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Operations in a Fuzzy-Valued Logic*

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A logical calculus is developed with propositions taking their truth values in the set of fuzzy sets of [0, 1]. This fuzzy-valued logic is an extension of already known multivalent logics, and the associated set theory is shown to be that of fuzzy sets of type 2 on a given universe. Various interpretative functions are given for the usual connectives of propositional calculus, using extended "max" and "min" operators. Examples of inference are provided and a compositional rule for fuzzy-valued fuzzy relations is suggested. Computations of truth values for composite propositions are shown to be very easy. It is hoped that such a logic will be helpful in the modelization of approximate reasoning, in natural language.

INTRODUCTION

In the framework of approximate reasoning, truth values of the propositions are rarely precisely known; binary logic is not well fitted to approximate reasoning because propositions are not always of black or white type, and even multivalent logicseems insufficient because truth values are linguistic rather than quantitative, as Zadeh (1978a) has already pointed out. Fuzzy-set theory seems particularly appropriate for modeling linguistic truth values considered as fuzzy sets of [0, 1] with such names as "true," "very true," ..., "false." The aim of a fuzzy-valued logic can be a more accurate representation of human thinking through natural language (Zadeh, 1978b).

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The first problem with such a logic is to define adequate interpretative functions for the usual connectives of propositional calculus; of course, those will be fuzzy functions, i.e., mappings from fuzzy subsets of [0, 1]. The set theory associated with the fuzzy-valued logic considered will be shown to be the theory of fuzzy sets of type 2 (Mizumoto and Tanaka, 1976).

The second problem is that of making easy inferences in such a logic; i.e., the computation of the values of interpretative functions must be practically tractable.

In the first section of this paper, we recall the two structures of the fuzzy subsets of [0, 1]. Section 2 provides a discussion of the different manipulations of fuzzy sets of type 2. Section 3 presents the interpretative functions in a fuzzy-valued logic and introduces a composition rule of inference.

The concept of a fuzzy-valued logic was introduced in Bellman and Zadeh (1977).

I. Fuzzy Subsets of [0, 1]: $\tilde{P}([0, 1])$

1. *Definitions*

A fuzzy subset A of [0, 1] is a mapping μ_A from [0, 1] to [0, 1]; i.e., a fuzzy subset of the real line whose support is a part of [0, 1]. A convex normalized fuzzy subset of [0, 1] will be called a fuzzy number of [0, 1].

(convex: $\forall (x, y, z) \in [0, 1]^3$, $x \leqslant y \leqslant z \rightarrow \mu_A(y) \geqslant \min(\mu_A(x), \mu_A(z))$;
normal: $\exists x \in [0, 1]$, $\mu_A(x) = 1$).

2. *First Structure of Lattice for the Fuzzy Subsets of* [0, 1]

[0, 1] is a pseudocomplemented distributive lattice for max and min operations; the pseudocomplementation is the complementation to 1. Thus, the mappings in [0, 1] have the same structure and in particular the fuzzy sets of [0, 1]. In this lattice "inf" = "intersection of fuzzy sets" and "sup" = "union of fuzzy

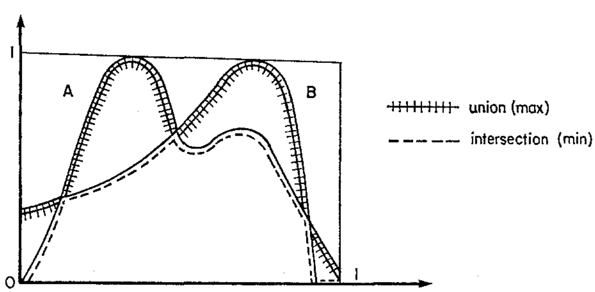

FIGURE 1

sets." (See Fig. 1.) The greatest and smallest elements in this lattice are respectively $\mathbb{1} = [0, 1]$ and $\mathbf{0} = \varnothing$.

3. *Second Structure for the Fuzzy Subsets of* [0, 1]

(a) *Definitions*

Max and min operations in [0, 1] can be extended to act on fuzzy subsets of [0, 1] using the extension principle (Zadeh, 1975); we can define $\widetilde{\max}(A, B)$ and $\widetilde{\min}(A, B)$ by their membership functions:

$$\mu_{\widetilde{\max}(A,B)}(z) = \sup_{z=\max(x,y)} \min(\mu_A(x), \mu_B(y)),$$

$$\mu_{\widetilde{\min}(A,B)}(z) = \sup_{z=\min(x,y)} \min(\mu_A(x), \mu_B(y)),$$

$$\widetilde{\max}(A; B) \in \tilde{P}[0, 1] \qquad \text{and} \qquad \widetilde{\min}(A, B) \in \tilde{P}[0, 1].$$

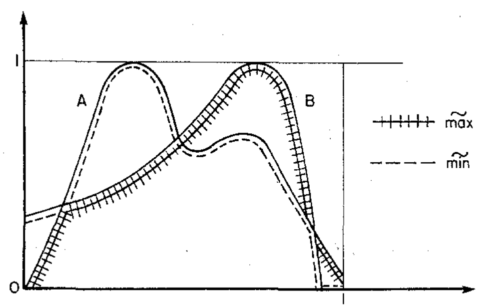

FIGURE 2

The practical rule for construction of $\widetilde{\max}(A, B)$ and of $\widetilde{\min}(A, B)$ is the following: For two convex fuzzy sets, take the leftmost part for the $\widetilde{\min}$ and the rightmost part for the $\widetilde{\max}$ of the pair of increasing parts of A and B and do the same for decreasing parts. For nonconvex fuzzy subsets, first decompose them into union of convex fuzzy subsets, use the above rule on each pair of convex fuzzy subsets, one from A and one from B, and last do the union of the elementary results (see Fig. 2. and Dubois and Prade, 1978, 1979).

