual letters of the word leaving only its pure shape.

Definition 29. Let $\sharp$ be an arbitrary symbol and $h_\sharp : \Sigma^* \to \{\sharp\}^*$ be the homomorphism that replaces every letter by $\sharp$.

For defining consistency formally, we extend the definition of an associated language for a formula F to a set of characteristic formulae a by

$$\mathcal{L}_{nA}^n(a) \stackrel{df}{=} \bigcap_{F' \in a} \mathcal{L}_{nA}^n(F') \cap \bigcap_{F' \notin a} \overline{\mathcal{L}_{nA}^n(F')}.$$

Using this definition, the intuition depicted above is formalised inductively as follows.

Definition 30. A subset $\Pi^{n+1}(F) \subseteq \Sigma_{nA}^{n+1}(F)$ for $n > 1$ is called *consistent* iff there is a consistent $\Pi^n \subseteq \bigcup_{a \in \Pi^{n+1}(F)} \bigcup_{G \in a} \Sigma_{nA}^n(G)$ such that

$$\bigcap_{a \in \Pi} h_\sharp(\mathcal{L}_{nA}^n(a) \cap (\Pi^n)^*) \neq \varnothing$$

The basic case is handled directly, every subset $\Pi^1(F) \subseteq \Sigma_{nA}^1(F)$ is consistent.

Combining consistency—the existence of interpretations satisfying the formulae for each slice that can be concatenated—and the property of $\mathcal{L}'$ that piecewise satisfaction of subformulae yields satisfaction of the whole formula, we define the language by

$$\mathcal{L}^{n+1}_{nA}(F) \stackrel{df}{=} \mathcal{L}'(F) \cap \left(\bigcup_{\substack{\Pi \subseteq \Sigma \\ \Pi \text{ is consistent}}} \Pi^* \right).$$

The correspondence between words in $\mathcal{L}^n_{nA}(F)$ and n-dimensional interpretations is established by the following definition and lemma.

Definition 31 (*word-interpretation correspondence*)**.** Let F be an n-dimensional $\mathsf{SC}_{\mathsf{nAlt}}$ formula, $\mathcal{I}$ be an n-dimensional interpretation, D an $(n-1)$-dimensional interval and $[0, f]$ a one-dimensional interval. We associate a word $w^{\mathcal{I}} = a^{\mathcal{I}}_0 \dots a^{\mathcal{I}}_{f-1} \in \Sigma^n_{nA}(F)^*$ to this interpretation such that for $n = 1$

$$a^{\mathcal{I}}_j \stackrel{df}{=} \{X | X \text{ is an observable occurring in } F \text{ and } \mathcal{I}, D \times [j, j+1] \models \lceil X \rceil\}$$

and, for $n > 1$,

$$a^{\mathcal{I}}_j \stackrel{df}{=} \left\{F' | F' \in Sub^{n-1}(F) \text{ and } \mathcal{I}, D \times [j, j+1] \models F' \right\},$$

respectively. Vice versa, we associate to a word $w \in \Sigma^n_{nA}(F)^*$ the set of interpretations and intervals $[w]$ such that

$$(\mathcal{I}, D) \in [w] \iff w = w^{\mathcal{I}}.$$

Lemma 32. *Let F be an n-dimensional $\mathsf{SC}_{\mathsf{nAlt}}$ formula.*

(α) *A subset $\Pi \subseteq \Sigma^n_{nA}(F)$ is consistent iff for every $w = a_0 \dots a_k \in \Pi^*$, there is an n-dimensional interpretation $\mathcal{I}$, an $(n-1)$-dimensional interval D such that $\mathcal{I}, D \times [j, j+1) \models G \iff G \in a_j$ holds for every $j \in [0, k]$ and every $G \in Sub^{n-1}(F)$.*
(β) *$\mathcal{I}, D \times [0, f] \models F$ implies $w^{\mathcal{I}} \in \mathcal{L}^n_{nA}(F)$.*
(γ) *$w \in \mathcal{L}^n_{nA}(F)$ implies $[w] \neq \emptyset$ and $\forall\, (\mathcal{I}, D) \in [w] : \mathcal{I}, D \times [0, |w|] \models F$*

An easy consequence is the following lemma.

Lemma 33. *$\mathcal{L}(F) \neq \emptyset$ iff F is satisfiable*

And as all these constructions can be done effectively this proves the following theorem.

Theorem 34. *Satisfiability and validity for $\mathsf{SC}_{\mathsf{nAlt}}$ are decidable.*

Like for $\mathsf{SC}_{\mathsf{fin}}$ the complexity is non-elementary due to the complementation of finite automata for each negation.

Example 35. These constructions are illustrated in Fig. 5c. Assume a formula

$$F \stackrel{df}{=} (F_1 \wedge \ell_{\vec{e}_1} = 1 \, \langle \vec{e}_1 \rangle \, F_1 \wedge \ell_{\vec{e}_1} = 1 \, \langle \vec{e}_1 \rangle \, F_1 \wedge \ell_{\vec{e}_1} = 1) \wedge$$
$$(F_2 \wedge \ell_{\vec{e}_1} = 1 \, \langle \vec{e}_1 \rangle \, F_3 \wedge \ell_{\vec{e}_1} = 1 \, \langle \vec{e}_1 \rangle \, F_3 \wedge \ell_{\vec{e}_1} = 1)$$

with

$$F_1 \stackrel{df}{=} \lceil X_1 \rceil \, \langle \vec{e}_2 \rangle \, \text{true}, \quad F_2 \stackrel{df}{=} \text{true} \, \langle \vec{e}_2 \rangle \, \lceil X_2 \rceil, \quad F_3 \stackrel{df}{=} \text{true} \, \langle \vec{e}_2 \rangle \, \lceil X_3 \rceil.$$

The word $(1,1,0)(1,0,1)(1,0,1)$ is in $\mathcal{L}'(F)$ and as the alphabet is consistent also in $\mathcal{L}(F)$. Therefore the models for F_1, F_2, F_3 can be combined to form a model for F.

