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Weighted zeta functions for quotients of regular coverings of graphs

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Abstract

Let G be a connected graph. We reformulate Stark and Terras' Galois Theory for a quotient H of a regular covering K of a graph G by using voltage assignments. As applications, we show that the weighted Bartholdi L-function of H associated to the representation of the covering transformation group of H is equal to that of G associated to its induced representation in the covering transformation group of K. Furthermore, we express the weighted Bartholdi zeta function of H as a product of weighted Bartholdi L-functions of G associated to irreducible representations of the covering transformation group of K. We generalize Stark and Terras' Galois Theory to digraphs, and apply to weighted Bartholdi L-functions of digraphs.

Keywords: Graph covering; Quotient graph; Voltage graph; Zeta function; L-function

1. Introduction

Graphs and digraphs treated here are finite and simple. Let $G = (V(G), E(G))$ be a connected graph with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges uv joining two vertices u and v. For $uv \in E(G)$, an arc (u, v) is the oriented edge from u to v. Let D be the symmetric

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digraph corresponding to G, i.e., the digraph obtained by replacing each edge of G by a pair of oppositely directed edges (arcs). Set $D(G) = A(D) = \{(u, v), (v, u) \mid uv \in E(G)\}$. For $e = (u, v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v, u)$ be the *inverse* of $e = (u, v)$.

A *path P of length n* in G is a sequence $P = (e_1, \dots, e_n)$ of n arcs such that $e_i \in D(G)$, $t(e_i) = o(e_{i+1})$ $(1 \leqslant i \leqslant n-1)$, where indices are treated mod n. Set $|P| = n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, P is called an *$(o(P), t(P))$-path*. If $e_i = (v_{i-1}, v_i)$ $(1 \leqslant i \leqslant n)$, then we set $P = (v_0, v_1, \dots, v_n)$. We say that a path $P = (e_1, \dots, e_n)$ has a *backtracking* if $e_{i+1}^{-1} = e_i$ for some i $(1 \leqslant i \leqslant n-1)$. A (v, w)-path is called a *v-cycle* (or *v-closed path*) if $v = w$. The *inverse cycle* of a cycle $C = (e_1, \dots, e_n)$ is the cycle $C^{-1} = (e_n^{-1}, \dots, e_1^{-1})$.

We introduce an equivalence relation between cycles. Two cycles $C_1 = (e_1, \dots, e_m)$ and $C_2 = (f_1, \dots, f_m)$ are called *equivalent* if there exists k such that $f_j = e_{j+k}$ for all j. The inverse cycle of C is in general not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *power* of B. A cycle C is *reduced* if C has no backtracking. Furthermore, a cycle C is *prime* if it is not a power of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of G at a vertex v of G.

Let G be a connected graph, and let $N(v) = \{w \in V(G) \mid vw \in E(G)\}$ for any vertex v in G. A graph H is called a *covering* of G with projection $\pi : H \to G$ if there is a surjection $\pi : V(H) \to V(G)$ such that $\pi|_{N(v')} : N(v') \to N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. When a finite group Π acts on a graph (digraph) G, the *quotient graph (digraph)* G/Π is a simple graph (digraph) whose vertices are the Π-orbits on $V(G)$, with two vertices adjacent in G/Π if and only if some two of their representatives are adjacent in G. A covering $\pi : H \to G$ is said to be a *regular covering* of G if there is a subgroup B of the automorphism group $Aut\,H$ of H acting freely on H such that the quotient graph H/B is isomorphic to G.

Let G be a graph and S_n the symmetric group on the set $\{1, 2, \dots, n\}$. Then a mapping $\alpha : D(G) \to S_n$ is called a *permutation voltage assignment* if $\alpha(v, u) = \alpha(u, v)^{-1}$ for each $(u, v) \in D(G)$. The pair (G, α) is called a *permutation voltage graph*. The *derived graph G^α* of the permutation voltage graph (G, α) is defined as follows:

$V(G^\alpha) = V(G) \times \Gamma$ and $((u, h), (v, k)) \in D(G^\alpha)$ if and only if $(u, v) \in D(G)$ and $k = \alpha(u, v)(h)$.

The *natural projection* $\pi_\alpha : G^\alpha \to G$ is defined by $\pi_\alpha(u, h) = u$. The graph G^α is called a *derived graph covering* of G with voltages in S_n or an *n-covering* of G. Note that the n-covering G^α is an n-fold covering of G. Furthermore, every n-fold covering of a graph G is an n-covering G^α of G for some permutation voltage assignment $\alpha : D(G) \to S_n$ (see [7,8]).

Let G be a connected graph and Γ a finite group. Then the mapping $\alpha : D(G) \to \Gamma$ is called an *ordinary voltage assignment* if $\alpha(v, u) = \alpha(u, v)^{-1}$ for each $(u, v) \in D(G)$. The pair (G, α) is called an *ordinary voltage graph*. The *derived graph G^α* of the ordinary voltage graph (G, α) is defined as follows:

$V(G^\alpha) = V(G) \times \Gamma$ and $((u, h), (v, k)) \in D(G^\alpha)$ if and only if $(u, v) \in D(G)$ and $k = h\alpha(u, v)$.

The *natural projection* $\pi_\alpha : G^\alpha \to G$ is defined by $\pi_\alpha(v, h) = v$ for all $(v, h) \in V(G) \times \Gamma$. The graph G^α is called a Γ*-covering* of G.

For each $c \in \Gamma$, let $\phi_c : G^\alpha \to G^\alpha$ denote the graph automorphism defined by the rules $\phi_c(u, a) = (u, ca)$ for each $(u, a) \in V(G^\alpha)$ and $\phi_c((u, a), (v, b)) = ((u, ca), (v, cb))$ for each $((u, a), (v, b)) \in D(G^\alpha)$. Then Γ acts freely on the left of the Γ-covering G^α. Thus, the covering transformation group of the Γ-covering G^α is isomorphic to Γ. The Γ-covering G^α is a $|\Gamma|$-fold regular covering of G. Every regular covering of G is a Γ-covering of G for some group Γ (see [7,8]).

Let G be a connected graph and $\alpha : D(G) \to \Gamma$ (or S_n) be an ordinary (or permutation) voltage assignment. For each path $P = (e_1, \dots, e_r)$ of G, set $\alpha(P) = \alpha(e_1) \cdots \alpha(e_r)$. This is called the *net voltage* of P.

Sunada [21,22] stated the Galois Theory for quotients of regular coverings of Riemannian manifolds, and presented a formula for the L-function of a quotient of a regular covering of a Riemannian manifold. Stark and Terras [20] developed the Galois Theory for quotients of regular coverings of graphs, and gave a formula for the (Artin) L-function of a quotient of a regular covering of a graph by using it.

Computation of zeta functions of graphs can be in general excessively difficult; one is greatly helped the presence of symmetry in the graph, especially if the graph is a covering of a simpler graph, from which one would like to “induce” the zeta function.

Another kind of “symmetry” may appear if one allows the zeta function to be weighted by the number of “backtrackings” that a path follows. In some case, paths without backtrackings are simpler to count (e.g. in cycles), while in the other cases general paths are easier to count (e.g. complete graphs).

Our main result is that the classical results for zeta functions of coverings (their expressions as L-functions over a representation of the Galois group of the coverings) hold, /mutatis mutandis/, for weighted zeta functions.

The main results of this paper are

Main Result 1. *Let G be a connected graph, $\mathbf{W}(G)$ a weighted matrix of G, Γ a finite group and $\alpha : D(G) \to \Gamma$ an ordinary voltage assignment. Then the weighted Bartholdi zeta function of the Γ-covering G^α of G is*

$$\zeta\big(G^\alpha, \tilde{w}, u, t\big) = \prod_\rho \zeta_G(w, u, t, \rho, \alpha)^{\deg \rho},$$

where ρ runs over all inequivalent irreducible representations of Γ, and $\tilde{w} = \tilde{w}_{G^\alpha}$.

Main Result 2. *Let G be a connected graph, T a spanning tree of G, $\mathbf{W}(G)$ a weighted matrix of G, Γ a finite group and $\alpha : D(G) \to \Gamma$ an ordinary voltage assignment. Furthermore, let $B \leqslant \Gamma$ be a subgroup of Γ and $H = G^\alpha / B$. Assume that G^α is connected and α is T-reduced. Then the weighted Bartholdi zeta function of the quotient H of the Γ-covering G^α of G is*

$$\zeta(H, \tilde{w}_H, u, t) = \zeta(G, w, u, t) \prod_{\rho \neq 1} \zeta_G(w, u, t, \rho, \alpha)^{m_\rho},$$

where ρ runs over all inequivalent irreducible representations of Γ except the identity representation $\mathbf{1}$, and each m_ρ $(\rho \neq \mathbf{1})$ is some nonnegative integer.

Main Result 3. *Let D be a connected digraph, $\mathbf{W}(D)$ a weighted matrix of D, Γ a finite group and $\alpha : A(D) \to \Gamma$ a symmetric ordinary voltage assignment. Then the weighted Bartholdi zeta function of the Γ-covering D^α of D is*

$$\zeta\left(D^\alpha, \tilde{w}_{D^\alpha}, u, t\right) = \prod_\rho \zeta_D(w, u, t, \rho, \alpha)^{\deg \rho},$$

where ρ runs over all inequivalent irreducible representations of Γ.