(b) *The Structure of* $\tilde{P}([0, 1])$ *with* $\widetilde{\max}$ *and* $\widetilde{\min}$

Mizumoto and Tanaka have recently shown the following results:

1. $\tilde{P}([0, 1])$ is a quasi lattice (Plonka 1967) for $\widetilde{\max}$ and $\widetilde{\min}$. More specifically:

—$\widetilde{\max}$ and $\widetilde{\min}$ are commutative and associative;

—$\widetilde{\max}(A, A) = A$; $\widetilde{\min}(A, A) = A$ (idempotence).

But $\widetilde{\max}\,(A, \widetilde{\min}(A, B)) \neq A$; $\widetilde{\min}(A, \widetilde{\max}(A, B)) \neq A$ (absorption laws do not hold).

Moreover: $1 \ominus \widetilde{\max}(A, B) = \widetilde{\min}(1 \ominus A,\ 1 \ominus B)$; $1 \ominus \widetilde{\min}(A, B) = \widetilde{\max}$ $(1 \ominus A, 1 \ominus B)$ (De Morgan laws), where $\ominus$ denotes the extended subtraction (Dubois and Prade, 1978), $1 \ominus A$ is the fuzzy subset of [0, 1] whose membership function is $\mu_{1 \ominus A}(x) = \mu_A(1 - x)$, and $1 \ominus A$ is the antonym of A, $1 \ominus A = \text{ant}(A)$ (Zadeh, 1978b). $1 \ominus A$ is only a pseudocomplement for A, because, generally, $\widetilde{\min}(A, 1 \ominus A) \neq 0$ and $\widetilde{\max}(A, 1 \ominus A) \neq 1$.

2. The set of fuzzy numbers of [0, 1], denoted $\tilde{N}([0, 1])$, is a pseudo-complemented distributive lattice; absorption laws hold in $\tilde{N}([0, 1])$. Moreover:

$$\mathbf{0} = 0 \qquad \text{and} \qquad \mathbb{1} = 1;$$

i.e., the lattice is complete.

The preceding results were proved by Mizumoto and Tanaka (1976). However, a property satisfied by max and min on [0, 1] no longer holds:

$$\widetilde{\max}[\widetilde{\min}(A, B), \widetilde{\min}(1 \ominus A, 1 \ominus B)] \neq \widetilde{\min}[\widetilde{\max}(A, 1 \ominus B), \widetilde{\max}(1 \ominus A, B)]$$

and

$$\widetilde{\min}[\widetilde{\max}(A, B), \widetilde{\max}(1 \ominus A, 1 \ominus B)] \neq \widetilde{\max}[\widetilde{\min}(A, 1 \ominus B), \widetilde{\min}(1 \ominus A, B)].$$

Using the distributivity, associativity, and commutativity properties on the left term of the first expression, we obtain

$$\widetilde{\min}[\widetilde{\max}(A, 1 \ominus B), \widetilde{\max}(1 \ominus A, B), \widetilde{\max}(1 \ominus A, A), \widetilde{\max}(1 \ominus B, B)],$$

which is generally different from the right term of the first expression; however, the equality holds as soon as $\max(A, 1 \ominus B)$ is equal to either $1 \ominus B$ or A,[1] using absorption. The second equality holds for the same conditions where $\widetilde{\max}$ is replaced by $\widetilde{\min}$.

N.B. We can think of extending the max and min operations on [0, 1] to $\tilde{P}$ ([0, 1]) using a "inf–max" composition. The membership function of what can be denoted $\underset{\sim}{\max}(A, B)$, the extended max by "inf–max" composition, is

$$\begin{aligned}\mu_{\underset{\sim}{\max}(A,B)}(z) &= \inf_{z=\max(x,y)} \max(\mu_A(x), \mu_B(y)) \\ &= \min[\inf_{x \leqslant z} \max(\mu_A(x), \mu_B(z)),\ \inf_{y \leqslant z} \max(\mu_A(z), \mu_B(y))]\end{aligned}$$

[1] and, of course, the same condition by changing A into B.

as soon as $\exists x \leqslant z$, $\mu_A(x) \leqslant \mu_B(z)$ and $\exists y \leqslant z$, $\mu_B(y) \leqslant \mu_A(z)$ (which is realized for fuzzy subsets of [0, 1] such that $\mu_A(0) = \mu_B(0) = 0$) we obtain $\underset{\sim}{\max}(A, B) = A \cap B$.

It is easy to check that $\underset{\sim}{\min}(A, B) = A \cap B$, where $\underset{\sim}{\min}$ is the extended min by "inf–max composition," as soon as $\mu_A(1) = \mu_B(1) = 0$.