Expressivity. Like in $\mathsf{SC}_{\mathsf{fin}}$ operators can be obtained in $\mathsf{SC}_{\mathsf{nAlt}}$. We illustrate the two-dimensional case here. At first we give definitions for formulae of type 1 which are to be used in the context of "$\wedge \, \ell_{\vec{e}_2} = 1$". We use the superscript ^{1} here to stress this restriction.

$$\begin{aligned}
\text{true}^1 &\stackrel{df}{\iff} \lceil 1 \rceil^1 \vee \neg \lceil 1 \rceil^1 \\
\ell^1_{\vec{e}_1} = 0 &\stackrel{df}{\iff} \neg \lceil 1 \rceil^1 \\
\ell^1_{\vec{e}_1} = 1 \wedge \neg \lceil\rceil &\stackrel{df}{\iff} \lceil 1 \rceil^1 \wedge \neg(\lceil 1 \rceil^1 \, \langle \vec{e}_1 \rangle \, \lceil 1 \rceil^1) \\
\ell^1_{\vec{e}_1} = k+1 &\stackrel{df}{\iff} (\ell^1_{\vec{e}_1} = k) \, \langle \vec{e}_1 \rangle \, (\ell^1_{\vec{e}_1} = 1) \\
\ell^1_{\vec{e}_1} > k &\stackrel{df}{\iff} (\ell^1_1 = k) \, \langle \vec{e}_1 \rangle \, \lceil 1 \rceil^1 \\
\textstyle\int^1 P = 0 &\stackrel{df}{\iff} \lceil \neg P \rceil^1 \vee \ell^1_{\vec{e}_1} = 0 \\
\textstyle\int^1 P = 1 &\stackrel{df}{\iff} \textstyle\int^1 P = 0 \, \langle \vec{e}_1 \rangle \, \lceil P \rceil^1 \wedge \ell^1_{\vec{e}_1} = 1 \, \langle \vec{e}_1 \rangle \textstyle\int^1 P = 0 \\
\textstyle\int^1 P = k+1 &\stackrel{df}{\iff} \textstyle\int^1 P = k \, \langle \vec{e}_1 \rangle \textstyle\int^1 P = 1
\end{aligned}$$

For formulae of type 2 the definitions are more complicated. At first "true" can be defined in the standard way.

$$\text{true} \stackrel{df}{\iff} (\lceil 1 \rceil \wedge \ell_{\vec{e}_2} = 1) \vee \neg(\lceil 1 \rceil \wedge \ell_{\vec{e}_2} = 1)$$

As $\ell_{\vec{e}_2}$ is nearly a primitive in $\mathsf{SC}_{\mathsf{nAlt}}$, it can be defined as follows:

$$\begin{aligned}
\ell_{\vec{e}_2} = 1 &\stackrel{df}{\iff} (\text{true}^1) \wedge \ell_{\vec{e}_2} = 1 \\
\ell_{\vec{e}_2} = k+1 &\stackrel{df}{\iff} (\ell_{\vec{e}_2} = k) \, \langle \vec{e}_2 \rangle \, (\ell_{\vec{e}_2} = 1)
\end{aligned}$$

The measure $\int P$ is non-zero iff there is a subinterval of length 1 on which the measure is non-zero. Therefore the measure can be defined using the type 1 formula $\int^1 P = 0$.

$$\int P = 0 \overset{df}{\iff} \neg \left(\text{true}\, \langle \vec{e}_2 \rangle \, \left(\left(\neg \left(\int^1 P = 0\right)\right) \wedge \ell_{\vec{e}_2} = 1\right) \, \langle \vec{e}_2 \rangle \, \text{true}\right)$$

Using the same idea, we can define $\int P = 1$.

$$\int P = 1 \overset{df}{\iff} \int P = 0 \, \langle \vec{e}_2 \rangle \, \left(\int^1 P = 1 \wedge \ell_{\vec{e}_2} = 1\right) \, \langle \vec{e}_2 \rangle \int P = 0$$

On an interval of length m the measure $\int P$ equals k iff it is equal to k_1 on the leftmost subinterval of length $m - 1$, is equal to k_2 on the rightmost subinterval of length 1 and $k = k_1 + k_2$.

$$\int P = k \overset{df}{\iff} \bigvee_{\substack{k_1, k_2 \in \mathbb{N}_0 \\ k_1 + k_2 = k}} \left(\int P = k_1 \, \langle \vec{e}_2 \rangle \int P = k_2\right)$$

6. Conclusion

In this paper, we investigated properties of a multi-dimensional extension of Duration Calculus. We showed that it is not axiomatisable and therefore not decidable. Nevertheless, we give an axiomatisation relative to an n-dimensional interval temporal logic. Tool support is crucial for applications of such a formalism in practice. For model-checking tools, decidable subsets play an important role. We presented two different subset for discrete spatial and temporal domains, one obtained by restricting the models to finite space while preserving infinite time and the other by restricting the class of formulae by excluding chop alternation.

In the future, we would like to apply this formalism to several case studies to derive a set of lightweight rules that make the handling of this formalism in practice easier. To give tool support, this should be accompanied by extending and implementing the decision procedures found so far.

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A. Proofs

Lemma 10. *Let $\mathcal{I}$ be an interpretation and $k \in \mathbb{N}$, $a, b \in \mathbb{T}$ and F_{grid} defined by*

$$F_{\text{grid}} \stackrel{df}{=} \ell_{\vec{e}_1} \geqslant 2 \wedge \ell_{\vec{e}_2} \geqslant 2 \wedge \tag{A.1a}$$
$$\lceil \bigstar \iff (\bigstar_1 \iff \bigstar_2) \rceil \wedge \tag{A.1b}$$
$$\Box_{\vec{e}_1}\big(((\lceil \bigstar_1 \rceil \langle \vec{e}_1 \rangle \ell_{\vec{e}_1} = 1 \Rightarrow \lceil \bigstar_1 \rceil \langle \vec{e}_1 \rangle \lceil \neg \bigstar_1 \rceil) \wedge \tag{A.1c}$$
$$(\lceil \neg \bigstar_1 \rceil \langle \vec{e}_1 \rangle \ell_{\vec{e}_1} = 1 \Rightarrow \lceil \neg \bigstar_1 \rceil \langle \vec{e}_1 \rangle \lceil \bigstar_1 \rceil)\big) \wedge \tag{A.1d}$$
$$\ell_{\vec{e}_1} \geqslant 1 \Rightarrow (\lceil \bigstar_1 \rceil \wedge \ell_{\vec{e}_1} = 1 \langle \vec{e}_1 \rangle \, \mathit{true}) \wedge \tag{A.1e}$$
$$\Box_{\vec{e}_2}\big(((\lceil \bigstar_2 \rceil \langle \vec{e}_2 \rangle \ell_{\vec{e}_2} = 1 \Rightarrow \lceil \bigstar_2 \rceil \langle \vec{e}_2 \rangle \lceil \neg \bigstar_2 \rceil) \wedge \tag{A.1f}$$
$$(\lceil \neg \bigstar_2 \rceil \langle \vec{e}_2 \rangle \ell_{\vec{e}_2} = 1 \Rightarrow \lceil \neg \bigstar_2 \rceil \langle \vec{e}_2 \rangle \lceil \bigstar_2 \rceil)\big) \wedge \tag{A.1g}$$
$$\ell_{\vec{e}_2} \geqslant 1 \Rightarrow (\lceil \bigstar_2 \rceil \wedge \ell_{\vec{e}_2} = 1 \langle \vec{e}_2 \rangle \, \mathit{true}) \tag{A.1h}$$