In Section 2, we reformulate Stark and Terras' Galois Theory for a quotient H of a regular covering K of a graph G by using voltage assignments. In Section 3, we show that the weighted Bartholdi L-function of H associated to the representation of the covering transformation group of H is equal to that of G associated to its induced representation in the covering transformation group of K. In Section 4, we express the weighted Bartholdi zeta function of H as a product of weighted Bartholdi L-functions of G associated to irreducible representations of the covering transformation group of K. In Section 5, we extend the results in Section 2 to digraphs. In Section 6, we extend the results in Section 3 to weighted Bartholdi L-functions of digraphs.

For a general theory of the representation of groups and graph coverings, the reader is referred to [18] and [8], respectively.

2. Quotients of regular coverings of graphs

Let G be a connected graph, Γ a finite group and $\alpha : D(G) \to \Gamma$ an ordinary voltage assignment. Furthermore, let $B \leqslant \Gamma$ be a subgroup of Γ. Then we consider the quotient graph $H = G^\alpha / B$. We write an action of Γ on G^α as follows:

$$c(u, a) = (u, ca) \quad \text{for each } (u, a) \in V\left(G^\alpha\right)$$

and

$$c\big((u, a), (v, b)\big) = \big((u, ca), (v, cb)\big) \quad \text{for each } \big((u, a), (v, b)\big) \in D\left(G^\alpha\right),$$

where $c \in \Gamma$. Then the quotient graph $H = G^\alpha / B$ is given as follows:

$$V\left(G^\alpha / B\right) = \left\{(u, Bg) \mid u \in V(G),\ g \in \Gamma\right\}$$

and

$$\big((u, Bg), (v, Bh)\big) \in D\left(G^\alpha / B\right) \quad \text{if and only if } (u, v) \in D(G) \text{ and } Bh = Bg\alpha(u, v).$$

By the theory of covering space (see [24]), G^α is a regular covering of the quotient graph $H = G^\alpha / B$. Thus, there exists an ordinary voltage assignment $\beta : D(H) \to B$ such that $H^\beta = G^\alpha$.

Now, since $V(H^\beta) = V(G^\alpha)$, we will identify the vertex (u, hg) of G^α with the vertex $((u, Bg), h)$ of H^β. Suppose that

$$\big(((v, Bg), h), ((w, Bg'), h')\big) = \big((v, hg), (w, h'g')\big) \in D\left(G^\alpha\right) = D\left(H^\beta\right)$$

for each $g, g' \in \Gamma, h, h' \in B$.

Proposition 1. *Let G be a connected graph, Γ a finite group and $\alpha : D(G) \to \Gamma$ an ordinary voltage assignment. Furthermore, let $B \leqslant \Gamma$ be a subgroup of Γ and $H = G^{\alpha}/B$. If $\beta : D(H) \to B$ is an ordinary voltage assignment such that $H^{\beta} = G^{\alpha}$, then*

$$\beta\big((v, Bg), \big(w, Bg'\big)\big) = g\alpha(v, w)g'^{-1}$$

for each $(v, w) \in D(G)$ and each $g, g' \in \Gamma$.

Proof. Let $(v, w) \in D(G)$. Then, $(((v, Bg), h), ((w, Bg'), k)) \in D(G^{\alpha}/B)$ if and only if $k = h\beta((v, Bg), (w, Bg'))$. Furthermore, $((v, hg), (w, kg')) \in D(G^{\alpha})$ if and only if $kg' = hg\alpha(v, w)$, i.e., $k = hg\alpha(v, w)g'^{-1}$. Therefore it follows that

$$h^{-1}k = \beta\big((v, Bg), \big(w, Bg'\big)\big) = g\alpha(v, w)g'^{-1}. \qquad \square$$

Next, let $|\Gamma| = n$ and $|B| = m$. Furthermore, let D be any prime cycle of G^{α} and $\pi = \pi_{\alpha} : G^{\alpha} \to G$ the natural projection. Then we have

$$\pi(D) = C^k,$$

where C is a prime cycle of G, and $k = ord(\alpha(C))$ be the order of $\alpha(C)$ in Γ.

Now, let $x = \alpha(C)$ and $f = n/k$. Furthermore, let

$$\Gamma = y_1\langle x\rangle \cup y_2\langle x\rangle \cup \cdots \cup y_f\langle x\rangle,$$

where $y_1 = 1$. Then we may have

$$\pi^{-1}(C) = D \cup y_2 D \cup \cdots \cup y_f D$$

and

$$Z(y_i D) = y_i\langle x\rangle y_i^{-1} \quad (1 \leqslant i \leqslant f),$$

where $Z(y_i D)$ is the stabilizer of $y_i D$ in Γ. Let C be a v-cycle and D a $(v, 1)$-cycle. Then $y_i D$ is a (v, y_i)-cycle.

Next, let $\pi_B : G^{\alpha} \to H$ be the natural projection. Then we have

$$\pi_B(y_i D) = y_i D/B = K_i^{d_i} \quad (1 \leqslant i \leqslant f),$$

where K_i is a prime cycle of H. Furthermore, K_i is a (v, By_i)-cycle and

$$|y_i D| = k|C| = d_i|K_i|.$$

Let $k = d_i f_i$ $(1 \leqslant i \leqslant f)$. By Proposition 1, we have

$$\beta(K_i) = y_i x^{f_i} y_i^{-1} \quad \text{and} \quad ord\big(\beta(K_i)\big) = d_i.$$

Note that $f_i = \min\{e \mid y_i x^e y_i^{-1} \in B\}$.

Now, let $K_1, K_2, \dots, K_s$ be all the distinct K_i's among $K_1, \dots, K_f$. Then $K_1, K_2, \dots, K_s$ are lifts of C in H. Let $\pi_B(y_j D) = K_j^{d_j}$. Then $\beta(K_j) = y_j x^{f_j} y_j^{-1}$ and $k = f_j d_j$ $(1 \leqslant j \leqslant s)$. Furthermore, let

$$\left|\pi_B^{-1}(K_j)\right| = m/d_j = l_j \quad (1 \leqslant j \leqslant s).$$

Then we have

$$l_1 + l_2 + \cdots + l_s = n/k = f.$$

Let $n = mt$. Then we have

$$f_1 + f_2 + \cdots + f_s = t.$$

Now, let $B_j = \langle \beta(K_j) \rangle$ $(j = 1, \dots, s)$ and

$$B = z_{j1} B_j \cup z_{j2} B_j \cup \cdots \cup z_{jl_j} B_j, \quad z_{j1} = 1.$$

Since $z_{jl}((v, By_j), 1) = z_{jl}(v, y_j) = (v, z_{jl} y_j)$, the lifts of K_j in G^α are

$$z_{j1} y_j D = y_j D, z_{j2} y_j D, \dots, z_{jl_j} y_j D,$$

and

$$Z(z_{jl} y_j D) = z_{jl} B_j z_{jl}^{-1} \quad (1 \leqslant l \leqslant l_j)$$

in H. Furthermore,

$$Z(z_{jl} y_j D) = z_{jl} y_j < \alpha(C) > y_j^{-1} z_{jl}^{-1} \quad (1 \leqslant l \leqslant l_j)$$

in Γ. Thus,

$$B y_j \langle \alpha(C) \rangle = z_{j1} y_j \langle \alpha(C) \rangle \cup \cdots \cup z_{jl_j} y_j \langle \alpha(C) \rangle \quad (1 \leqslant j \leqslant s).$$

Therefore it follows that $\Gamma = B y_1 \langle \alpha(C) \rangle \cup \cdots \cup B y_s \langle \alpha(C) \rangle$ is the set of all the distinct double cosets of B and $\langle \alpha(C) \rangle$ in Γ. By [18,23], we have

$$l_j = \left| B \colon B \cap y_j \langle \alpha(C) \rangle y_j^{-1} \right| \quad (1 \leqslant j \leqslant s).$$

Proposition 2. *Let G be a connected graph, Γ a finite group and $\alpha : D(G) \to \Gamma$ an ordinary voltage assignment. Furthermore, let $B \leqslant \Gamma$ be a subgroup of Γ and $H = G^\alpha / B$. For any prime cycle C in G, the number of lifts of C in H is equal to the cardinality of the set $B \setminus \Gamma / \langle \alpha(C) \rangle$ of distinct double cosets of B and $\langle \alpha(C) \rangle$ in Γ, and the length of each lift of C in H is of form*

$$|C| \cdot \frac{k}{m} \cdot \left| B \colon B \cap y_j \langle \alpha(C) \rangle y_j^{-1} \right|,$$

where $y_1 = 1, y_2, \dots, y_s$ are the representatives of distinct double cosets of B and $\langle \alpha(C) \rangle$ in Γ.

Since $|\Gamma/B| = n/m = t$, H is a t-covering of G, that is, there exists a permutation voltage assignment $\phi : D(G) \to S_t$ such that $H = G^{\phi}$. For any arc $(v, w) \in D(G)$, $((v, Bg_i), (w, Bg_j)) \in D(H)$ if and only if $(v, w) \in D(G)$ and $Bg_j = Bg_i\alpha(v, w)$, where $\Gamma/B = \{Bg_1 = B, Bg_2, \ldots, Bg_t\}$.

We identify i with Bg_i $(1 \leqslant i \leqslant t)$, and so let

$$\phi(v, w)(i) = j \quad \text{if } Bg_j = Bg_i\alpha(v, w).$$

Then it is clear that

$$H = G^{\phi}.$$

By [8, Theorem 2.4.3] implies that, for any prime cycle C in G, the number of lifts of C in H is equal to that of cycles of $\phi(C)$, and the length of each lift of C in H is $i|C|$ for some i such that $c_i \neq 0$, where $(c_1, c_2, \ldots, c_t)$ is the cycle type of $\phi(C)$.