Since $\underset{\sim}{\max}(A, B) = \underset{\sim}{\min}(A, B) = A \cap B$ in most cases, it is impossible to obtain a lattice structure with these two operators; "inf–max" composition can be given up.

II. From Usual Fuzzy sets to Fuzzy Sets of Type 2

Let U be a universe.

1. *Extensions of Classical Set Theory*

A classical set A is usually understood as a mapping from $U \to \{0, 1\}$. This mapping is called the characteristic function of A. The set of subsets of U, $P(U)$ has a structure of Boolean lattice, as $\{0, 1\}$. Fuzzy set theory can be built by extending the values of the characteristic function to [0, 1]. Then, union and intersection of A and B are defined as $\{(\max(\mu_A(x), \mu_B(x)), x)\}$ and $\{(\min(\mu_A(x), \mu_B(x)), x)\}$, respectively, and denoted $A \cup B$ and $A \cap B$. The complementation $\{(1 - \mu_A(x), x)\}$ is denoted $\bar{A}$.

Another extension of classical set theory can be obtained by replacing $\{0, 1\}$ by $P(\{0, 1\})$; thus $\mu_A(x)$ is a classical set of $\{0, 1\}$. A is said to be a classical set of type 2. The union, the intersection, and the complementation of such sets can be defined using the union and the intersection in $P(\{0, 1\})$, denoted by $\mathbb{V}$ and $\mathbb{A}$, respectively.

$\mu_{A \mathbb{V} B}(x)(x) = \mu_A(x) \vee \mu_B(x)$, where "$\vee$" denotes the union in $P(\{0, 1\})$.

$\mu_{A \mathbb{A} B}(x) = \mu_A(x) \wedge \mu_B(x)$, where "$\wedge$" denotes the intersection in $P(\{0, 1\})$.

$\mu_{\neg A}(x) = \neg\mu_A(x) = \{0, 1\} \backslash \mu_A(x)$, where "$\neg$" denotes the complementation in $P(\{0, 1\})$.

Since $P(\{0, 1\})$ has a structure of Boolean lattice with $\vee$ and $\wedge$, the set $P_2(U)$ of classical subsets of type 2 of U is also a Boolean lattice with $\mathbb{V}$ and $\mathbb{A}$ whereas the set $\tilde{P}(U)$ of fuzzy subsets of U has a structure of pseudo-complemented distributive lattice (with $\cup$ and $\cap$) owing to that of [0, 1]. Note that in $P(U)$, $\mathbb{V}$, $\mathbb{A}$, and $\neg$ coincide with $\cup$, $\cap$ and $-$; i.e., these operators coincide when applied to classical subsets of U, and yield back the usual union, intersection, and complementation of classical subsets. Last, $P_2(U)$ and $\tilde{P}(U)$ both include $P(U)$, but $P_2(U) \backslash P(U)$ and $\tilde{P}(U) \backslash P(U)$ are disjoint sets.

2. *Fuzzy Subsets of Type* 2

Fuzzy subsets of type 2 can be obtained as an extension of fuzzy subsets or of classical subsets of type 2. First, we extend [0, 1] in $[0, 1]^{[0,1]}$, max in $\widetilde{\max}$, min in $\widetilde{\min}$, "1 −" in "1 ⊖," in the sense of Zadeh's principle (Section I, 3a). The operators of union, intersection, and complementation of fuzzy sets of type 2 can then be defined as

$$\tilde{\mu}_{A \sqcup B}(x) = \widetilde{\max}(\tilde{\mu}_A(x), \tilde{\mu}_B(x)),$$

$$\tilde{\mu}_{A \sqcap B}(x) = \widetilde{\min}(\tilde{\mu}_A(x), \tilde{\mu}_B(x)),$$

$$\tilde{\mu}_{\overline{A}}(x) = 1 \ominus \tilde{\mu}_A(x).$$

Here, membership values are fuzzy subsets of [0, 1], which is why we place "$\sim$" above them.

Second, we can extend $P(\{0, 1\}) = \{0, 1\}^{\{0,1\}}$ in $[0, 1]^{[0,1]} = \tilde{P}([0, 1])$. "$\vee$", "$\wedge$", "$\neg$" in $P(\{0, 1\})$ become the fuzzy union $\cup$, intersection $\cap$, complementation "$-$" in $\tilde{P}([0, 1])$. The latter operators can be used to define a union, an intersection and complementation on $\tilde{P}_2(U)$, the set of fuzzy subsets of type two of $\cup$.