Then:

$$\mathcal{I}, [0,k] \times [a,b] \models F_{\text{grid}}$$

if and only if $k \geqslant 2$, $b - a \geqslant 2$, *and for all* $i \in \mathbb{N}$, $i \leqslant k$ *and arbitrary* $a' \leqslant b' \in [a,b]$ *the following holds.*

$$\alpha) \quad \mathcal{I}, [i, i+1] \times [a', b'] \models \begin{cases} \lceil \bigstar_1 \rceil & \textit{if } i \textit{ is even} \\ \lceil \neg \bigstar_1 \rceil & \textit{otherwise} \end{cases}$$

$$\beta) \quad \mathcal{I}, [a', b'] \times [i, i+1] \models \begin{cases} \lceil \bigstar_2 \rceil & \textit{if } i \textit{ is even} \\ \lceil \neg \bigstar_2 \rceil & \textit{otherwise} \end{cases}$$

$$\gamma) \quad \mathcal{I}, [i, i+1] \times [j, j+1] \models \begin{cases} \lceil \bigstar \rceil & \textit{if } i,j \textit{ are both even or both odd} \\ \lceil \neg \bigstar \rceil & \textit{otherwise} \end{cases}$$

Proof

"**only if**" To prove α), we proceed by induction on i.

Case $i = 0$. This case is clear by (A.1a).

Case $i \rightsquigarrow i+1$. Without loss of generality, assume i is even, the other case is similar.

$$\begin{aligned}
\{\text{ By (IH)}\} \quad & \mathcal{I}, [i-1, i] \times [a', b'] \models \lceil \neg \bigstar_1 \rceil \\
& \Rightarrow \mathcal{I}, [i-1, i+1] \times [a', b'] \models \lceil \neg \bigstar_1 \rceil \langle \vec{e}_1 \rangle \ell_{\vec{e}_1} = 1 \\
\{F_{\text{grid}} A.1d\} & \Rightarrow \mathcal{I}, [i-1, i+1] \times [a', b'] \models \lceil \neg \bigstar_1 \rceil \langle \vec{e}_1 \rangle \lceil \bigstar_1 \rceil \\
\{\bigstar_1 \wedge \neg \bigstar_1 \equiv \text{false}\} \Rightarrow \quad & \mathcal{I}, [i, i+1] \times [a', b'] \models \lceil \bigstar_1 \rceil
\end{aligned}$$

Case β) is analogue to case α). For case γ) assume without loss of generality i and j to be even, the other cases are similar.

$$\begin{aligned}
\{\text{ By } \alpha) \text{ and } \beta)\} \quad & \mathcal{I}, [i-1, i] \times [j, j+1] \models \lceil \bigstar_1 \rceil \wedge \lceil \bigstar_2 \rceil \\
\{A.1b\} & \Rightarrow \mathcal{I}, [i-1, i] \times [j, j+1] \models \lceil \bigstar \rceil
\end{aligned}$$

"if" The Condition (A.1a) follows from the assumption on the interval and Conditions (A.1b), (A.1e) and (A.1h) are direct consequences of α), β), and γ). To prove (A.1c) assume

$$\begin{aligned}
& \mathcal{I},[a_1,b_1]\times[a,b]\models\lceil\bigstar_1\rceil\,\langle\vec{e}_1\rangle\,\ell_{\vec{e}_1}=1.\\
& \Rightarrow \mathcal{I},[a_1,b_1-1]\times[a,b]\models\lceil\bigstar_1\rceil\\
\left\{\begin{array}{l}\exists i\in\mathbb{N}:\\ i\leqslant a_1\leqslant b_1-1\leqslant i+1\end{array}\right\} & \Rightarrow \mathcal{I},[a_1,i+1]\times[a,b]\models\lceil\bigstar_1\rceil\\
\{\alpha)\} & \Rightarrow \mathcal{I},[i+1,i+2]\times[a,b]\models\lceil\neg\bigstar_1\rceil\\
\{b_1\leqslant i+2\} & \Rightarrow \mathcal{I},[i+1,b_1]\times[a,b]\models\lceil\neg\bigstar_1\rceil\\
& \Rightarrow \mathcal{I},[a_1,b_1]\times[a,b]\models\lceil\bigstar_1\rceil\,\langle\vec{e}_1\rangle\,\lceil\neg\bigstar_1\rceil
\end{aligned}$$

The other cases are proven similarly. □

Lemma 15. *For an H-triple* $(\mathcal{I}_{ITL^n},\mathcal{V},[b_1,f_1]\times[b_2,f_2])$*, a state assertion* π*, and an SC-Interpretation as defined in equation 1 there is a finite partition* $[m_1^1,m_1^2]\times[m_2^1,m_2^2],\ldots,[m_1^{n-1},m_1^n]\times[m_2^{o-1},m_2^o]$ *of the two-dimensional interval* $[b_1,f_1]\times[b_2,f_2]$ *such that for every point* $(x,y)\in[m_1^i,m_1^{i+1})\times[m_2^j,m_2^{j+1})$ *holds*

$$\mathcal{I}_{ITL^n},\mathcal{V},[m_1^i,m_1^{i+1})\times[m_2^j,m_2^{j+1})\models_{ITL^n}\lceil v_{[\pi]}\rceil\vee\lceil v_{[\neg\pi]}\rceil \tag{A.2}$$

and

$$\mathcal{I}_{SC}[\![\pi]\!]\big((x,y)\big)=\begin{cases}1 & \text{if } \mathcal{I},\mathcal{V},[m_1^i,m_1^{i+1})\times[m_2^j,m_2^{j+1})\models_{ITL^n}\lceil v_{[\pi]}\rceil\\ 0 & \text{if } \mathcal{I},\mathcal{V},[m_1^i,m_1^{i+1})\times[m_2^j,m_2^{j+1})\models_{ITL^n}\lceil v_{[\neg\pi]}\rceil.\end{cases} \tag{A.3}$$

Proof. We prove this lemma by induction on the structure of π.

Case 1. Observable X.
This case is clear from the definition of $\mathcal{I}_{SC}$ and Lemma 14.

Case 2. $\neg\pi$.
By the induction hypothesis the lemma holds for π and we use the same partition. Let $(x,y)\in[m_1^j,m_1^{j+1})\times[m_2^i,m_2^{i+1})$. From the induction hypothesis we obtain

$$\mathcal{I}_{ITL^n},\mathcal{V},[m_1^i,m_1^{i+1})\times[m_2^j,m_2^{j+1})\models_{ITL^n}\lceil v_{[\pi]}\rceil\vee\lceil v_{[\neg\pi]}\rceil.$$

As $\pi\equiv\neg\neg\pi$, we obtain $v_{[\pi]}=v_{[\neg(\neg\pi)]}$ and therefore

$$\mathcal{I}_{ITL^n},\mathcal{V},[m_1^i,m_1^{i+1})\times[m_2^i,m_2^{i+1})\models_{ITL^n}\lceil v_{[\neg\neg\pi)]}\rceil\vee\lceil v_{[\neg\pi]}\rceil.$$

(1) If $\mathcal{I}_{ITL^n},\mathcal{V},[m_1^i,m_1^{i+1})\times[m_2^i,m_2^{i+1})\models_{ITL^n}\lceil v_{[\neg\neg\pi]}\rceil$ then by the hypothesis $\mathcal{I}_{SC}[\![\pi]\!](x,y)=1$ and by definition of negation we obtain $\mathcal{I}_{SC}[\![\neg\pi]\!](x,y)=1-\mathcal{I}_{SC}[\![\pi]\!](x,y)=0$ as required.