Corollary 1.

$$\sum_{i=1}^{t} c_i = \bigl|B \setminus \Gamma/\langle\alpha(C)\rangle\bigr|.$$

If $c_i \neq 0$, then

$$i = \frac{k}{m} \cdot \bigl|B\colon\ B \cap y_j\langle\alpha(C)\rangle y_j^{-1}\bigr| \quad \textit{for some } y_j.$$

3. Weighted Bartholdi L-functions of quotients of regular coverings

Let G be a connected graph. We say that a path $P = (e_1, \ldots, e_n)$ has a *bump* at $t(e_i)$ if $e_{i+1} = e_i^{-1}$ $(1 \leqslant i \leqslant n)$. The *cyclic bump count* $cbc(\pi)$ of a cycle $\pi = (\pi_1, \ldots, \pi_n)$ is

$$cbc(\pi) = \bigl|\bigl\{i = 1, \ldots, n \bigm| \pi_i = \pi_{i+1}^{-1}\bigr\}\bigr|,$$

where $\pi_{n+1} = \pi_1$.

Let $V(G) = \{v_1, \ldots, v_n\}$. Then we consider an $n \times n$ matrix $\mathbf{W} = (w_{ij})_{1 \leqslant i,j \leqslant n}$ with ij entry the complex variable $w_{ij} \neq 0$ if $(v_i, v_j) \in D(G)$, and $w_{ij} = 0$ otherwise. The matrix $\mathbf{W} = \mathbf{W}(G)$ is called the *weighted matrix* of G. For each path $P = (v_{i_1}, \ldots, v_{i_r})$ of G, the *norm* $w(P)$ of P is defined as follows: $w(P) = w_{i_1 i_2} w_{i_2 i_3} \cdots w_{i_{r-1} i_r}$. Furthermore, let $w(v_i, v_j) = w_{ij}$, $v_i, v_j \in V(G)$ and $w(e) = w_{ij}$, $e = (v_i, v_j) \in D(G)$.

A *representation* ϕ of a group Γ over $\mathbf{C}$ is a homomorphism into the group $GL(r, \mathbf{C})$ of invertible $r \times r$ matrices over $\mathbf{C}$. We say that r is the *degree* of ϕ (see [18]). Furthermore, let ρ be a unitary representation of Γ and d its degree. The *weighted Bartholdi L-function* of G associated with ρ and α is defined by

$$\zeta_G(w, u, t, \rho, \alpha) = \prod_{[C]} \det\bigl(\mathbf{I}_d - w(C)\rho\bigl(\alpha(C)\bigr)u^{cbc(C)}t^{|C|}\bigr)^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of G (see [17]). If $\rho = \mathbf{1}$ (the identity representation of Γ), then the weighted Bartholdi L-function of G is called the *weighted Bartholdi zeta function* of G, denoted by $\zeta(G, w, u, t) = \zeta_G(w, u, t, \mathbf{1}, \alpha)$.

If $u = 0$, then the weighted Bartholdi L-function of G is the weighted L-function of G (see [15]). If $w(v_i, v_j) = 1$ for each $(v_i, v_j) \in D(G)$, then the weighted Bartholdi L-function of G is the Bartholdi L-function of G (see [14]). Furthermore, in the case that $u = 0$ and $w(v_i, v_j) = 1$ for each $(v_i, v_j) \in D(G)$, then the weighted Bartholdi L-function of G is the L-function of G (see [13,20]).

In the case of $\rho = \mathbf{1}$, the weighted L-function, the Bartholdi L-function and the L-function of G is the weighted, Bartholdi and Ihara zeta function of G, respectively (see [1,10,15]).

The *(Ihara) zeta function* of a graph G is defined to be a function of $u \in \mathbf{C}$ with $|u|$ sufficiently small, by

$$\mathbf{Z}(G, u) = \mathbf{Z}_G(u) = \prod_{[C]} \left(1 - u^{|C|}\right)^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G (see [10]).

Zeta functions of graphs started from zeta functions of regular graphs by Ihara [10]. In [10], he showed that their reciprocals are explicit polynomials. A zeta function of a regular graph G associated with a unitary representation of the fundamental group of G was developed by Sunada [21,22]. Hashimoto [9] treated multivariable zeta functions of bipartite graphs. Bass [2] generalized Ihara's result on the zeta function of a regular graph to an irregular graph, and showed that its reciprocal is again a polynomial:

$$\mathbf{Z}(G, u)^{-1} = \left(1 - u^2\right)^{r-1} \det\left(\mathbf{I} - u\mathbf{A}(G) + u^2(\mathbf{D} - \mathbf{I})\right),$$

where r and $\mathbf{A}(G)$ are the Betti number and the adjacency matrix of G, respectively, and $\mathbf{D} = (d_{ij})$ is the diagonal matrix with $d_{ii} = \deg v_i$ where $V(G) = \{v_1, \ldots, v_n\}$.

Stark and Terras [19] gave an elementary proof of Bass' Theorem, and discussed three different zeta functions of any graph. Furthermore, various proofs of Bass' Theorem were given by Foata and Zeilberger [6], Kotani and Sunada [12].

Sato [17] gave a determinant expression of the weighted Bartholdi L-function of G, and showed that the weighted Bartholdi zeta function of a regular covering of G is a product of weighted Bartholdi L-functions of G.

Let B be a subgroup of a finite group Γ and $\sigma : B \to GL(W)$ a representation of B. Then the *induced representation from B up to Γ*, denoted $\sigma^* = \mathrm{Ind}_B^\Gamma \sigma$ is a group homomorphism $\sigma^* : \Gamma \to GL(V)$, where

$$V = \left\{ f : \Gamma \to W \;\middle|\; f(hg) = \sigma(h) f(g),\ h \in B,\ g \in \Gamma \right\}.$$

The representation $\sigma^*(g)$ is then defined on $f \in V$ by

$$\left(\sigma^*(g) f\right)(x) = f(xg), \quad x, g \in \Gamma.$$

The *character* χ_π of a representation π of Γ is defined by $\chi_\pi(g) = \mathrm{Tr}\,\pi(g)$ for all $g \in \Gamma$. We use the following result (see [18]).

Lemma 1. *Let B be a subgroup of a finite group Γ and $\sigma : B \to GL(W)$ a representation of B. Let $\sigma^* = \mathrm{Ind}_B^\Gamma \sigma$. Then we have the following formula that relates the characters of the two representations:*

$$\chi_{\sigma^*}(g) = \frac{1}{|B|} \sum_{x \in \Gamma} \tilde{\chi}_\sigma\big(xgx^{-1}\big),$$

where

$$\tilde{\chi}_\sigma(x) = \begin{cases} \chi_\sigma(x) & \text{if } x \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Let G be a connected graph, Γ a finite group and $\alpha : D(G) \to \Gamma$ an ordinary voltage assignment. In the Γ-covering G^α, set $v_g = (v, g)$ and $e_g = (e, g)$, where $v \in V(G)$, $e \in D(G)$, $g \in \Gamma$. For $e = (u, v) \in D(G)$, the arc e_g emanates from u_g and terminates at $v_{g\alpha(e)}$. Note that $e_g^{-1} = (e^{-1})_{g\alpha(e)}$.

Let $\mathbf{W} = \mathbf{W}(G)$ be a weighted matrix of G. Then we define the *weighted matrix* $\tilde{\mathbf{W}} = \mathbf{W}(G^\alpha) = (\tilde{w}(u_g, v_h))$ of G^α *derived from* $\mathbf{W}$ as follows:

$$\tilde{w}(u_g, v_h) := \begin{cases} w(u, v) & \text{if } (u, v) \in D(G) \text{ and } h = g\alpha(u, v), \\ 0 & \text{otherwise.} \end{cases}$$

Set $\tilde{w}_{G^\alpha} = \tilde{w}$. Furthermore, for an n-covering G^ϕ of G by a permutation voltage assignment $\phi : D(G) \to S_n$, the *weighted matrix* $\tilde{\mathbf{W}} = \mathbf{W}(G^\phi)$ of G^ϕ *derived from* $\mathbf{W}$ is defined similarly.

By Propositions 1, 2 and Lemma 1, we give a formula for the weighted Bartholdi L-function of a graph G associated to the representation of a finite group Γ induced from a representation of a subgroup of Γ.

Theorem 1. *Let G be a connected graph, $\mathbf{W}(G)$ a weighted matrix of G, Γ a finite group and $\alpha : D(G) \to \Gamma$ an ordinary voltage assignment. Furthermore, let $B \leqslant \Gamma$ be a subgroup of Γ and $H = G^\alpha / B$. Assume that G^α is connected. Let σ be any representation of B and $\sigma^* = \mathrm{Ind}_B^\Gamma \sigma$ the representation of Γ induced from σ. Let $\beta : D(H) \to B$ be an ordinary voltage assignment such that $H^\beta = G^\alpha$. Then we have*

$$\zeta_H(\tilde{w}_H, u, t, \sigma, \beta) = \zeta_G\big(w, u, t, \sigma^*, \alpha\big).$$

Proof. The proof is an analogue of that of Theorem 8 in Stark and Terras [20].

At first, set $\tilde{w} = \tilde{w}_H$. Since $\log\det(\mathbf{I} - \mathbf{B}) = \mathrm{Tr}(\log(\mathbf{I} - \mathbf{B}))$,

$$\begin{aligned} \log \zeta_G\big(w, u, t, \sigma^*, \alpha\big) &= -\sum_{[C]} \log\det\big(\mathbf{I} - w(C)\sigma^*\big(\alpha(C)\big)u^{cbc(C)}t^{|C|}\big) \\ &= \sum_{[C]} \sum_{s=1}^\infty \frac{1}{s} \mathrm{Tr}\big(\sigma^*\big(\alpha(C)^s\big)\big) w(C)^s u^{cbc(C)s} t^{|C|s} \\ &= \sum_{[C]} \sum_{s=1}^\infty \frac{1}{s} \chi_{\sigma^*}\big(\alpha(C)^s\big) w(C)^s u^{cbc(C)s} t^{|C|s}, \end{aligned}$$

where $\chi_{\sigma^*} = \mathrm{Tr}\,\sigma^*$.