$$\tilde{\mu}_{A \mathbin{\vee\!\!\!\vee} B}(x) = \tilde{\mu}_A(x) \cup \tilde{\mu}_B(x)$$

$$\tilde{\mu}_{A \mathbin{\wedge\!\!\!\wedge} B}(x) = \tilde{\mu}_A(x) \cap \tilde{\mu}_B(x)$$

$$\tilde{\mu}_{\neg A}(x) = [\overline{\tilde{\mu}_A(x)}]$$

$\tilde{P}_2(U)$ with $\mathbin{\vee\!\!\!\vee}$, $\mathbin{\wedge\!\!\!\wedge}$, $\neg$ is a pseudocomplemented distributive lattice because $\tilde{P}([0, 1])$ with $\cup$, $\cap$, $-$, is such. $\tilde{P}_2(U)$ with $\sqcup$, $\sqcap$, $\overline{}$ is only a quasilattice because $\tilde{P}([0, 1])$ with $\widetilde{\max}$, $\widetilde{\min}$, $1 \ominus$, is such (see Section I, 2b). However, since the set of fuzzy numbers $\tilde{N}([0, 1])$ with $\widetilde{\max}$, $\widetilde{\min}$, $1 \ominus$, is a pseudocomplemented distributive lattice, such is the subset of $\tilde{P}_2(U)$ made of fuzzy subsets of type 2 of U whose characteristic functions map in $\tilde{N}([0, 1])$. Note that the latter fuzzy subsets are the most interesting from an interpretative point of view since the fuzzy membership value of x means "approximately $\mu_A(x)$." From now on, they will be called FNVS (fuzzy number-valued subset). The combination of FNVS's using $\sqcup$, $\sqcap$, $\overline{}$ yields FNVS's. This not true for $\mathbin{\vee\!\!\!\vee}$, $\mathbin{\wedge\!\!\!\wedge}$, $\neg$. Hence in the manipulation of FNVS's, the operators $\sqcup$, $\sqcap$, $\overline{}$ are more convenient; they canonically extend the fuzzy union, intersection, and complementation. $\mathbin{\vee\!\!\!\vee}$, $\mathbin{\wedge\!\!\!\wedge}$, $\neg$ have no evident interpretation; neither has $P_2(U)$ from which they derive.

3. *Subset Inclusion*

Set inclusion in the sense of $\sqcup$, $\sqcap$ can be defined as

$$A \sqsubset B \leftrightarrow \begin{cases} A \sqcup B = B, \\ A \sqcap B = A, \end{cases}$$

which gives for the membership values:

$$\forall x \in U, \qquad \widetilde{\min}[\tilde{\mu}_A(x), \tilde{\mu}_B(x)] = \tilde{\mu}_A(x)$$

and

$$\widetilde{\max}[\tilde{\mu}_A(x), \tilde{\mu}_B(x)] = \tilde{\mu}_B(x),$$

and subset equality is defined as $\forall x\ \tilde{\mu}_A(x) = \tilde{\mu}_B(x)$.

However, this definition may seem too rigid and we can relax it using approximate inequality and equality between $\tilde{\mu}_A(x)$ and $\tilde{\mu}_B(x)$ (Dubois and Prade, 1980).

The possibility of having $\tilde{\mu}_A(x)$ greater than $\tilde{\mu}_B(x)$ is

$$v(\tilde{\mu}_A(x) \geqslant \tilde{\mu}_B(x)) = \max_{\substack{u \geqslant w \\ (u,w) \in [0,1]^2}} \min[\mu_{\tilde{\mu}_A(x)}(u), \mu_{\tilde{\mu}_B(x)}(w)],$$

where $\mu_{\tilde{\mu}_A(x)}$ is the membership function of $\tilde{\mu}_A(x)$. When A and B are FNVSs, call mean value of $\tilde{\mu}_A(x)$ the real number $\mu_A(x)$ such that $\mu_{\tilde{\mu}_A(x)}(\mu_A(x)) = 1$, then

$$\begin{aligned} v(\tilde{\mu}_A(x) \geqslant \tilde{\mu}_B(x)) &= 1 \qquad \text{if} \quad \mu_A(x) \geqslant \mu_B(x) \\ &= \text{height}\ (\tilde{\mu}_A(x) \cap \tilde{\mu}_B(x)) \qquad \text{if} \quad \mu_B(x) \geqslant \mu_A(x), \end{aligned}$$

also called consistency by Zadeh (1978). We can decide that $\tilde{\mu}_A(x)$ is greater than $\tilde{\mu}_B(x)$ as soon as the consistency is under a given threshold. If it is true for any x, then B is approximately included in A.

Note that we could have defined the inclusion consistent with $\bigvee\!\!\!\!\bigvee$ and $\bigwedge\!\!\!\!\bigwedge$ as $A \Subset B \leftrightarrow \tilde{\mu}_A(x) \subset \tilde{\mu}_B(x)\ \forall x \in U$. However, when A and B are FNVSs and if we consider only the fuzzy subsets of U : $\mathring{A} = \{(\mu_A(x), x)\}$ and $\mathring{B} = \{(\mu_B(x), x)\}$ where $\mu_A(x)$ and $\mu_B(x)$ are the mean values of $\tilde{\mu}_A(x)$ and $\tilde{\mu}_B(x)$, then if mean values are unique,

$$A \sqsubset B \text{ reduces to } \mathring{A} \subset \mathring{B}$$

and

$$A \Subset B \text{ reduces to } \mathring{A} = \mathring{B},$$

so that $\Subset$ is even more rigid than $\sqsubset$.

III. Fuzzy-Valued Logic

1. *Truth Values of a Proposition*

In a fuzzy-valued logic, the truth value of a proposition P will be a fuzzy set of [0, 1]; here only fuzzy numbers will be considered. Predicates are related to fuzzy sets of type 2, since the proposition "x is F" is valued by a fuzzy number.