(2) If $\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [m_1^i, m_1^{i+1}) \times [m_2^i, m_2^{i+1}) \models_{\mathsf{ITL}^n} \lceil v_{[\neg\pi]} \rceil$ then by the hypothesis $\mathcal{I}_{\mathsf{SC}}[\![\pi]\!](x, y) = 0$ and we obtain $\mathcal{I}_{\mathsf{SC}}[\![\neg\pi]\!](x, y) = 1 - \mathcal{I}_{\mathsf{SC}}[\![\pi]\!](x, y) = 1$ as required.

Case 3. $\pi_1 \vee \pi_2$.
Applying the induction hypothesis on π_1 and π_2, we obtain two partitions. Let $[m_1^{1'}, m_1^{2'}] \times [m_2^{1'}, m_2^{2'}], \ldots [m_1^{n'-1'}, m_1^{n''}] \times [m_2^{o'-1'}, m_2^{o''}]$ be a common refinement of both partitions. By Lemma 13 (3) this is also a valid partition for π_1 and π_2. On every interval one of the following cases holds.

Case 3.1. At least one disjunct π_1 or π_2 is true throughout the interval, i.e. $\lceil v_{[\pi_1]} \rceil$ or $\lceil v_{[\pi_2]} \rceil$. Without loss of generality, we assume

$$\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [m_1^{i'}, m_1^{i+1'}) \times [m_2^{j'}, m_2^{j+1'}) \models_{\mathsf{ITL}^n} \lceil v_{[\pi_1]} \rceil.$$

Applying the induction hypothesis yields for each
$(x, y) \in [m_1^{i'}, m_1^{i+1'}) \times [m_2^{j'}, m_2^{j+1'})$

$$\mathcal{I}_{\mathsf{SC}}[\![\pi_1]\!](\mathcal{V}, (x, y)) = 1$$

and by the definition of the semantics of state assertions in SC we obtain $\mathcal{I}_{\mathsf{SC}}[\![\pi_1 \vee \pi_2]\!](\mathcal{V}, [m_1^{i'}, m_1^{i+1'}) \times [m_2^{j'}, m_2^{j+1'})) = 1$. The ITL^n evaluation is derived as follows:
{ By Lemma 13(3)}
$\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [m_1^{i'}, m_1^{i+1'}) \times [m_2^{j'}, m_2^{j+1'}) \models_{\mathsf{ITL}^n} v_{[\pi_1]} \leqslant v_{[\pi_1 \vee \pi_2]}$
{ and by Lemma 13(2)}
$\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [m_1^{i'}, m_1^{i+1'}) \times [m_2^{j'}, m_2^{j+1'}) \models_{\mathsf{ITL}^n} v_{[\pi_1 \vee \pi_2]} \leqslant \ell$
{ By the assumption holds }
$\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [m_1^{i'}, m_1^{i+1'}) \times [m_2^{j'}, m_2^{j+1'}) \models_{\mathsf{ITL}^n} v_{[\pi_1]} = \ell.$
{ Combining the equations we obtain}
$\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [m_1^{i'}, m_1^{i+1'}) \times [m_2^{j'}, m_2^{j+1'}) \models_{\mathsf{ITL}^n} v_{[\pi_1 \vee \pi_2]} = \ell$
{ By definition we conclude}
$\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [m_1^{i'}, m_1^{i+1'}) \times [m_2^{j'}, m_2^{j+1'}) \models_{\mathsf{ITL}^n} \lceil v_{[\pi_1 \vee \pi_2]} \rceil.$

Case 3.2. Both π_1 and π_2 are false throughout the interval, i.e.

$$\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [m_1^{i'}, m_1^{i+1'}) \times [m_2^{j'}, m_2^{j+1'}) \models_{\mathsf{ITL}^n} \lceil v_{[\neg\pi_1]} \rceil \wedge \lceil v_{[\neg\pi_2]} \rceil$$

Applying the induction hypothesis yields

$$\mathcal{I}_{\mathsf{SC}}[\![\pi_1]\!](\mathcal{V}, [m_1^{i'}, m_1^{i+1'}) \times [m_2^{j'}, m_2^{j+1'})) = 0$$

and

$$\mathcal{I}_{\mathsf{SC}}[\![\pi_2]\!](\mathcal{V}, [m_1^{i'}, m_1^{i+1'}) \times [m_2^{j'}, m_2^{j+1'})) = 0$$

and hence

$$\mathcal{I}_{\mathsf{SC}}[\![\pi_1 \vee \pi_2]\!](\mathcal{V}, [m_1^{i'}, m_1^{i+1'}) \times [m_2^{j'}, m_2^{j+1'})) = 0.$$

The derivation of the ITL^n evaluation proceeds as follows:

$$\begin{gathered}
\{\text{ By the assumptions}\} \\
\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [m_1^{i'}, m_1^{i+1'}) \times [m_2^{j'}, m_2^{j+1'}) \models_{\mathsf{ITL}^n} v_{[\pi_1]} = 0 \text{ and} \\
\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [m_1^{i'}, m_1^{i+1'}) \times [m_2^{j'}, m_2^{j+1'}) \models_{\mathsf{ITL}^n} v_{[\pi_2]} = 0 \\
\{\text{ and by Lemma 13 holds}\} \\
\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [m_1^{i'}, m_1^{i+1'}) \times [m_2^{j'}, m_2^{j+1'}) \models_{\mathsf{ITL}^n} v_{[\pi_1 \vee \pi_2]} \geqslant 0 \text{ and} \\
\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [m_1^{i'}, m_1^{i+1'}) \times [m_2^{j'}, m_2^{j+1'}) \models_{\mathsf{ITL}^n} v_{[\pi_1 \wedge \pi_2]} \geqslant 0 \\
\{\text{ By } \mathcal{H}_4\} \\
\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [m_1^{i'}, m_1^{i+1'}) \times [m_2^{j'}, m_2^{j+1'}) \models_{\mathsf{ITL}^n} v_{[\pi_1 \vee \pi_2]} = 0 \\
\{\text{ and by Lemma 13 (13) holds}\} \\
\mathcal{I}_{\mathsf{ITL}^n}, \mathcal{V}, [m_1^{i'}, m_1^{i+1'}) \times [m_2^{j'}, m_2^{j+1'}) \models_{\mathsf{ITL}^n} \lceil v_{[\neg(\pi_1 \vee \pi_2)]} \rceil \qquad \square
\end{gathered}$$