By Lemma 1, we have

$$\log \zeta_G\big(w, u, t, \sigma^*, \alpha\big) = \sum_{[C]} \sum_{s \geqslant 1} \frac{1}{s|B|} \sum_{g \in \Gamma} \tilde{\chi}_\sigma\big(g\alpha(C)^s g^{-1}\big) w(C)^s u^{cbc(C)s} t^{|C|s}.$$

Let C be a v-cycle of G and let D be the lift of C in G^α which is a $(v, 1)$-cycle, where $v \in V(G)$. Furthermore, let $x = \alpha(C)$ and

$$\Gamma = y_1\langle x\rangle \cup y_2\langle x\rangle \cup \cdots \cup y_f\langle x\rangle,$$

where $y_1 = 1$ and $f = |\Gamma|/|\langle\alpha(C)\rangle|$. Then we have

$$\log \zeta_G\big(w, u, t, \sigma^*, \alpha\big) = \sum_{[C]} \sum_{s \geqslant 1} \frac{1}{s|B|} \sum_{i=1,\, y_i x^s y_i^{-1} \in B}^{f} k\chi_\sigma\big(y_i x^s y_i^{-1}\big) w(C)^s u^{cbc(C)s} t^{|C|s},$$

where $k = ord(x)$.

Now, let

$$B \setminus \Gamma/\langle x\rangle = \big\{By_1\langle x\rangle, \ldots, By_r\langle x\rangle\big\},$$

and let $l_j = |B\colon\ B \cap y_j\langle x\rangle y_j^{-1}|$ $(1 \leqslant j \leqslant r)$. Note that l_j is the number of right cosets of $\langle x\rangle$ of Γ contained in $By_j\langle x\rangle$ $(1 \leqslant j \leqslant r)$. By the fact that $\chi_\sigma(hxh^{-1}) = \chi_\sigma(x)$, $h, x \in B$, we have

$$\log \zeta_G\big(w, u, t, \sigma^*, \alpha\big) = \sum_{[C]} \sum_{s \geqslant 1} \frac{1}{s|B|} \sum_{j=1,\, y_j x^s y_j^{-1} \in B}^{r} l_j k\chi_\sigma\big(y_j x^s y_j^{-1}\big) w(C)^s u^{cbc(C)s} t^{|C|s}.$$

Next, let $\beta : D(H) \to B$ be an ordinary voltage assignment such that $H^\beta = G^\alpha$. Let $\pi_\beta(y_j D) = K_j^{d_j}$ and $f_j = k/d_j$ $(1 \leqslant j \leqslant r)$, where K_j is a prime cycle of H. Since D is a $(v, 1)$-cycle in G^α, $y_j D$ is a (v, y_j)-cycle in G^α, and so K_j is a (v, By_j)-cycle in H. Then we have

$$\beta(K_j) = y_j x^{f_j} y_j^{-1} \quad (1 \leqslant j \leqslant r).$$

But, $y_j x^s y_j^{-1} \in B$ if and only if $s = f_j s'$ for some s'. Furthermore, note that $cbc(K_j) = f_j cbc(C)$ for each $j = 1, \ldots, r$. Thus,

$$\begin{aligned}\log \zeta_G\big(w, u, t, \sigma^*, \alpha\big) &= \sum_{[C]} \sum_{s' \geqslant 1} \sum_{j=1}^{r} \frac{l_j k}{f_j s'|B|} \chi_\sigma\big(y_j x^{f_j s'} y_j^{-1}\big) w(C)^{f_j s'} u^{cbc(C) f_j s'} t^{|C| f_j s'} \\ &= \sum_{[C]} \sum_{[K_j]} \sum_{s' \geqslant 1} \frac{k l_j}{f_j s'|B|} \chi_\sigma\big(\beta(K_j)^{s'}\big) \tilde{w}(K_j)^{s'} u^{cbc(K_j)s'} t^{|K_j|s'}.\end{aligned}$$

Since $kl_j = f_j|B|$ for each $j = 1, \dots, r$, we have

$$\begin{aligned}\log \zeta_G\big(w, u, t, \sigma^*, \alpha\big) &= \sum_{[C]} \sum_{[K_j]} \sum_{s' \geqslant 1} \frac{1}{s'} \chi_\sigma\big(\beta(K_j)^{s'}\big) \tilde{w}(K_j)^{s'} u^{cbc(K_j)s'} t^{|K_j|s'} \\ &= \sum_{[C]} \sum_{[K_j]} \sum_{s' \geqslant 1} \frac{1}{s'} \operatorname{Tr}\big(\sigma\big(\beta(K_j)^{s'}\big)\big) \tilde{w}(K_j)^{s'} u^{cbc(K_j)s'} t^{|K_j|s'} \\ &= \sum_{[C]} \sum_{[K_j]} \log \det\big(\mathbf{I} - \tilde{w}(K_j)\sigma\big(\beta(K_j)\big) u^{cbc(K_j)} t^{|K_j|}\big)^{-1}.\end{aligned}$$

Hence,

$$\begin{aligned}\zeta_G\big(w, u, t, \sigma^*, \alpha\big) &= \prod_{[C]} \prod_{[K_j]} \det\big(\mathbf{I} - \tilde{w}(K_j)\sigma\big(\beta(K_j)\big) u^{cbc(K_j)} t^{|K_j|}\big)^{-1} \\ &= \prod_{[K]} \det\big(\mathbf{I} - \tilde{w}(K)\sigma\big(\beta(K)\big) u^{cbc(K)} t^{|K|}\big)^{-1} \\ &= \zeta_H(\tilde{w}, u, t, \sigma, \beta),\end{aligned}$$

where $[K]$ runs over all equivalence classes of prime cycles in H. □

In the case of $\sigma = \mathbf{1}$, we obtain a decomposition formula for the weighted Bartholdi zeta function of a regular covering of G by a product of weighted Bartholdi L-functions of G (see [17]). Let $\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$ be the block diagonal sum of square matrices $\mathbf{M}_1, \dots, \mathbf{M}_s$:

$$\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s = \begin{bmatrix} \mathbf{M}_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{M}_s \end{bmatrix}.$$

For a positive integer f, the matrix $f \circ \mathbf{B}$ is the block diagonal sum of f copies of a square matrix $\mathbf{B}$.

Corollary 2. *Let G be a connected graph, $\mathbf{W}(G)$ a weighted matrix of G, Γ a finite group and $\alpha : D(G) \to \Gamma$ an ordinary voltage assignment. Then we have*

$$\zeta\big(G^\alpha, \tilde{w}, u, t\big) = \prod_\rho \zeta_G(w, u, t, \rho, \alpha)^{\deg \rho},$$

where ρ runs over all inequivalent irreducible representations of Γ, and $\tilde{w} = \tilde{w}_{G^\alpha}$.

Proof. Let $B = \{1\}$ be the trivial subgroup of Γ and $\Gamma = \Gamma/B$. Furthermore, let $\mathbf{1}$ be the trivial representation of B and $\rho = \operatorname{Ind}_B^\Gamma \mathbf{1}$ the representation of Γ induced from $\mathbf{1}$. Then ρ is the right regular representation of Γ. In Theorem 1, we let $\sigma = \mathbf{1}$, $\sigma^* = \rho$ and $\beta = 1$. Therefore, it follows that

$$\zeta\big(G^{\alpha}, \tilde{w}, u, t\big) = \zeta_G(w, u, t, \rho, \alpha) = \prod_{[C]} \det\big(\mathbf{I} - w(C)\rho\big(\alpha(C)\big)u^{cbc(C)}t^{|C|}\big)^{-1}.$$

Since ρ is the right regular representation of Γ, we have

$$\rho = (1) \oplus f_2 \circ \rho_2 \oplus \cdots \oplus f_t \circ \rho_t,$$

where $\rho = 1, \rho_2, \ldots, \rho_t$ are all inequivalent irreducible representations of Γ and f_i the degree of ρ_i $(f_1 = 1)$. Hence,

$$\begin{aligned}\zeta\big(G^{\alpha}, \tilde{w}, u, t\big) &= \prod_{i=1}^{t} \prod_{[C]} \det\big(\mathbf{I} - w(C)\rho_i\big(\alpha(C)\big)u^{cbc(C)}t^{|C|}\big)^{-f_i} \\ &= \prod_{i=1}^{t} \zeta_G(w, u, t, \rho_i, \alpha)^{f_i}. \qquad \square\end{aligned}$$

4. Weighted Bartholdi zeta functions of quotients of regular coverings

Let G be a connected graph, and let $\pi_H : H \to G$ and $\pi_K : K \to G$ be two finite coverings of G. Then H and K are called *isomorphic*, denoted $H \cong K$, if there exists an isomorphism $\Phi : H \to K$ such that $\pi_K \Phi = \pi_H$.