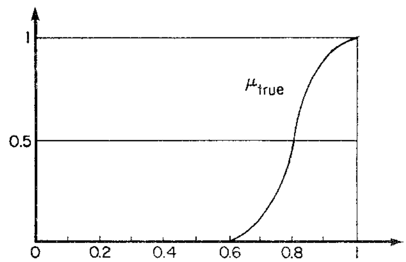

Figure 3

Figure 3 shows the truth value "true," considered by Zadeh (1978a).

$$\mu_{\text{true}} = S(0.6, 0.8, 1.0),$$

where

$$\begin{aligned} S(\alpha, \beta, \gamma; u) &= 0; & u &\leqslant \alpha, \\ &= 2\left[\frac{u-\alpha}{\gamma-\alpha}\right]^2; & \alpha &\leqslant u \leqslant \beta = \frac{\alpha+\gamma}{2}, \\ &= 1 - 2\left[\frac{u-\gamma}{\gamma-\alpha}\right]^2; & \beta &\leqslant u \leqslant \gamma, \\ &= 1; & u &\geqslant \gamma. \end{aligned}$$

We write $v(P)$ = "true" and more generally $v(P) = \tau$.

2. *Interpretative Functions for Connectives*

To define the interpretative functions of a fuzzy valued logic, we extend, according to Zadeh's principle, the interpretative functions of two multivalent logics, which mainly differ by the connectives related to the implication.

In Dubois and Prade (1977), these logical calculi are presented; they can be called Zadeh's multivalent logic which underlies fuzzy-set theory, and Dienes Rescher logic, which is the fuzzification of binary propositional calculus, according to Zadeh's extension principle (see Gaines, 1976). The first one is nothing but Luckaziewicz's (1930) $L_{\aleph_1}$.

(a) *Interpretative Functions Belonging to Both Logics*

· Negation:

$$v(\overline{P}) = 1 \ominus v(P) = \operatorname{ant}[v(P)].$$

· And:

$$v(P \sqcap Q) = \widetilde{\min}(v(P), v(Q)).$$

· Or:

$$v(P \sqcup Q) = \widetilde{\max}(v(P), v(Q)).$$

· " | " (Shaffer):

$$v(P \mid Q) = \widetilde{\max}(1 \ominus v(P), 1 \ominus v(Q)) = \operatorname{ant}[v(P \sqcap Q)].$$

· " ↓ " (Pierce):

$$v(P \downarrow Q) = \widetilde{\min}(1 \ominus v(P), 1 \ominus v(Q)) = \operatorname{ant}[v(P \sqcup Q)].$$

EXAMPLES. Consider the linguistic truth values: "true" with $\mu_{\text{true}} = S(\alpha, (\alpha + 1)/2, 1)$, $\alpha \in [0, 1]$, "false" = ant("true"), "dubious" with $\mu_{\text{dubious}} = S(\beta, (\beta + 0.5)/2, 0.5)$, on [0, 0.5] and $\mu_{\text{dubious}} = \operatorname{ant}(S(\beta, (\beta + 0.5)/2, 0.5))$ in [0.5, 1], $\beta \in [0, 0.5]$, see Fig. 4.

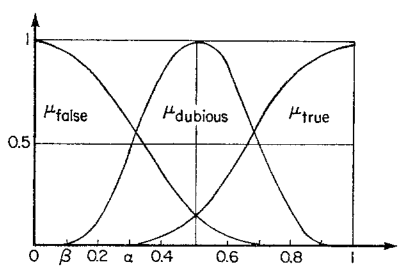

FIGURE 4

Using modifiers such as "very" or "more or less," we can define "very true" ($\mu_{\text{very true}} = S^2(\alpha, (\alpha + 1)/2, 1)$), "more or less true" ($\mu_{\text{more or less true}} = S^{1/2}(\alpha, (\alpha + 1)/2, 1)$), and so on.

· P is "true," Q is "dubious": "P and Q" is $\widetilde{\min}$ (true, dubious) $\simeq$ "dubious."

· P is "more or less dubious," Q is "very true": "P or Q" is $\widetilde{\max}$ (more or less dubious, very true) $\simeq$ "very true."

These equalities hold strictly when $\beta \leqslant \alpha$.

· P is dubious: $\overline{P}$ is ant(dubious) = dubious.

Note that "$\overline{P}$ is dubious" does not have the same meaning as "P is not dubious" —which means "P is either true or false."

N.B. The meaning of "not true and not false" is approximately that of "dubious." Indeed $\mu_{\text{not true and not false}} = \min(1 - \mu_{\text{true}}, 1 - \mu_{\text{false}})$ has a maximum ($\neq 1$) for $u = 0.5$ as μ_{dubious} · $\cup$, $\cap$, —, can be used in $\tilde{P}([0, 1])$ to build new linguistic values (provided some normalization); on the contrary, $\widetilde{\max}$, $\widetilde{\min}$, $1 \ominus$ must be used to evaluate composite fuzzy propositions.