Lemma 22. *Let F be an n-dimensional $\mathsf{SC}_{\mathit{fin}}$ Formula, $[0,t]$ a temporal interval and $D = [b_1, f_1] \times \cdots \times [b_{n-1}, f_{n-1}]$ a spatial interval. Then the following holds:*

(α) *Let $\mathcal{I}$ be an interpretation satisfying F on the interval $D \times [0,t]$, then the word $\sigma \overset{df}{=} c_0 \ldots c_{t-1} \in \Sigma_F(D)^*$ representing $\mathcal{I}$ with*

$$c_k(\vec{x}) \overset{df}{=} \{X \in Obs \mid \mathcal{I}(X)(\vec{x}, k) = 1\}$$

for $k \in \{0, \ldots, t-1\}$ is in $\mathcal{L}_D(F)$.

(β) *Let $\sigma = c_0 \cdots c_{t-1} \in \mathcal{L}_D(F)$, then the corresponding interpretation $\mathcal{I}$ defined by*

$$\mathcal{I}(X)(\vec{x}, t) \overset{df}{=} \begin{cases} 1 & \text{if } X \in c_t(\vec{x}) \\ 0 & \text{otherwise} \end{cases}$$

satisfies $\mathcal{I}, D \times [0,t] \models F$.

Proof. We proceed inductively on the structure of F.

Case $\lceil \pi \rceil$. (α) Let $\mathcal{I}$ be an interpretation such that $\mathcal{I}, D \times [0,t] \models \lceil \pi \rceil$. Then—by definition—$\mathcal{I}[\![\pi]\!](\vec{x}, t') = 1$ and $f_k > b_k$ holds for all $\vec{x} \in D^-$, $t' \in [0,t)$ and $k \in \{1, \ldots n\}$. By definition of $\Vdash$ we obtain $c_k(\vec{x}) \Vdash \pi$ for all $\vec{x} \in [b_1, f_1) \times \cdots \times [b_{n-1}, f_{n-1})$ and therefore $\sigma \in \mathcal{L}_D(\lceil \pi \rceil)$

(β) Let $\sigma = c_0 \ldots c_{n-1} \in \mathcal{L}_D(\lceil\pi\rceil)$. The definition of $\mathcal{L}_D(\lceil\pi\rceil)$ and $\Vdash$ immediately yields $\mathcal{I}, D \times [0,t] \models \lceil\pi\rceil$ as required.

Case $F_1 \wedge F_2$. (α) Suppose that $\mathcal{I}, D \times [0,t] \models F_1 \wedge F_2$. By definition of $\wedge$ we obtain $\mathcal{I}, D \times [0,t] \models F_1$ and $\mathcal{I}, D \times [0,t] \models F_2$ and by the induction hypothesis and definition 21 $\sigma \in \mathcal{L}_D(F_1) \cap \mathcal{L}_D(F_2) = \mathcal{L}_D(F_1 \wedge F_2)$.

(β) Let $\sigma \in \mathcal{L}_D(F_1 \wedge F_2) = \mathcal{L}_D(F_1) \cap \mathcal{L}_D(F_2)$, by the induction hypothesis we obtain $\mathcal{I}, D \times [0,t] \models F_1 \wedge F_2$.

Case $\neg F_1$. (α) Suppose that $\mathcal{I}, D \times [0,t] \models \neg F_1$. By definition $\mathcal{I}, D \times [0,t] \not\models F_1$. By the induction hypothesis for (β) $\sigma \notin \mathcal{L}_D(F_1)$ and henceforth we obtain $\sigma \in \Sigma_F(D)^* \setminus \mathcal{L}_D(F_1) = \mathcal{L}_D(\neg F_1)$.

(β) Let $\sigma \in \mathcal{L}_D(\neg F_1) = \Sigma_F(D)^* \setminus \mathcal{L}_D(F_1)$. By the induction hypothesis for (α) holds $\mathcal{I}, D \times [0,t] \not\models F_1$ and henceforth $\mathcal{I}, D \times [0,t] \models \neg F_1$.

Case $F_1 \langle \vec{e}_t \rangle F_2$(temporalchop). ($\alpha$) Assume $\mathcal{I}, D \times [0,t] \models F_1 \langle \vec{e}_t \rangle F_2$. By definition of chop, there is an $t' \in [0,t]$ such that $\mathcal{I}, D \times [0,t'] \models F_1$ and $\mathcal{I}, D \times [t',t] \models F_2$. The interpretation $\mathcal{I}'$ which is obtained by left-shifting $\mathcal{I}$ by t' satisfies $\mathcal{I}', D \times [0, t-t'] \models F_2$. The induction hypothesis yields $\sigma_1 \in \mathcal{L}_D(F_1)$ and $\sigma_2 \in \mathcal{L}_D(F_2)$ and therefore $\sigma_1\sigma_2 \in \mathcal{L}_D(F_1) \circ \mathcal{L}_D(F_2)$.

(β) Let $\sigma \in \mathcal{L}_D(F_1 \langle \vec{e}_t \rangle F_2)$. By definition $\sigma = \sigma_1\sigma_2$ for $\sigma_1 \in \mathcal{L}_D(F_1)$ and $\sigma_2 \in \mathcal{L}_D(F_2)$. By the induction hypothesis the two corresponding interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ satisfy $\mathcal{I}_1, D \times [0,t_1] \models F_1$ and $\mathcal{I}_2, D \times [0,t_2] \models F_2$. By definition of $\mathcal{I}$ holds

$$\mathcal{I}(X)(\vec{x},t) = \begin{cases} \mathcal{I}_1(X)(\vec{x},t) & \text{if } t \le t_1 \\ \mathcal{I}_2(X)(\vec{x},t-t_1) & \text{otherwise} \end{cases}.$$

and so $\mathcal{I}, D \times [0, t_1 + t_2] \models F_1 \langle \vec{e}_t \rangle F_2$

Case $F_1 \langle \vec{e}_i \rangle F_2$(spatialchop). ($\alpha$) Assume $\mathcal{I}, D \times [0,t] \models F_1 \langle \vec{e}_i \rangle F_2$. By definition there is an $r \in [b_i, f_i]$ such that $\mathcal{I}, (D \prec_i r) \times [0,t] \models F_1$ and $\mathcal{I}, (D \succ_i r) \times [0,t] \models F_2$. By the induction hypothesis the words σ_1 and σ_2 obtained from $\mathcal{I}$ on the spatial intervals $D \prec_i r$ and $D \succ_i r$ satisfy

$$\sigma_1 \in \mathcal{L}_{D \prec_i r}(F_1) \text{ and } \sigma_2 \in \mathcal{L}_{D \succ_i r}(F_2).$$

As σ is derived from $\mathcal{I}$ for space D and σ_1 is derived from the same interpretation for a space $(D \prec_i r) \subseteq D$, they describe the same spatial configurations for $(D \prec_i r) \subseteq D$, i.e., $h_{D \to D \prec_i r}(\sigma) = \sigma_1$. For σ_2 we obtain $h_{D \to D \succ_i r}(\sigma) = \sigma_2$, respectively.
Therefore $\sigma \in h^{-1}_{D \to D \prec_i r}(\sigma_1) \cap h^{-1}_{D \to D \succ_i r}(\sigma_2)$ as required.