Let G be a connected graph, S_n the symmetric group on $N = \{1, 2, \ldots, n\}$ and $\alpha : D(G) \to S_n$ a permutation voltage assignment. Furthermore, let T be a spanning tree of G rooted at a vertex $v \in V(G)$. Then the *T-voltage* α_T of α is defined as follows: $\alpha_T(x, y) = \alpha(P_x)\alpha(x, y)\alpha(P_y)^{-1}$ for each $(x, y) \in D(G)$, where P_x is the unique path from v to x in T, etc. Note that $\alpha_T(e) = 1$ for each $e \in D(T)$. By [8, Theorem 2.5.4], we have $G^{\alpha} \cong G^{\alpha_T}$. Thus,

$$\zeta\big(G^{\alpha}, \tilde{w}_{G^{\alpha}}, u, t\big) = \zeta\big(G^{\alpha_T}, \tilde{w}_{G^{\alpha_T}}, u, t\big)$$

for a weighted matrix $\mathbf{W}(G)$. A permutation voltage assignment β is called *T-reduced* if $\beta(e) = 1$ for each $e \in D(T)$. For an ordinary voltage assignment, we can treat similarly.

We present a decomposition formula for the weighted Bartholdi zeta function of a quotient of a regular covering of a graph.

Theorem 2. *Let G be a connected graph, T a spanning tree of G, $\mathbf{W}(G)$ a weighted matrix of G, Γ a finite group and $\alpha : D(G) \to \Gamma$ an ordinary voltage assignment. Furthermore, let $B \leqslant \Gamma$ be a subgroup of Γ and $H = G^{\alpha}/B$. Assume that G^{α} is connected and α is T-reduced. Then we have*

$$\zeta(H, \tilde{w}_H, u, t) = \zeta(G, w, u, t) \prod_{\rho \neq 1} \zeta_G(w, u, t, \rho, \alpha)^{m_\rho},$$

where ρ runs over all inequivalent irreducible representations of Γ except the identity representation $\mathbf{1}$, and each m_ρ $(\rho \neq \mathbf{1})$ is some nonnegative integer.

Proof. Let $k = |\Gamma/B|$. Since H is a k-covering of G, that is, there exists a permutation voltage assignment $\phi : D(G) \to S_k$ such that $H = G^{\phi}$. In fact, for any arc $(v, w) \in D(G)$, ϕ is given as follows:

$$\phi(v, w)(i) = j \quad \text{if } Bg_j = Bg_i \alpha(v, w),$$

where $i, j = 1, \ldots, k$ and $\Gamma/B = \{Bg_1 = B, Bg_2, \ldots, Bg_k\}$.

Let T be a spanning tree of G rooted at $v \in V(G)$. Since α is T-reduced, ϕ is T-reduced. Let $\Sigma = \langle\{\phi(u, z) \mid (u, z) \in D(G) \setminus D(T)\}\rangle$ be the subgroup of S_k generated by $\{\phi(u, z) \mid (u, z) \in D(G) \setminus D(T)\}$. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_r$ be the irreducible representations of Σ, and f_i the degree of ρ_i for each i, where $f_1 = 1$.

Let $P : \Sigma \to GL(k, \mathbf{C})$ be the permutation representation of Σ such that $P(\gamma) = \mathbf{P}_{\gamma}$, and m_i the multiplicity of ρ_i in P for each $i = 1, \ldots, r$, that is, P is equivalent to a representation $m_1 \circ 1 \oplus m_2 \circ \rho_2 \oplus \cdots \oplus m_r \circ \rho_r$. Since G^{α} is connected, $H = G^{\phi}$ is connected, i.e., $m_1 = 1$ (see [18]).

By [3, Theorem 7], we have

$$\zeta\left(G^{\phi}, \tilde{w}_H, u, t\right) = \zeta(G, w, u, t) \prod_{i=2}^{r} \zeta_G(w, u, t, \rho_i, \phi)^{m_i}.$$

But, we consider the following permutation representation of Γ:

$$\Theta(g)(i) = j \quad \text{if } Bg_j = Bg_i g,$$

where $g \in \Gamma$ and $i, j = 1, \ldots, k$. By Lemma 4 of [20], Θ is a faithful representation of Γ, i.e.,

$$\left\langle\left\{\Theta(g) \mid g \in \Gamma\right\}\right\rangle \cong \Gamma.$$

Since G^{α} is connected, the local subgroup $\langle\{\alpha(C) \mid C\colon v\text{-cycle}\}\rangle$ of Γ generated by $\{\alpha(C) \mid C\colon v\text{-cycle}\} = \{\alpha(e) \mid e \in D(G) \setminus D(T)\}$ is equal to Γ by [8, Theorem 2.5.1]. Thus, we have

$$\left\langle\left\{\Theta(g) \mid g \in \Gamma\right\}\right\rangle = \Sigma,$$

i.e.,

$$\Sigma \cong \Gamma.$$

Therefore the result follows. □

Next, we shall give the values of m_{ρ} by using Theorem 1.

Theorem 3. *Let G be a connected graph, $\mathbf{W}(G)$ a weighted matrix of G, Γ a finite group and $\alpha : D(G) \to \Gamma$ an ordinary voltage assignment. Furthermore, let $B \leqslant \Gamma$ be a subgroup of Γ and $H = G^{\alpha}/B$. Assume that G^{α} is connected.*

Let $\sigma_1 = 1, \sigma_2, \ldots, \sigma_m$ be the irreducible representations of B, and h_i the degree of σ_i for each i, where $h_1 = 1$. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_n$ be the irreducible representations of Γ, and f_j the degree of ρ_j for each j, where $f_1 = 1$. Let

$$\sigma_i^* = k_{i1} \circ \rho_1 \oplus k_{i2} \circ \rho_2 \oplus \cdots \oplus k_{in} \circ \rho_n \quad (1 \leqslant i \leqslant m).$$

Then we have

$$m_j = f_j - k_{2j}h_2 - \cdots - k_{mj}h_m \quad (1 \leqslant j \leqslant n),$$

where $m_1 = 1$.

Proof. Let $\phi : D(G) \to S_d$ be a permutation voltage assignment such that $H = G^\phi$. Furthermore, let $\beta : D(H) \to B$ be an ordinary voltage assignment such that $H^\beta = G^\alpha$. By Corollary 2, we have

$$\zeta\big(G^\alpha, \tilde{w}_{G^\alpha}, u, t\big) = \zeta(G, w, u, t) \prod_{j=2}^{n} \zeta_G(w, u, t, \rho_j, \alpha)^{f_j}$$

and

$$\zeta\big(G^\alpha, \tilde{w}_{G^\alpha}, u, t\big) = \zeta\big(H^\beta, \tilde{w}_{G^\alpha}, u, t\big) = \zeta(H, \tilde{w}_H, u, t) \prod_{i=2}^{m} \zeta_H(\tilde{w}_H, u, t, \sigma_i, \beta)^{h_i}.$$

But, by Theorem 1, we have

$$\zeta_H(\tilde{w}_H, u, t, \sigma_i, \beta) = \zeta_G\big(w, u, t, \sigma_i^*, \alpha\big).$$

Set

$$\sigma_i^* = k_{i1} \circ \rho_1 \oplus k_{i2} \circ \rho_2 \oplus \cdots \oplus k_{in} \circ \rho_n \quad (1 \leqslant i \leqslant m).$$

Then we have

$$\begin{aligned}\zeta(G, w, u, t) \prod_{j=2}^{n} \zeta_G(w, u, t, \rho_j, \alpha)^{f_j} &= \zeta(H, \tilde{w}_H, u, t) \prod_{i=2}^{m} \prod_{j=1}^{n} \zeta_G(w, u, t, \rho_j, \alpha)^{k_{ij}h_i} \\ &= \zeta(H, \tilde{w}_H, u, t) \prod_{j=1}^{n} \prod_{i=2}^{m} \zeta_G(w, u, t, \rho_j, \alpha)^{k_{ij}h_i}.\end{aligned}$$

Thus,

$$\zeta(H, \tilde{w}_H, u, t) = \zeta(G, w, u, t)^{1-k_{21}h_2-\cdots-k_{m1}h_m} \prod_{j=2}^{n} \zeta_G(w, u, t, \rho_j, \alpha)^{f_j-k_{2j}h_2-\cdots-k_{mj}h_m}.$$

Therefore it follows that

$$m_j = f_j - k_{2j}h_2 - \cdots - k_{mj}h_m \quad (1 \leqslant j \leqslant n).$$

But, since H is connected, $m_1 = 1$. $\square$

Corollary 3. *Let Γ be a finite group and $B \leqslant \Gamma$ a subgroup of Γ. Let $\sigma_1 = 1, \sigma_2, \dots, \sigma_m$ be the irreducible representations of B, and h_i the degree of σ_i for each i, where $h_1 = 1$. Furthermore, let $\rho_1 = 1, \rho_2, \dots, \rho_n$ be the irreducible representations of Γ, and f_j the degree of ρ_j for each j, where $f_1 = 1$. Set*

$$\sigma_i^* = k_{i1} \circ \rho_1 \oplus k_{i2} \circ \rho_2 \oplus \cdots \oplus k_{in} \circ \rho_n \quad (1 \leqslant i \leqslant m).$$

Then we have

$$k_{i1} = 0 \quad (2 \leqslant i \leqslant m)$$

and

$$f_j \geqslant k_{2j} h_2 + \cdots + k_{mj} h_m \quad (2 \leqslant j \leqslant n).$$

Proof. By the facts that $m_1 = 1$ and $m_j \geqslant 0$ $(2 \leqslant j \leqslant n)$. □

5. Examples

We give an example.

Let G be a connected graph with $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(G) = \{v_1v_2, v_2v_3, v_3v_1, v_3v_4, v_4v_5, v_5v_3\}$, and $\Gamma = S_3$ the symmetric group on $\{1, 2, 3\}$. Furthermore, let $\alpha : D(G) \to \Gamma$ be the ordinary voltage assignment such that $\alpha(v_1, v_2) = (12)$, $\alpha(v_3, v_4) = (23)$ and $\alpha(v_1, v_3) = \alpha(v_2, v_3) = \alpha(v_3, v_5) = \alpha(v_4, v_5) = 1$.

