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Frank Separation of Two Convex Sets and Hahn–Banach Theorem

J. VANGELDÈRE

Institut de Mathématique, University of Liège, Belgium

Submitted by Ky Fan

First we give some elementary properties of the core of a subset relative to a linear subspace. Then we prove a theorem on the frank separation of two convex sets. This theorem admits as particular cases the known theorems on the frank separation and introduces new cases. Finally, we provide a very general version of the Hahn–Banach theorem in an analytic form.

INTRODUCTION

Throughout this paper E denotes a nonzero vector space over the field $\mathbb{R}$ of real numbers. We use notations introduced elsewhere [2, 3, 6, 11, 12]. If a, b are two points of E, $(a\colon b)$ denotes the set of points $\{a + \lambda(b - a) : \lambda \in \mathbb{R}\}$. Then if a, b are distinct, $(a\colon b)$ is the straight line determined by these points. The subsets of $(a\colon b)$ characterized by $0 \leqslant \lambda \leqslant 1$, $0 < \lambda \leqslant 1$, $0 \leqslant \lambda < 1$, $0 < \lambda < 1$, $0 \leqslant \lambda$, and $0 < \lambda$ are represented by $[a\colon b]$, $]a\colon b]$, $[a\colon b[$, $]a\colon b[$, $[a\colon b)$, and $]a\colon b)$, respectively.

If A is a subset of E, ${}^lA$ and ${}^cA$ are the *affine and convex hulls* of A, respectively, that is, the smallest affine variety and the smallest convex set containing A.

A straight line $(a\colon b)$ *inserts* a point x into a subset A when there exist two distinct points c and d of $(a\colon b)$ such that $x \in]c\colon d[\subset A$. The *intrinsic core* of A is the set of all points x such that all straight lines of ${}^lA$ through x insert x into A and is denoted by ${}^iA$; it is also called the *core* of A when ${}^lA = E$.

If T is a nonvoid affine variety, T_0 represents the linear subspace of E parallel to T; in particular, if A is nonvoid, $({}^lA)_0$ is the linear subspace of E parallel to ${}^lA$.

Whatever the subsets A and B may be, ${}^l(A + B) = {}^lA + {}^lB$ [6, p. 540]. Then if A and B are nonvoid, $({}^l(A + B))_0 = ({}^lA)_0 + ({}^lB)_0$; moreover, if T, U are two nonvoid affine varieties and if α, β are two nonzero reals, $(\alpha T + \beta U)_0 = T_0 + U_0$.

Let A be a subset and V be a linear subspace of E. The *core of A relative to V* is the set of all points x satisfying the following property: For any $v \in V$, there exists a number $\eta > 0$ such that $x + \alpha v \in A$ ($\alpha \in \mathbb{R}$ and $|\alpha| < \eta$). This set is denoted by ${}^{i(V)}A$.

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When $V = ({}^lA)_0$, ${}^{i(V)}A$ is the intrinsic core of A. If $V = \{0\}$, ${}^{i(V)}A = A$. This notion is not new; it was introduced in a slightly more general manner by Klee [8, pp. 238–239], but to date it does not seem to have been employed very often.

LEMMA 1. *Let V, W be two linear subspaces and A be a subset of E.*

(a) *If $V \subset W$, then ${}^{i(W)}A \subset {}^{i(V)}A$.*

(b) *If A is convex, ${}^{i(V)}A \cap {}^{i(W)}A \subset {}^{i(V+W)}A$.*

Proof. (a) is immediate.

(b) follows from the convexity of A and the equality

$$x + \alpha(v + w) = \tfrac{1}{2}(x + 2\alpha v) + \tfrac{1}{2}(x + 2\alpha w).$$

LEMMA 2. *For every linear subspace V and all subsets A, B of E,*

$${}^{i(V)}(A \cap B) = {}^{i(V)}A \cap {}^{i(V)}B.$$

Proof. Obvious.

LEMMA 3. *Whatever the nonzero real λ and the linear subspace V may be,* ${}^{i(V)}(\lambda A) = \lambda\, {}^{i(V)}A$.

Proof. Obvious.

LEMMA 4. *Whatever the point x and the linear subspace V may be,* ${}^{i(V)}(x + A) = x + {}^{i(V)}A$.

Proof. Obvious.

LEMMA 5. *Let A be a convex subset and V be a linear subspace of E.*

(a) *If $a \in {}^{i(V)}A$ and $b \in A$, then $[a: b[\; \subset {}^{i(V)}A$.*

(b) ${}^{i(V)}({}^{i(V)}A) = {}^{i(V)}A$.

Proof. (a) Let $x = a + \epsilon(b - a)$, $0 \leqslant \epsilon < 1$, be an arbitrary point of $[a: b[$ and any point v of V. Since $a \in {}^{i(V)}A$, there exists $\eta > 0$ such that $a + \alpha v \in A$ when $|\alpha| < \eta$. Then $x \in {}^{i(V)}A$ results from the convexity of A and from the fact that $x + \alpha v = a + (\alpha/(1 - \epsilon))\, v + \epsilon(b - a - (\alpha/(1 - \epsilon))\, v)$.