(β) Let $\sigma \in \mathcal{L}_D(F_1 \langle \vec{e}_i \rangle F_i)$. There is an $r \in [b_i, f_i]$ such that

$$\sigma \in h^{-1}_{D \to D \prec_i r}(\mathcal{L}_{D \prec_i r}(F_1)) \text{ and } \sigma \in h^{-1}_{D \to D \succ_i r}(\mathcal{L}_{D \succ_i r}(F_2)).$$

Hence

$$\sigma_1 = h_{D \to D \prec_i r}(\sigma) \in \mathcal{L}_{D \prec_i r}(F_1) \text{ and } \sigma_2 h_{D \to D \succ_i r}(\sigma) \in \mathcal{L}_{D \succ_i r}(F_2).$$

Applying the induction hypothesis, yields that the two interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ obtained from σ_1 and σ_2 satisfy $\mathcal{I}_1, (D \prec_i r) \times [0,t] \models F_1$ and $\mathcal{I}_2, (D \succ_i r) \times [0,t] \models F_2$. The function $\mathcal{I}$ obtained from σ coincides with $\mathcal{I}_1$ on $(D \prec_i r) \times [b_n, f_n]$ and with $\mathcal{I}_2$ on $(D \succ_i r) \times [0,t]$ and therefore $\mathcal{I}, D \times [0,t] \models F_1 \langle \vec{e}_i \rangle F_2$. □

Lemma 27. *Let F be an SC_{nAlt} formula of type $\delta(F) = n$.*

(α) *Assume $\mathcal{I}^k$ for $k \geqslant n$ to be a k-dimensional interpretation and D a k-dimensional interval such that $\mathcal{I}^k, D \models F$. Define $\mathcal{I}^{k\to k+1}$ as a $k+1$-dimensional interpretation by $\mathcal{I}^{k\to k+1}(X)(\vec{x}, y) \stackrel{df}{=} \mathcal{I}^k(\vec{x})$. Then any interval $[b,f]$ satisfies $\mathcal{I}^{k\to k+1}, D \times [b,f] \models F$.*

(β) *Conversely, assume $\mathcal{I}^{k+1}, D \times [b, b+1] \models F$ for $b \in \mathbb{N}$, then for an interpretation $\mathcal{I}^{k+1\to k}(X)(\vec{x}) \stackrel{df}{=} \mathcal{I}^{k+1}(\vec{x}, b)$ we obtain $\mathcal{I}^{k+1\to k}, D \models F$.*

(γ) *Let $\mathcal{I}, \mathcal{I}'$ be two n-dimensional interpretations, $[0,f]$ a one-dimensional interval and D, D' two $(n-1)$-dimensional intervals such that*

$$\mathcal{I}, D \times [j, j+1] \models F^{n-1} \wedge \ell_{\vec{e}_n} = 1 \ \text{ iff } \mathcal{I}', D' \times [j, j+1] \models F^{n-1} \wedge \ell_{\vec{e}_n} = 1$$

for all subformulae F^{n-1} of type $(n-1)$ occurring in F and all $j \in [0, f)$. Then $\mathcal{I}, D \times [0, f] \models F$ iff $\mathcal{I}', D' \times [0, f] \models F$

Proof

(α) Proof by structural induction.

Case $\lceil \pi \rceil$. From the assumption we deduce $\mathcal{I}[\![\pi]\!](\vec{x}) = 1$ for all $\vec{x} \in D$. By construction of $\mathcal{I}^{k\to k+1}$ we obtain for all $\vec{x'} \in D \times [b,f]$ that $\mathcal{I}^{k\to k+1}[\![\pi]\!](\vec{x'}) = 1$ and therefore $\mathcal{I}^{k\to k+1}, D \times [b,f] \models \lceil \pi \rceil$ as required.

Case $F_1 \wedge F_2, \neg F_1$. These cases are clear from the definition.

Case $F_1 \langle \vec{e}_i \rangle F_2, i \leqslant k$. By definition of chop there is an r such that

$$\mathcal{I}^k, D \prec_i r \models F_1 \ \text{ and } \mathcal{I}^k, D \succ_i r \models F_2.$$

Applying the induction hypothesis, yields

$$\mathcal{I}^{k\to k+1}, (D \prec_i r) \times [b,f] \models F_1 \ \text{ and } \mathcal{I}^{k\to k+1}, (D \succ_i r) \times [b,f] \models F_2.$$

Henceforth, $\mathcal{I}^{k\to k+1}, D \times [b,f] \models F_1 \langle \vec{e}_i \rangle F_2$.

(β) Similar to (α), note that $\mathcal{I}$ is assumed to be constant on $[n, n+1)$ for all $n \in \mathbb{N}$.

(γ) Proof by structural induction.

Case $F^{n-1} \wedge \ell_{\vec{e}_n} = 1$. This case is obvious from the assumptions.

Case $F_1^n \wedge F_2^n$.

$$\begin{array}{rrl} & & \mathcal{I}, D \times [0,f] \models F_1^n \wedge F_2^n \\ & \iff & \mathcal{I}, D \times [0,f] \models F_1^n \text{ and } \mathcal{I}, D \times [0,f] \models F_2^n \\ \{\text{ By (IH)}\} & \iff & \mathcal{I}', D' \times [0,f] \models F_1^n \text{ and } \mathcal{I}', D' \times [0,f] \models F_2^n \\ & \iff & \mathcal{I}', D' \times [0,f] \models F_1^n \wedge F_2^n \end{array}$$

Case $\neg F_1^n$. Follows directly from the definition and induction hypothesis.

Case $F_1^n \langle \vec{e}_n \rangle F_2^n$.