Let $B = \{1, (12)\} \cong \mathbf{Z}_2$ and $H = G^\alpha / B$. Then we have $S_3/B = B \cup B(13) \cup B(23)$. Since B is not a normal subgroup of Γ, H is not regular covering of G. By Proposition 1, the ordinary voltage assignment $\beta : D(H) \to B$ such that $H^\beta = G^\alpha$ is given as follows: $\beta((v_1, B), (v_2, B)) = \beta((v_1, B(23)), (v_2, B(13))) = \beta((v_1, B(13)), (v_2, B(23))) = \beta((v_3, B(13)), (v_4, B(13))) = (12)$ and $\beta(e) = 1$ for each e of H except the above four arcs and their inverse arcs.

Next, let

$$D = \big((v_3, 1), \big(v_4, (23)\big), \big(v_5, (23)\big), \big(v_3, (23)\big), (v_4, 1), (v_5, 1), (v_3, 1)\big)$$

and $\pi_\alpha : G^\alpha \to G$ the natural projection. Then we have $\pi_\alpha(D) = C^2$ and $C = (v_3, v_4, v_5, v_3)$. Furthermore, $\alpha(C) = (23)$ and $ord(\alpha(C)) = 2$.

Let $x = (23)$. Then we have

$$\langle x \rangle = \big\{1, (23)\big\}, \qquad S_3 = \langle x \rangle \cup (12)\langle x \rangle \cup (13)\langle x \rangle.$$

Thus, lifts of C in G^α are D, $(12)D$, $(13)D$.

For the $\pi_B : G^\alpha \to H$, we have

$$\pi_B(D) = K, \qquad \pi_B\big((12)D\big) = K, \qquad \pi_B\big((23)D\big) = L,$$

where $K = ((v_3, B), (v_4, B(23)), (v_5, B(23)), (v_3, B(23)), (v_4, B), (v_5, B), (v_3, B))$ and $L = ((v_3, B(13)), (v_4, B(13)), (v_5, B(13)), (v_3, B(13)))$. Furthermore, $\beta(K) = 1$, $\beta(L) = (12)$.

Now, let $B_1 = \langle \beta(K) \rangle$ and $B_2 = \langle \beta(L) \rangle$. Then $B = B_1 \cup (12)B_1$ and $B = B_2$. Thus, lifts of K in G^α are D, $(12)D$; the lift of L in G^α is $(13)D$. Furthermore,

$$B\langle x \rangle = \langle x \rangle \cup (12)\langle x \rangle \quad \text{and} \quad B(13)\langle x \rangle = (13)\langle x \rangle.$$

Therefore, the set of distinct double cosets of B and $\langle \alpha(C) \rangle$ in Γ is given by

$$B \setminus \Gamma / \big\langle \alpha(C) \big\rangle = \big\{ B\langle x \rangle, B(13)\langle x \rangle \big\}.$$

Now, we construct a permutation voltage assignment $\phi : D(G) \to S_3$ such that $G^\phi = H$. Let $g_1 = 1$, $g_2 = (13)$, $g_3 = (23)$. Then we have $\Gamma = Bg_1 \cup Bg_2 \cup Bg_3$. Furthermore, we have

$$Bg_i \cdot 1 = Bg_i \quad (i = 1, 2, 3);$$
$$Bg_1 \cdot (12) = Bg_1, \qquad Bg_2 \cdot (12) = Bg_3, \qquad Bg_3 \cdot (12) = Bg_2;$$
$$Bg_1 \cdot (23) = Bg_3, \qquad Bg_2 \cdot (23) = Bg_2, \qquad Bg_3 \cdot (23) = Bg_1.$$

Since $\phi(v, w)(i) = j$ if $Bg_j = Bg_i \alpha(v, w)$, $\phi(v_1, v_2) = (23)$, $\phi(v_3, v_4) = (13)$ and $\phi(e) = 1$ for any $e \neq (v_1, v_2), (v_3, v_4) \in D(G)$. It is clear that $G^\phi = H$.

Next, the characters of $B \cong \mathbf{Z}_2$ are given as follows: $\chi_1 = 1$, $\chi_2((12)^i) = (-1)^i$ $(i = 0, 1)$. Furthermore, S_3 has three irreducible representations $\rho_1 = 1$, ρ_2 (the sign representation) and ρ_3 with degrees $f_1 = f_2 = 1$ and $f_3 = 2$, respectively. The representation ρ_3 is given by

$$\rho_3(1) = \mathbf{I}_2, \qquad \rho_3\big((123)\big) = \begin{bmatrix} \eta & 0 \\ 0 & \eta^2 \end{bmatrix}, \qquad \rho_3\big((132)\big) = \begin{bmatrix} \eta^2 & 0 \\ 0 & \eta \end{bmatrix},$$
$$\rho_3\big((12)\big) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \rho_3\big((23)\big) = \begin{bmatrix} 0 & \eta \\ \eta^2 & 0 \end{bmatrix}, \qquad \rho_3\big((13)\big) = \begin{bmatrix} 0 & \eta^2 \\ \eta & 0 \end{bmatrix},$$

where $\eta = \exp \frac{2\pi\sqrt{-1}}{3} = \frac{-1+\sqrt{-3}}{2}$ (see [18]).

Let $\chi_i^* = \mathrm{Ind}_B^\Gamma \chi_i$ be the representation of Γ induced from χ_i. Then by Theorem 3.13 in [11], we have

$$\chi_1^* = \rho_1 + \rho_3 \quad \text{and} \quad \chi_2^* = \rho_2 + \rho_3.$$

By Theorem 1, we have

$$\zeta_H(\tilde{w}_H, u, t, \chi_1, \beta) = \zeta_G\big(w, u, t, \chi_1^*, \alpha\big) = \zeta_G(w, u, t, \rho_1, \alpha)\zeta_G(w, u, t, \rho_3, \alpha) \tag{1}$$

and

$$\zeta_H(\tilde{w}_H, u, t, \chi_2, \beta) = \zeta_G\big(w, u, t, \chi_2^*, \alpha\big) = \zeta_G(w, u, t, \rho_2, \alpha)\zeta_G(w, u, t, \rho_3, \alpha). \tag{2}$$

Furthermore, by Corollary 2, we have

$$\zeta\big(G^{\alpha}, \tilde{w}, u, t\big) = \zeta_H(\tilde{w}_H, u, t, \chi_1, \beta)\zeta_H(\tilde{w}_H, u, t, \chi_2, \beta)$$
$$= \zeta_G(w, u, t, \rho_1, \alpha)\zeta_G(w, u, t, \rho_2, \alpha)\zeta_G(w, u, t, \rho_3, \alpha)^2. \quad (3)$$

But, since $\zeta_H(\tilde{w}_H, u, t, \chi_1, \beta) = \zeta(H, \tilde{w}_H, u, t)$, it follows that

$$\zeta(H, \tilde{w}_H, u, t) = \zeta_G(w, u, t, \rho_1, \alpha)\zeta_G(w, u, t, \rho_3, \alpha) = \zeta(G, w, u, t)\zeta_G(w, u, t, \rho_3, \alpha). \quad (4)$$

In the case that $w(e^{-1}) = w(e)^{-1}$ and $|w(e)| = 1$ for each $e \in D(G)$, by Theorem 6 in [17], we have

$$\zeta_G(w, u, t, \rho, \alpha)^{-1} = \big(1 - (1-u)^2 t^2\big)^{(l-n)d} \det\Big(\mathbf{I}_{nd} - t \sum_{h \in \Gamma} \rho(h) \bigotimes \mathbf{W}_h$$
$$+ (1-u)t^2\Big(\mathbf{I}_d \bigotimes \mathbf{D} - (1-u)\mathbf{I}_{nd}\Big)\Big), \quad (5)$$

where $n = |V(G)|$, $l = |E(G)|$, $d = \deg \rho$, and $\mathbf{W}_h = (w_{uv}^{(h)})$ is given by

$$w_{uv}^{(h)} := \begin{cases} w(u, v) & \text{if } (u, v) \in D(G) \text{ and } \alpha(u, v) = h, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\mathbf{W}(G) = \begin{bmatrix} 0 & a & b & 0 & 0 \\ \bar{a} & 0 & c & 0 & 0 \\ \bar{b} & \bar{c} & 0 & p & q \\ 0 & 0 & \bar{p} & 0 & r \\ 0 & 0 & \bar{q} & \bar{r} & 0 \end{bmatrix}.$$

Then the matrices $\mathbf{W}_h$ $(h \in S_3)$ are given as follows:

$$\mathbf{W}_1 = \begin{bmatrix} 0 & 0 & b & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ \bar{b} & \bar{c} & 0 & 0 & q \\ 0 & 0 & 0 & 0 & r \\ 0 & 0 & \bar{q} & \bar{r} & 0 \end{bmatrix}, \qquad \mathbf{W}_{(12)} = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ \bar{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\mathbf{W}_{(23)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 \\ 0 & 0 & \bar{p} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\mathbf{W}_{(13)} = \mathbf{W}_{(123)} = \mathbf{W}_{(132)} = \mathbf{0}_3.$$

By (5), we have

$$
\begin{aligned}
\zeta_G(w,u,t,\rho_1,\alpha)^{-1} &= \zeta(G,w,u,t)^{-1}\\
&= \bigl(1-(1-u)^2t^2\bigr)\det\Bigl(\mathbf{I}_5 - t\sum_{h\in S_3}\mathbf{W}_h + (1-u)t^2\bigl(\mathbf{D}-(1-u)\mathbf{I}_5\bigr)\Bigr)\\
&= \bigl(1-(1-u)^2t^2\bigr)\det\left(\begin{bmatrix} x & -at & -bt & 0 & 0\\ -\bar{a}t & x & -ct & 0 & 0\\ -\bar{b}t & -\bar{c}t & y & -pt & -qt\\ 0 & 0 & -\bar{p}t & x & -rt\\ 0 & 0 & -\bar{q}t & -\bar{r}t & x \end{bmatrix}\right)\\
&= \bigl(1-(1-u)^2t^2\bigr)\bigl(x^2-t^2\bigr)\\
&\quad\times\bigl(x^2y-4t^2x-t^2y-(a\bar{b}c+\bar{a}b\bar{c}+p\bar{q}r+\bar{p}q\bar{r})t^3\bigr),
\end{aligned}
$$

where $x=1+(1-u^2)t^2$ and $y=1+(3-2u-u^2)t^2$.