(b) *Dienes–Rescher Connectives*

· Implication:

$$v(P \Rightarrow Q) = \widetilde{\max}(1 \ominus v(P), v(Q)).$$

· Equivalence:

$$v(P \Leftrightarrow Q) = \widetilde{\min}(\widetilde{\max}(1 \ominus v(P), v(Q)), \widetilde{\max}(1 \ominus v(Q), v(P))).$$

· Exclusive or:

$$v(P \text{ ex } Q) = \widetilde{\max}(\widetilde{\min}(1 \ominus v(P), v(Q)), \widetilde{\min}(1 \ominus v(Q), v(P))).$$

N.B. Note that the properties stated in Section I, 3, b, which no longer hold in $\tilde{P}([0, 1])$, produce alternative definitions for $\Leftrightarrow$ and "ex". However, these expressions are generally approximately equal.

· Tautology:

$$v(\dot{P}) = \widetilde{\max}(v(P), 1 \ominus v(P)).$$

· Contradiction:

$$v(\mathring{P}) = \widetilde{\min}(v(P), 1 \ominus v(P)).$$

Those two definitions are consistent with

$$v(P \Rightarrow \mathring{P}) = \overline{P}; \qquad v(\dot{P} \Rightarrow P) = v(P).$$

EXAMPLE

P is dubious, Q is true: "$P \Rightarrow Q$" is $\widetilde{\max}$(ant(dubious), true) $\simeq$ true

"$Q \Rightarrow P$": $\widetilde{\max}$(ant(true), dubious) $\simeq$ dubious

"$P \Leftrightarrow Q$" is dubious

(c) *Zadeh's Connectives*

· Implication:

$$v(P \to Q) = \widetilde{\min}(1, 1 \ominus v(P) \oplus v(Q)),$$

where $\ominus$, $\oplus$ denote respectively extended subtraction and extended addition (Dubois and Prade, 1978)[2]. This expression is the extension of $\min(1, 1 - p + q)$ $(p, q) \in [0, 1]^2$ (Luckaziewicz, 1930) to fuzzy arguments.

· Equivalence:

$$\begin{aligned} v(P \leftrightarrow Q) &= \widetilde{\min}(\widetilde{\min}(1, 1 \ominus v(P) \oplus v(Q)), \widetilde{\min}(1, 1 \ominus v(Q) \oplus v(P))) \\ &= 1 \ominus \widetilde{\max}(0, v(Q) \ominus v(P), v(P) \ominus v(Q)) \\ &= 1 \ominus | v(P) \ominus v(Q) |. \end{aligned}$$

· Exclusive or:

$$v(P \text{ ex } Q) = v(\overline{P \leftrightarrow Q}) = | v(P) \ominus v(Q) |.$$

· Tautology—contradiction:

$$v(\dot{P}) = 1; \quad v(\mathring{P}) = 0.$$

The following formulas still hold:

$$\begin{aligned} v(\overline{P}) &= v(P \to \mathring{P}), \\ v(P) &= v(\dot{P} \to P). \end{aligned}$$

All the preceding expressions yield elements of $\tilde{N}([0, 1])$, whenever $(v(P), v(Q)) \in \tilde{N}^2([0, 1])$. The absolute value of a fuzzy number A is defined as

$$\begin{aligned} \mu_{|A|}(x) &= \max(\mu_A(x), \mu_A(-x)), \qquad x \geqslant 0, \\ &= 0, \qquad x < 0. \end{aligned}$$

Example

P is dubious, Q is true:

$$\begin{aligned} v(P \to Q) &= \widetilde{\min}(1, \text{ant(dubious)} \oplus \text{true}) \\ &= \widetilde{\min}(1, \text{dubious} \oplus \text{true}). \end{aligned}$$

As soon as $\alpha + \beta \geqslant 1$ $v(P \to Q) = 1$; however, generally, $v(P \to Q) \subset$ true, which is the result of Dienes–Rescher implication.

[2] $A \oplus B$ and $A \ominus B$ are defined, for any A, B in $\tilde{P}(\mathbb{R})$, by $\mu_{A \perp B}(z) = \sup_{x,y} \min(\mu_A(x), \mu_B(y))$ under the constraint $x + y = z$ for $\perp = \oplus$ and $x - y = z$ for $\perp = \ominus$.

3. *Relation between Extended Zadeh's Logic and Fuzzy Sets of Type* 2

Fuzzy subsets of type 2 with $\sqcup$, $\sqcap$, $\urcorner$ corresponds to both of the above extended logics. However, if we want equivalently to relate Zadeh's implication $\min(1, 1 \ominus v(P) \oplus v(Q))$ with inclusion "$\sqsubset$," we must restrict this inclusion in the following way:

$$A \sqsubset B \text{ iff } \begin{cases} A \wedge B = \varnothing \\ A \sqcap B = A \end{cases} \qquad \text{equivalent to } \begin{cases} A \wedge B = \varnothing \\ A \sqcup B = B \end{cases}$$

which means, respectively:

$$\begin{cases} \mu_A \cap \mu_B = 0, \\ \underset{\sim}{\min}(\mu_A, \mu_B) = \mu_A, \end{cases} \qquad \text{and} \qquad \begin{cases} \mu_A \cap \mu_B = 0, \\ \underset{\sim}{\max}(\mu_A, \mu_B) = \mu_B . \end{cases}$$

Then

$$v(P \to Q) = 1 \text{ is equivalent to } A \sqsubset B,$$

where P means "X is A," Q means "X is B," and $X \in U$. With the definition of Section II, 3 for $\sqsubseteq$, we have only

$$v(P \to Q) = 1 \text{ implies } A \sqsubseteq B.$$

In the example of Fig. 5, $\mu_A \cap \mu_B = 0$ as soon as $\alpha + \beta \geqslant 1$.