"**only if**" Assuming $\mathcal{I}, D \times [0,f] \models F_1^n \langle \vec{e}_n \rangle F_2^n$, there is an $m \in [0,f]$ such that $\mathcal{I}, D \times [0,m] \models F_1^n$ and $\mathcal{I}, D \times [m,f] \models F_2^n$. Let $\mathcal{I}^{\leftarrow m}$ and $\mathcal{I}'^{\leftarrow m}$ be the functions obtained from $\mathcal{I}$, respectively $\mathcal{I}'$ by left-shifting by m in the nth dimension. Then $\mathcal{I}^{\leftarrow m}, D \times [0, f-m] \models F_2^n$ and for all $j \in [0, f-m)$ and all subformulae F^{n-1}
$\mathcal{I}^{\leftarrow m}, D \times [j, j+1] \models F^{n-1} \wedge \ell_{\vec{e}_n} = 1$
iff
$\mathcal{I}'^{\leftarrow m}, D' \times [j, j+1] \models F^{n-1} \wedge \ell_{\vec{e}_n} = 1$
Applying the induction hypothesis on this yields
$\mathcal{I}'^{\leftarrow m}, D' \times [0, f-m] \models F_2^n$
and therefore $\mathcal{I}', D' \times [0,f] \models F_1^n \langle \vec{e}_n \rangle F_2^n$

"**if**" similar. □

Lemma 32. *Let F be an n-dimensional* SC_{nAlt} *formula.*

(α) *A subset* $\Pi \subseteq \Sigma_{nA}^n(F)$ *is consistent iff for every* $w = a_0 \ldots a_k \in \Pi^*$ *there is an n-dimensional interpretation* $\mathcal{I}$*, an* $(n-1)$*-dimensional interval D such that* $\mathcal{I}, D \times [j, j+1) \models G \iff G \in a_j$ *holds for every* $j \in [0,k]$ *and every* $G \in Sub^{n-1}(F)$.
(β) $\mathcal{I}, D \times [0,f] \models F$ *implies* $w^{\mathcal{I}} \in \mathcal{L}_{nA}^n(F)$.
(γ) $w \in \mathcal{L}_{nA}^n(F)$ *implies* $[w] \neq \varnothing$ *and* $\forall (\mathcal{I}, D) \in [w] : \mathcal{I}, D \times [0, |w|] \models F$

Proof. We simultaneously prove (α)-(γ) by induction on the structure of F.

Case $\delta(F) = 1$

"**only if**" (α) As all $\Pi \subset \Sigma_{nA}^1$ are consistent, let Π be be an arbitrary subset and $w = a_0 \ldots a_k \in \Pi^*$. The interpretation $\mathcal{I}$ given by

$$\mathcal{I}[\![X]\!](i) = \begin{cases} 1 \text{ if } X \in a_i \\ 0 \text{ otherwise} \end{cases}$$

has the required property.

"**if**" As every subset of Σ_{nA}^1 is consistent there is nothing to show.

(β, γ) These cases are clear from the classical construction for duration calculus.

Case $F^n \wedge \ell_{\vec{e}_{n+1}} = 1$

(α) The proof of (α) is essentially the same for all the other cases.

"only if" Let $\Pi^{n+1} \subseteq \Sigma_{nA}^{n+1}(F^n \wedge \ell_{\vec{e}_{n+1}} = 1)$ be a consistent alphabet and $w = a_0 \ldots a_k \in (\Pi^{n+1})^*$. By definition 30 there is a consistent alphabet Π^n and for each letter a_i, $0 \leqslant i \leqslant k$ there is a word $u_i = u_i(0)...u_i(l) \in \mathcal{L}(a_i) \cap (\Pi^n)^*$ such that all words have the same length, i.e., $|u_i| = l$. The concatenation of all $u_0 \ldots u_k$ is still a word in $(\Pi^n)^*$ and therefore the induction hypothesis is applicable yielding an n-dimensional interpretation $\mathcal{I}'$ and an $(n-1)$-dimensional interval D' such that for all subformulae G occurring in $u_1 \ldots u_k$ the condition $\mathcal{I}', D' \times [i \cdot l + j, i \cdot l + j + 1] \models G' \iff G' \in u_i(j)$ holds. The interpretation $\mathcal{I}''$ obtained from "folding" $\mathcal{I}'$ defined by $\mathcal{I}''(X)(\vec{x}, i, j) = \mathcal{I}'(X)(\vec{x}, i \cdot l + j)$ still satisfies

$$\mathcal{I}'', D' \times [i, i+1] \times [j, j+1] \models G' \iff G' \in u_i(j). \tag{A.4}$$

We show that this interpretation $\mathcal{I}''$ satisfies for all subformulae G of type n

$$\mathcal{I}'', D' \times [i, i+1] \times [0, i] \models G \iff G \in a_i. \tag{A.5}$$

Applying the induction hypothesis for (β) on each $u_i \in \mathcal{L}(a_i)$, we obtain interpretations and intervals $(\mathcal{I}_i, D_i)$ satisfying for all subformulae G' of type $n-1$

$$\mathcal{I}_i, D_i \times [j, j+1] \models G' \iff G' \in u_i(j).$$

and additionally all formulae given in a_i. Using Lemma 27 (γ), we then obtain that $\mathcal{I}''$ satisfies all formulae given in a_i, thus Eq. (A.5) and hence $\mathcal{I}''$ is the required interpretation.

"if" Let Π^{n+1} be an alphabet, $w = a_0 \ldots a_k \in (\Pi^{n+1})^*$ be a word, $\mathcal{I}$ be the interpretation and D the n-dimensional interval satisfying for all subformula G that $\mathcal{I}, D \times [j, j+1) \models G \iff G \in a_j$. Then for every j the restrictions given by $\mathcal{I}_j^{n+1 \to n}(X)(\vec{x}) = \mathcal{I}(\vec{x}, j)$ satisfy $\mathcal{I}_j^{n+1 \to n}, D \models G \iff G \in a_j$ due to Lemma 27 (β). Then all words $w^{\mathcal{I}_j^{n+1 \to n}}$ have the same length by construction and are in $\mathcal{L}(a_j)$ by the induction hypothesis (β). Defining Π' to be the common alphabet of all words $w^{\mathcal{I}^{n+1 \to n}(X)_j}$ and applying the induction hypothesis for part (α) yields consistency of Π' from which we can conclude the consistency of Π.

(β) Let $\mathcal{I}, D \times [0, f] \models F^n \wedge \ell_{\vec{e}_n+1} = 1$. From $\ell_{\vec{e}_{n+1}} = 1$ we obtain $f = 1$. Therefore $w^{\mathcal{I}} = a_0^{\mathcal{I}}$ and $F^n \in a_0^{\mathcal{I}}$ and $w^{\mathcal{I}} \in \mathcal{L}'(F^n \wedge \ell_{\vec{e}_{n+1}} = 1)$ due to definition 31. The consistency of $\Pi = \{a_0^{\mathcal{I}}\}$ is a consequence of (α).