Next, we have

$$
\begin{aligned}
\zeta_G(w,u,t,\rho_2,\alpha)^{-1} &= \bigl(1-(1-u)^2t^2\bigr)\det\Bigl(\mathbf{I}_5 - t\sum_{h\in S_3}\rho_2(h)\mathbf{W}_h + (1-u)t^2\bigl(\mathbf{D}-(1-u)\mathbf{I}_5\bigr)\Bigr)\\
&= \bigl(1-(1-u)^2t^2\bigr)\det\left(\begin{bmatrix} x & at & -bt & 0 & 0\\ \bar{a}t & x & -ct & 0 & 0\\ -\bar{b}t & -\bar{c}t & y & pt & -qt\\ 0 & 0 & \bar{p}t & x & -rt\\ 0 & 0 & -\bar{q}t & -\bar{r}t & x \end{bmatrix}\right)\\
&= \bigl(1-(1-u)^2t^2\bigr)\bigl(x^2-t^2\bigr)\\
&\quad\times\bigl(x^2y-4t^2x-t^2y+(a\bar{b}c+\bar{a}b\bar{c}+p\bar{q}r+\bar{p}q\bar{r})t^3\bigr).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\zeta_G(w,u,t,\rho_3,\alpha)^{-1}\\
&= \bigl(1-(1-u)^2t^2\bigr)^2\det\Bigl(\mathbf{I}_{10} - t\sum_{h\in S_3}\rho_3(h)\bigotimes\mathbf{W}_h + (1-u)t^2\Bigl(\mathbf{I}_2\bigotimes\mathbf{D}-(1-u)\mathbf{I}_{10}\Bigr)\Bigr)\\
&= \bigl(1-(1-u)^2t^2\bigr)^2\\
&\quad\times\det\left(\begin{bmatrix}
x & 0 & -bt & 0 & 0 & 0 & -at & 0 & 0 & 0\\
0 & x & -ct & 0 & 0 & -\bar{a}t & 0 & 0 & 0 & 0\\
-\bar{b}t & -\bar{c}t & y & 0 & -qt & 0 & 0 & 0 & -\eta pt & 0\\
0 & 0 & 0 & x & -rt & 0 & 0 & -\eta\bar{p}t & 0 & 0\\
0 & 0 & -\bar{q}t & -\bar{r}t & x & 0 & 0 & 0 & 0 & 0\\
0 & -at & 0 & 0 & 0 & x & 0 & -bt & 0 & 0\\
-\bar{a}t & 0 & 0 & 0 & 0 & 0 & x & -ct & 0 & 0\\
0 & 0 & 0 & -\eta^2pt & 0 & -\bar{b}t & -\bar{c}t & y & 0 & -qt\\
0 & 0 & -\eta^2\bar{p}t & 0 & 0 & 0 & 0 & 0 & x & -rt\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{q}t & -\bar{r}t & x
\end{bmatrix}\right)
\end{aligned}
$$

$$= \big(1-(1-u)^2t^2\big)^2\big(x^2-t^2\big)^2\big\{\big(x^2y-yt^2-3xt^2\big)\big(x^2y-yt^2-4xt^2\big) - \big(1+(p\bar{q}r)^2+(\bar{p}q\bar{r})^2-(a\bar{b}c+\bar{a}b\bar{c})(p\bar{q}r+\bar{p}q\bar{r})+(a\bar{b}c+\bar{a}b\bar{c})^2\big)t^6 - x^3y+xyt^2+4x^2t^2-t^4\big\}.$$

Therefore, we can calculate $\zeta(H,\tilde{w}_H,u,t)$, $\zeta_H(\tilde{w}_H,u,t,\chi_2,\beta)$ and $\zeta(G^\alpha,\tilde{w},u,t)$ by (1), (2), (3) and (4).

6. Generalization to digraphs

Deng, Sato and Wu [5] generalized the notion of a covering of a graph to general digraphs. Let D be a connected simple digraph. A digraph H is called a *covering* of D with projection $\pi : H \to D$ if there is a surjection $\pi : V(H) \to V(D)$ such that both $\pi|_{N_H^+(v')} : N_H^+(v') \to N_D^+(v)$ and $\pi|_{N_H^-(v')} : N_H^-(v') \to N_D^-(v)$ are bijections for all vertices $v \in V(D)$ and $v' \in \pi^{-1}(v)$, where $N_D^+(v) = \{e \in A(D) \mid o(e) = v\}$ and $N_D^-(v) = \{e \in A(D) \mid t(e) = v\}$, etc. The projection $\pi : H \to D$ is an *n-fold covering* of D if π is n-to-one.

Let D be a connected digraph and S_n the symmetric group on $N = \{1,2,\dots,n\}$. Let $A(D)$ be the set of arcs in D. Then a mapping $\alpha : A(D) \to S_n$ is called a *permutation voltage assignment*. The pair (D,α) is called a *permutation voltage digraph*. The *derived digraph* D^α of the permutation voltage digraph (D,α) is defined as follows: $V(D^\alpha) = V(D) \times N = \{u_i \mid u \in V(D),\ i \in N\}$, $A(D^\alpha) = A(D) \times N = \{e_i \mid e \in A(D),\ i \in N\}$ and e_i is an arc from u_j to v_k if and only if $e \in A(D)$ and $k = \alpha(e)(j)$. The digraph D^α is called an *n-covering* of D. The *natural projection* $\pi_\alpha : D^\alpha \to S_n$ is defined by $\pi_\alpha(u,h) = u$. The n-covering D^α is an n-fold covering of D. Note that an n-covering of the symmetric digraph D corresponding to a graph G is an n-covering of G. Furthermore, the following fact is shown by Deng, Sato and Wu [5].

Theorem 4 *(Deng, Sato and Wu). Let $\pi : \tilde{D} \to D$ be an n-fold covering of a connected digraph D. Then there exists a permutation voltage assignment $\alpha : A(D) \to S_n$ such that the n-covering D^α is isomorphic to $\tilde{D}$.*

A permutation voltage assignment $\alpha : A(D) \to S_n$ of D is called *symmetric* if $\alpha(w,v) = \alpha(v,w)^{-1}$ for each $(v,w) \in A(D)$ such that $(w,v) \in A(D)$. Similarly to the proof of Theorem 4, we obtain the following result.

Corollary 4. *Let $\pi : \tilde{D} \to D$ be an n-fold covering of a connected digraph D such that the preimage of each symmetric arcs in D consists of n symmetric arcs of $\tilde{D}$. Then there exists a symmetric permutation voltage assignment $\alpha : A(D) \to S_n$ such that the n-covering D^α is isomorphic to $\tilde{D}$.*

Dend and Wu [4] generalized the notion of a covering of a graph by an ordinary voltage assignment to general digraphs. Let D be a digraph and let Γ be a group such that, for each $g \in \Gamma$, there is a digraph isomorphism $\phi_g : D \to D$ and the following three conditions hold:

1. If 1 is the unit of Γ, then $\phi_1 : D \to D$ is the identity isomorphism.
2. $\phi_{gh} = \phi_g \circ \phi_h$ for all $g, h \in \Gamma$.
3. For any $g \neq 1 \in \Gamma$, there is no vertex v of D such that $\phi_g(v) = v$ and no arc e of D such that $\phi_g(e) = e$.

Then we say that the group Γ *acts freely* on the left of the digraph D.

Let $\pi : H \to D$ be an n-fold digraph covering, and let Γ be a group of order n which acts freely on H such that the following two conditions hold:

1. For any two vertices u and v of H such that $\pi(u) = \pi(v)$, there is an element $g \in \Gamma$ such that $\phi_g(u) = v$.
2. For any two arcs e and f of H such that $\pi(e) = \pi(f)$, there is an element $g \in \Gamma$ such that $\phi_g(e) = f$.

Then $\pi : H \to D$ is called a *regular covering*, and Γ is called the *covering transformation group* for $\pi : H \to D$.

Let D be a connected digraph and Γ a finite group. Then a mapping $\alpha : A(D) \to \Gamma$ is called an *ordinary voltage assignment*. The pair (D, α) is called an *ordinary voltage digraph*. The *derived digraph* D^α of the ordinary voltage digraph (D, α) is defined as follows: $V(D^\alpha) = V(D) \times \Gamma = \{u_g \mid u \in V(D),\ g \in \Gamma\}$, $A(D^\alpha) = A(D) \times \Gamma = \{e_g \mid e \in A(D),\ g \in \Gamma\}$ and e_g is an arc from u_g to v_h if and only if $e \in A(D)$ and $h = g\alpha(e)$. The digraph D^α is called a *Γ-covering* of D. The *natural projection* $\pi_\alpha : D^\alpha \to D$ is defined by $\pi_\alpha(u, h) = u$. Note that the Γ-covering D^α is a regular $|\Gamma|$-fold covering of D. Furthermore, every regular n-fold covering of a connected digraph D is a Γ-covering D^α of D for some group Γ and some ordinary voltage assignment $\alpha : A(D) \to \Gamma$ (see [4]). Note that a Γ-covering of the symmetric digraph corresponding to a graph G is a Γ-covering of G.