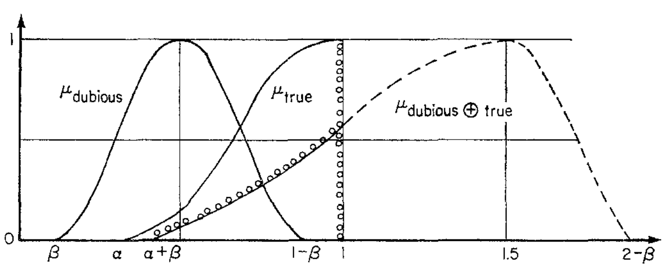

FIG. 5. ∘∘∘, $v(P \to Q)$.

CONCLUSION: INFERENCES IN A FUZZY-VALUED LOGIC

In a multivalued logic, the problem of inference from "more or less true" premises can be stated as:

Given $v(P) = \tau$ and $v(P \to Q) = 1 - \Sigma$, where τ and $\Sigma \in [0, 1]$, find $v(Q) \in [0, 1]$.

Using the logic underlying fuzzy-set theory we may define

$$v(P \to Q) = \min(1, 1 - v(P) + v(Q)).$$

This is Luckasiewicz's implication, which ensures

$$v(P \to Q) = 1 \text{ is equivalent to } v(Q) \geqslant v(P).$$

In the inference problem, $v(Q)$ is the solution of

$$\min(1, 1 - \tau + v(Q)) = 1 - \Sigma.$$

That is to say,

$$\text{For } \Sigma = 0\text{: } v(Q) \in [\tau, 1],$$
$$\text{For } \Sigma > 0 \text{ and } \tau \geqslant \Sigma\text{: } v(Q) = \tau - \Sigma.$$

$\tau \geqslant \Sigma$ is a necessary condition for the problem to make sense. It means that "P cannot be more false than $P \to Q$." When $\tau < \Sigma$, $v(Q)$ cannot be calculated.

In a fuzzy-valued logic, the problem of inferring fuzzy-valued propositions from fuzzy premises can be stated as:

Given $v(P) = \tau$ and $v(P \to Q) = 1 \ominus \Sigma$, where $(\tau, \Sigma) \in \tilde{N}^2([0, 1])$, the problem is to find $v(Q) \in \tilde{N}([0, 1])$. As is done in the nonfuzzy case, we could think of finding $v(Q)$ as a solution of the equation

$$1 \ominus \Sigma = \widetilde{\min}(1, 1 \ominus \tau \oplus v(Q)), \tag{1}$$

which implicitly defines $v(Q)$.

However, in fuzzy equations implicit definitions of variables are not equivalent to the corresponding explicit definitions, which are usually the only valid ones. As a matter of fact, from the above equation, for $\Sigma \neq 0$, we see that the more precise τ is, the more vague $v(Q)$ is, and conversely, which contradicts Zadeh's inference rule (1978a) according to which from a fuzzy proposition P we can infer P' if the possibility distribution induced by P is contained in the possibility distribution induced by P'. For example, let $\tau =$ "very true"; assume we find $v(Q) =$ "true" for an appropriate value of $1 \ominus \Sigma$; now let $\tau =$ "true": by using Eq. (1), $v(Q)$ can no longer be "true," but is "very true," which contradicts Zadeh's inference rule. To be consistent with it, we must directly fuzzify the nonfuzzy result of the above equation.

In order to justify this result, we must consider Eq. (1) as an extended equation in the sense of Zadeh's extension principle, i.e., Equation (1) implicitly defines $v(Q)$ by

$$\mu_{v(Q)}(z) = \sup_{x,t} \min(\mu_\tau(t), \mu_\Sigma(x)) \qquad \forall z \in [0, 1]$$

under the constraint: $1 - x = \min(1, 1 - t + z)$. This constraint is equivalent to

$$z = t - x \qquad \text{for } x > 0 \qquad (t \geqslant x \text{ because } z \geqslant 0)$$

and

$$t \leqslant z \qquad \text{for } x = 0.$$

Hence

$$\mu_{v(Q)}(z) = \max(\sup_{\substack{t-x=z \\ x>0}} \min(\mu_\tau(t), \mu_\Sigma(x)), \sup_{t \leqslant z} \min(\mu_\tau(t), \mu_\Sigma(0))).$$

Denoting by $\bar{t}$ the most possible value of $\tau(\mu_\tau(\bar{t}) = 1)$ and by $t^* \leqslant \bar{t}$ the value of τ such that

$$\mu_\tau(t^*) = \mu_\Sigma(0),$$

we infer that

$$\begin{aligned} \mu_{v(Q)}(z) &= \max(\mu_{\tau \ominus \Sigma}(z), \mu_\tau(z)) \qquad \text{for} \quad z \in [0, t^*] \\ &= \max(\mu_{\tau \ominus \Sigma}(z), \mu_\Sigma(0)) \qquad \text{for} \quad z \in [t^*, 1] \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

(see Fig. 6).