(γ) Let $w \in \mathcal{L}_{nA}^{n+1}(F^n \wedge \ell_{\vec{e}_{n+1}}{=}1)$. By definition $w = a$ and $F^n \in a$ holds. As $\mathcal{L}^{n+1}(F^n \wedge \ell_{\vec{e}_{n+1}} = 1)$ is non-empty, $\{a\}$ is consistent and (α) yields an interpretation $\mathcal{I}^n$ and interval D such that $\mathcal{I}^n, D \models F^n$. Therefore $[w] \neq \varnothing$. Any interpretation $\mathcal{I}^n$ satisfying $\mathcal{I}^n, D \models F^n$ can be extended by Lemma 27 (α) to $\mathcal{I}^{n\to n+1}$ such that $\mathcal{I}^{n\to n+1}, D \times [0,1] \models F^n \wedge \ell_{\vec{e}_n} = 1$.

Case $F_1^{n+1} \wedge F_2^{n+1}$. ($\beta$) Let $\mathcal{I}, D \times [0, f] \models F_1^{n+1} \wedge F_2^{n+1}$. By definition of conjunction and the induction hypothesis, we obtain $w^{\mathcal{I}} \in \mathcal{L}_{nA}^{n+1}(F_1^{n+1})$ and $w^{\mathcal{I}} \in \mathcal{L}_{nA}^{n+1}(F_2^{n+1})$ and consistency. Therefore holds $w^{\mathcal{I}} \in \mathcal{L}_{nA}^{n+1}(F_1^{n+1} \wedge F_1^{n+1})$ as required.

(γ) Let $w \in \mathcal{L}_{nA}^{n+1}(F_1^{n+1} \wedge F_2^{n+1})$. By construction $w \in \mathcal{L}_{nA}^{n+1}(F_1^{n+1})$ and $w \in \mathcal{L}_{nA}^{n+1}(F_2^{n+1})$. Applying the induction hypothesis yields $[w] \neq \varnothing$ and for every interpretation $\mathcal{I}$ and interval D corresponding to w the relation $\mathcal{I}, D \times [0, |w|] \models F_1^{n+1} \wedge F_2^{n+1}$ holds as required.

Case $\neg F_1^{n+1}$. (β) Let $\mathcal{I}, D \times [0, f] \models \neg F_1^{n+1}$, so $\mathcal{I}, D \times [0, f] \not\models F_1^{n+1}$. Applying the induction hypothesis for (γ), we obtain $w^{\mathcal{I}} \notin \mathcal{L}_{nA}^{n+1}(F_1^{n+1})$ and therefore $w^{\mathcal{I}} \in \mathcal{L}'(\neg F_1^{n+1})$. By definition 31 the righthand side of (α) is satisfied yielding consistency. Hence, $w^{\mathcal{I}} \in \mathcal{L}_{nA}^{n+1}(\neg F_1^{n+1})$.

(γ) Let $w \in \mathcal{L}_{nA}^{n+1}(\neg F_1^{n+1})$. At first we show $[w] \neq \varnothing$. Assume $[w] = \varnothing$. Then for all $(\mathcal{I}, D)$ with $w^{\mathcal{I}} = w \mathcal{I}, D \times [0, |w|] \not\models \neg F_1$ and therefore $\mathcal{I}, D \times [0, |w|] \models F_1$. But then, applying the induction hypothesis yields $w \in \mathcal{L}_{nA}^{n+1}(F)$ contradicting $w \in \mathcal{L}_{nA}^{n+1}(\neg F_1^{n+1})$. Therefore $[w] \neq \varnothing$. The second proposition is a direct consequence of the induction hypothesis for (β).

Case $F_1^{n+1} \langle \vec{e}_{n+1} \rangle F_2^{n+1}$. ($\beta$) Let $\mathcal{I}, D \times [0, f] \models F_1^{n+1} \langle \vec{e}_{n+1} \rangle F_2^{n+1}$. By definition there is an $m \in [0, f]$ such that $\mathcal{I}, D \times [0, m] \models F_1^{n+1}$ and $\mathcal{I}, D \times [m, f] \models F_2^{n+1}$. The interpretation $\mathcal{I}'$ defined by left-shifting $\mathcal{I}$ by m along dimension $(n + 1)$ satisfies $\mathcal{I}', D \times [0, f - m] \models F_2^{n+1}$. Applying the induction hypothesis twice yields two words $w_1^{\mathcal{I}} \in \mathcal{L}(F_1^{n+1})$ and $w_2^{\mathcal{I}'} \in \mathcal{L}(F_2^{n+1})$ corresponding to $\mathcal{I}$ on $D \times [0, m]$ and $\mathcal{I}'$ on $D \times [0, f - m]$, respectively. By construction $w^{\mathcal{I}} = w_1^{\mathcal{I}} w_2^{\mathcal{I}'}$ holds and henceforth $w^{\mathcal{I}} \in \mathcal{L}'(F_1^{n+1} \langle \vec{e}_{n+1} \rangle F_2^{n+1})$. Consistency is a consequence of (α).

(γ) Let $w = a_0 \ldots a_{|w|-1} \in \mathcal{L}_{nA}^{n+1}(F_1^{n+1} \langle \vec{e}_{n+1} \rangle F_2^{n+1})$. By definition $w = w_1 w_2$ such that $w_1 \in \mathcal{L}_{nA}^{n+1}(F_1^{n+1})$ and $w_2 \in \mathcal{L}_{nA}^{n+1}(F_2^{n+1})$. The induction hypothesis yields two pairs $(\mathcal{I}_1, D_1)$ and $(\mathcal{I}_2, D_2)$ such that

$$\mathcal{I}_1, D_1 \times [0, |w_1|] \models F_1^{n+1} \text{ and} \tag{A.6}$$

$$\mathcal{I}_2, D_1 \times [0, |w_2|] \models F_2^{n+1} \tag{A.7}$$

From consistency of the alphabet and (α) we obtain an interpretation $\mathcal{I}'$ and n-dimensional interval D such that for all $j \in [0, |w| - 1]$ and for all subformulae F^n of type n holds $\mathcal{I}', D \times [j, j+1] \models F^n \iff F^n \in a_j$. By Definition 31

$$\mathcal{I}_1, D_1 \times [j, j+1] \models F^n \iff F^n \in a_j \text{ for } j \in [0, |w_1|) \text{ and} \tag{A.8}$$

$$\mathcal{I}_2, D_2 \times [j - |w_1|, j - |w_1| + 1] \models F^n \iff F^n \in a_j \text{ for } j \in [|w_1|, |w_1| + |w_2|). \tag{A.9}$$

Applying Lemma 27 (γ) on equations (A.6), (A.7),(A.8) and (A.10), we conclude $\mathcal{I}', D \times [0, |w_1|] \models F_1^{n+1}$ and $\mathcal{I}', D \times [|w_1|, |w_1| + |w_2|] \models F_2^{n+1}$. Therefore $\mathcal{I}', D \times [0, |w|] \models F_1^{n+1} \langle \vec{e}_{n+1} \rangle F_2^{n+1}$. The second part follows from Lemma 27 (γ). □

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