An ordinary voltage assignment $\alpha : A(D) \to S_n$ of D is called *symmetric* if $\alpha(w, v) = \alpha(v, w)^{-1}$ for each $(v, w) \in A(D)$ such that $(w, v) \in A(D)$. Similarly to the proof of [4, Lemma 2.2], we obtain the following corollary.

Corollary 5. *Let $\pi : \tilde{D} \to D$ be an n-fold regular covering of a connected digraph D such that the preimage of each symmetric arcs in D consists of n symmetric arcs of $\tilde{D}$. Then there exist a finite group Γ and a symmetric ordinary voltage assignment $\alpha : A(D) \to \Gamma$ such that the Γ-covering D^α is isomorphic to $\tilde{D}$.*

For quotients of regular coverings of digraphs, we have analogue of the properties for quotients of regular coverings of graphs stated in Section 2.

Proposition 3. *Let D be a connected digraph, Γ a finite group and $\alpha : A(D) \to \Gamma$ be a symmetric ordinary voltage assignment. Furthermore, let $B \leqslant \Gamma$ be a subgroup of Γ and $H = D^\alpha/B$. If $\beta : A(H) \to B$ is an ordinary voltage assignment such that $H^\beta = D^\alpha$, then*

$$\beta\big((v, Bg), \big(w, Bg'\big)\big) = g\alpha(v, w)g'^{-1}$$

for each $(v, w) \in A(D)$ and each $g, g' \in \Gamma$.

Proposition 4. *Let D be a connected digraph, Γ a finite group and $\alpha : A(D) \to \Gamma$ be a symmetric ordinary voltage assignment. Furthermore, let $B \leqslant \Gamma$ be a subgroup of Γ and $H = D^\alpha/B$. For any prime cycle C in D, the number of lifts of C in H is equal to the cardinality of the set $B \setminus \Gamma/\langle\alpha(C)\rangle$ of distinct double cosets of B and $\langle\alpha(C)\rangle$ in Γ, and the length of each lift of C in H is of form*

$$|C| \cdot \frac{ord(\alpha(C))}{|B|} \cdot \left|B\colon\ B \cap y_j \langle \alpha(C) \rangle y_j^{-1}\right|,$$

where $y_1 = 1, y_2, \ldots, y_s$ are the representatives of distinct double cosets of B and $\langle \alpha(C) \rangle$ in Γ.

7. Weighted Bartholdi L-functions of digraphs

Let D be a connected graph with n vertices $v_1, \ldots, v_n$ and m arcs. Then we consider an $n \times n$ matrix $\mathbf{W} = \mathbf{W}(D) = (w_{ij})_{1 \leqslant i,j \leqslant n}$ with ij entry the complex variable $w_{ij} \neq 0$ if $(v_i, v_j) \in A(D)$, and $w_{ij} = 0$ otherwise. The matrix $\mathbf{W} = \mathbf{W}(D)$ is called the *weighted matrix* of D. Furthermore, let $w(v_i, v_j) = w_{ij}$, $v_i, v_j \in V(D)$ and $w(e) = w_{ij}$, $e = (v_i, v_j) \in A(D)$. For each path $P = (e_1, \ldots, e_r)$ of G, the *norm* $w(P)$ of P is defined as follows: $w(P) = w(e_1) \cdots w(e_r)$.

Let Γ be a finite group and $\alpha : A(D) \to \Gamma$ a symmetric ordinary voltage assignment. For each path $P = (e_1, \ldots, e_r)$ of D, set $\alpha(P) = \alpha(e_1) \cdots \alpha(e_r)$. This is called the *net voltage* of P. Furthermore, let ρ be a representation of Γ and d its degree.

The *weighted Bartholdi L-function* of D associated with ρ and α is defined by

$$\zeta_D(w, u, t, \rho, \alpha) = \prod_{[C]} \det\left(\mathbf{I}_d - w(C)\rho\left(\alpha(C)\right)u^{cbc(C)}t^{|C|}\right)^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of D (see [3,16]). If $\rho = \mathbf{1}$, then the weighted Bartholdi L-function of D is called the *weighted Bartholdi zeta function* of D, denoted by $\zeta(D, w, u, t) = \zeta_D(w, u, t, \mathbf{1}, \alpha)$.

The following theorem holds similarly to the proof of Theorem 1.

Theorem 5. *Let D be a connected digraph, $\mathbf{W}(D)$ a weighted matrix of D, Γ a finite group and $\alpha : A(D) \to \Gamma$ a symmetric ordinary voltage assignment. Furthermore, let $B \leqslant \Gamma$ be a subgroup of Γ and $H = D^\alpha / B$. Assume that D^α is connected. Let σ be any representation of B and $\sigma^* = \mathrm{Ind}_B^\Gamma \sigma$ the representation of Γ induced from σ. Let $\beta : D(H) \to B$ be an ordinary voltage assignment such that $H^\beta = D^\alpha$. Then we have*

$$\zeta_H(\tilde{w}_H, u, t, \sigma, \beta) = \zeta_D\left(w, u, t, \sigma^*, \alpha\right).$$

Corollary 6. *Let D be a connected digraph, $\mathbf{W}(D)$ a weighted matrix of D, Γ a finite group and $\alpha : A(D) \to \Gamma$ a symmetric ordinary voltage assignment. Then we have*

$$\zeta\left(D^\alpha, \tilde{w}_{D^\alpha}, u, t\right) = \prod_\rho \zeta_D(w, u, t, \rho, \alpha)^{\deg \rho},$$

where ρ runs over all inequivalent irreducible representations of Γ.

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References

[1] L. Bartholdi, Counting paths in graphs, Enseign. Math. 45 (1999) 83–131.

[2] H. Bass, The Ihara–Selberg zeta function of a tree lattice, Internat. J. Math. 3 (1992) 717–797.

[3] Y.B. Choe, J.H. Kwak, Y.S. Park, I. Sato, Bartholdi zeta and L-functions of weighted digraph, their coverings and products, Adv. Math. 213 (2007) 865–886.

[4] A. Deng, Y. Wu, Characteristic polynomials of digraphs having a semi-free action, Linear Algebra Appl. 408 (2005) 189–206.

[5] A. Deng, I. Sato, Y. Wu, Characteristic polynomials of ramified uniform covering digraphs, Linear Algebra Appl. 423 (2007) 386–407.

[6] D. Foata, D. Zeilberger, A combinatorial proof of Bass's evaluations of the Ihara–Selberg zeta function for graphs, Trans. Amer. Math. Soc. 351 (1999) 2257–2274.

[7] J.L. Gross, T.W. Tucker, Generating all graph coverings by permutation voltage assignments, Discrete Math. 18 (1977) 273–283.

[8] J.L. Gross, T.W. Tucker, Topological Graph Theory, Wiley–Interscience, New York, 1987.

[9] K. Hashimoto, Zeta functions of finite graphs and representations of p-adic groups, in: Adv. Stud. Pure Math., vol. 15, Academic Press, New York, 1989, pp. 211–280.

[10] Y. Ihara, On discrete subgroups of the two by two projective linear group over p-adic fields, J. Math. Soc. Japan 18 (1966) 219–235.

[11] N. Iwahori, Representation Theory of Symmetric Groups and General Linear Groups, Iwanami Shoten, Tokyo, 1978 (in Japanese).

[12] M. Kotani, T. Sunada, Zeta functions of finite graphs, J. Math. Sci. Univ. Tokyo 7 (2000) 7–25.

[13] H. Mizuno, I. Sato, Zeta functions of graph coverings, J. Combin. Theory Ser. B 80 (2000) 247–257.

[14] H. Mizuno, I. Sato, Bartholdi zeta functions of graph coverings, J. Combin. Theory Ser. B 89 (2003) 27–41.

[15] H. Mizuno, I. Sato, Weighted zeta functions of graphs, J. Combin. Theory Ser. B 91 (2004) 169–183.

[16] H. Mizuno, I. Sato, Weighted Bartholdi zeta functions of digraphs, Far East J. Math. Sci. 27 (2007) 323–344.

[17] I. Sato, Weighted Bartholdi zeta functions of graphs, European J. Combin. 27 (2006) 644–657.

[18] J.-P. Serre, Linear Representations of Finite Group, Springer-Verlag, New York, 1977.

[19] H.M. Stark, A.A. Terras, Zeta functions of finite graphs and coverings, Adv. Math. 121 (1996) 124–165.

[20] H.M. Stark, A.A. Terras, Zeta functions of finite graphs and coverings, part II, Adv. Math. 154 (2000) 132–195.

[21] T. Sunada, L-functions in geometry and some applications, in: Lecture Notes in Math., vol. 1201, Springer-Verlag, New York, 1986, pp. 266–284.

[22] T. Sunada, Fundamental Groups and Laplacians, Kinokuniya, Tokyo, 1988 (in Japanese).

[23] M. Suzuki, Group Theory, vol. 1, Iwanami Shoten, Tokyo, 1977 (in Japanese).

[24] S. Suzuki, Linear Topology of Surfaces, vol. 2, Maki Shoten, Tokyo, 1987 (in Japanese).