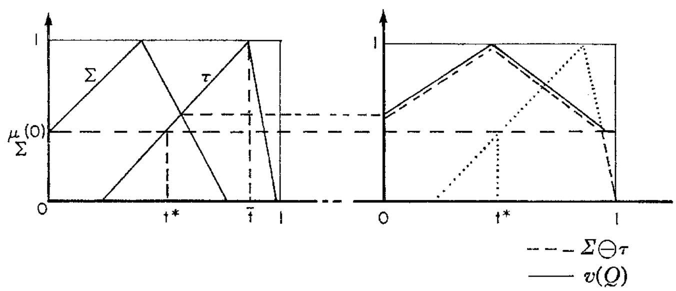

FIG. 6. ---, $\tau \ominus \Sigma$; ——, $v(Q)$.

If $\Sigma = 0$, then

$$\begin{aligned} t^* = \bar{t}, \text{ and } \mu_{v(Q)}(z) &= \mu_\tau(z) \qquad \text{for} \quad z \leqslant \bar{t} \\ &= 1 \qquad \text{for} \quad z \geqslant \bar{t}, \end{aligned}$$

i.e., $v(Q)$ is the fuzzy interval $(\tau, 1]$.

If $\mu_\Sigma(0) = 0$, i.e., $\mu_{v(P \to Q)}(1) = 0$, then

$$\mu_{v(Q)} = \mu_{\tau \ominus \Sigma}(z) \qquad \text{for} \quad z \in [0, 1];$$

in other words,

$$v(Q) = \widetilde{\max}(0, \tau \ominus \Sigma).$$

The possibility of calculating $v(Q)$ is evaluated by

$$p = \sup_{z \in [0,1]} \mu_{\tau \ominus \Sigma}(z).$$

If $p = 0$, the inference problem is impossible.

Actually, owing to Zadeh's extension principle, the results of the multivalued logic case have been generalized.

Another statement of this problem can be:

$$v(P) = \tau.$$

If

$$v(P) = \alpha \text{ then } v(Q) = \beta; \ (\tau, \alpha, \beta) \in \tilde{N}^3([0, 1]).$$

Is it possible to deduce $v(Q)$?

If we assume that $P \rightarrow Q$ is a "fuzzy theorem," i.e., its truth value is constant, we can deduce $v(Q)$ by solving the equation

$$\min(1, 1 \ominus \tau \oplus v(Q)) = v(P \rightarrow Q) = \min(1, 1 \ominus \alpha \oplus \beta)$$

in the sense of the extension principle.

Note that this inference works similarly as a rule of three. If $P \rightarrow Q$ is not a theorem, we can deduce $v(Q)$ on the condition that it is possible to infer $v(P) = \alpha$ from $v(P) = \tau$, that is, $\widetilde{\min}(1, 1 \ominus \tau \oplus \alpha) = 1$ (example: τ = very α); then $v(Q) = \beta$ can be asserted.

Last, Zadeh's compositional rule of inference (Zadeh, 1978a) can also be extended to

$$\text{If } (X, Y) \text{ is } F \text{ and } (Y, Z) \text{ is } G,$$

where F and G are now fuzzy-valued relations (i.e., fuzzy subsets of type 2 of $U \times V$ and $V \times W$, respectively), then we can infer if V is finite.

$$(X, Z) \text{ is } F \tilde{\circ} G,$$

where $F \tilde{\circ} G$ is defined as

$$\tilde{\mu}_{F \tilde{\circ} G}(u, w) = \widetilde{\max_v} \ \widetilde{\min}(\tilde{\mu}_F(u, v), \tilde{\mu}_G(v, w)).$$

Generalized modus ponens for fuzzy-valued propositions will then be

X is F,

If X is G then Y is H.

We deduce

Y is K,

where, generalizing the formula of Bellman and Zadeh (1977), K is defined as

$$\tilde{\mu}_K(v) = \widetilde{\max_u}\ \widetilde{\min}(\tilde{\mu}_F(u), \widetilde{\min}(1, 1 \ominus \tilde{\mu}_G(u) \oplus \tilde{\mu}_H(v)).$$

In such a fuzzy-valued logic, the degree of truth is expressed roughly by the mean value of the linguistic truth value τ while the vagueness of τ is expressed by its "spread." In particular, $\widetilde{\max}$, $\widetilde{\min}$, $1\ominus$, and $\cup$, $\cap$ — do not play the same role in that context; for instance, given two predicates P and Q, whose truth values are σ, τ, respectively:

$P \Subset Q$, i.e., $\sigma \subset \tau$ means the truth value of P is more precise than that of Q, and the degrees of truth are practically the same.

$$P \sqsubseteq Q, \text{ i.e., } \begin{cases} \widetilde{\max}(\sigma, \tau) = \tau \\ \sigma \cap \tau = \varnothing \end{cases}$$

means that Q is truer than P. Note that "very true" in Zadeh's sense is *not truer* than "true" but only *more precise*. But, in natural language, "very" usually does not relate only to precision.

In conclusion, we can note that in a chain of approximate inferences, truth and precision progress in the same sense; conclusions are always less precise and less true than premises; $\tau \ominus \Sigma$ is smaller than τ and also more fuzzy.

In other respects, practical computation of truth values can be made easier using results in fuzzy real algebra (Dubois and Prade, 1979).

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