(b) is a direct consequence of (a) and of the definitions.

LEMMA 6. *Let V, W be two linear subspaces and A, B be two subsets of E.*

(a) ${}^{i(V)}A + {}^{i(W)}B \subset {}^{i(V+W)}(A + B)$.

(b) *If A and B are convex, if $V + W = ({}^lA)_0 + ({}^lB)_0$, and if ${}^{i(V)}A \neq \varnothing$ and ${}^{i(W)}B \neq \varnothing$, then ${}^{i(V)}A + {}^{i(W)}B = {}^{i(V+W)}(A + B)$.*

Proof. If $x \in {}^{i(V)}A$, $y \in {}^{i(W)}B$, $v \in V$, and $w \in W$, there exists $\eta > 0$ such that $x + \alpha v \in A$ and $b + \alpha w \in B$ when $|\alpha| < \eta$. Consequently, $x + y + \alpha(v + w) \in A$ when $|\alpha| < \eta$ and the proof of (a) is complete.

Concerning (b), there exist $a \in {}^{i(V)}A$, $b \in {}^{i(W)}B$ and we just have to verify that an arbitrary point u of ${}^{i(V+W)}(A + B)$ belongs to ${}^{i(V)}A + {}^{i(W)}B$. The simple case $V + W = \{0\}$ may be omitted.

Since $V + W = ({}^{l}A)_0 + ({}^{l}B)_0 = ({}^{l}(A + B))_0$, we have ${}^{i(V+W)}(A + B) = {}^{i}(A + B)$, $a + b \in {}^{i}(A + B)$, and $u \in {}^{i}(A + B)$. The straight line $(a + b: u)$ (it can be supposed that $u \neq a + b$) inserts u into $A + B$. There exist $m \in A$, $n \in B$ such that $u \in]a + b: m + n[$. Thus, it is possible to find a real ϵ $(0 < \epsilon < 1)$ such that $u = m + \epsilon(a - m) + n + \epsilon(b - n)$ and by Lemma 5 the proof is complete.

LEMMA 7. *If A is a convex set and V is a finite-dimensional linear subspace such that $V \subset ({}^{l}A)_0$, then ${}^{i(V)}A \neq \varnothing$.*

Proof. Let a be an arbitrary point of A (such a point exists since $V \subset ({}^{l}A)_0$). The linear subspace spanned by $A - a$ coincides with $({}^{l}A)_0$ and every point of V can be written as a linear combination of finitely many points of $A - a$. In particular, a basis for V introduces only a finite subset F of $A - a$. The linear subspace W spanned by F includes V and admits as a basis a maximal linearly independent subset $\{b_1, b_2, \ldots, b_k\}$ of F. The polytope P determined by the $k + 1$ points a, $a + b_1, \ldots, a + b_k$ is then included in A and its intrinsic core is nonvoid (for instance, the barycenter of a polytope belongs to its intrinsic core). Now ${}^{i}P \subset {}^{i(W)}A$; thus, ${}^{i(W)}A \neq \varnothing$ and a fortiori ${}^{i(V)}A \neq \varnothing$.

FRANK SEPARATION OF TWO CONVEX SETS

Recall that two nonvoid subsets A and B of E are *frankly separated* by a hyperplane H (respectively, by a nonnull linear form f) when H (respectively, a hyperplane parallel to the kernel of f) separates A and B and does not include $A \cup B$. The subsets A and B are called *frankly separated* when there exists a hyperplane (or equivalently, a nonnull linear form) which frankly separates them [2, p. 23; 3, p. 474].

THEOREM 1. *Let A, B be two convex sets and V, W be two linear subspaces of E such that $V + W = ({}^{l}A)_0 + ({}^{l}B)_0$, ${}^{i(V)}A \neq \varnothing$, and ${}^{i(W)}B \neq \varnothing$. Then A and B are frankly separated if and only if ${}^{i(V)}A \cap {}^{i(W)}B = \varnothing$.*

Taking account of Lemmas 3 and 6 and of the fact that $({}^{l}(A - B))_0 = ({}^{l}A)_0 + ({}^{l}B)_0 = ({}^{l}(A + B))_0$, we have ${}^{i(V)}A - {}^{i(W)}B = {}^{i(V+W)}(A - B) = {}^{i}(A - B)$. Consequently, the nonemptiness of ${}^{i(V)}A$ and ${}^{i(W)}B$ implies that ${}^{i}(A - B) \neq \varnothing$; hence, the condition ${}^{i(V)}A \cap {}^{i(W)}B = \varnothing$ is equivalent to

$0 \notin {}^{i}(A - B)$. Then the result follows from the fact that the following three properties are equivalent: A and B are frankly separated, $\{0\}$ and $A - B$ are frankly separated, 0 does not belong to ${}^{i}(A - B)$ [3, 1.2 and 1.3, p. 475].

THEOREM 2. *Let A, B be two convex sets and V, W be two linear subspaces of E such that $V + W = ({}^{l}A)_0 + ({}^{l}B)_0$. If A and B are frankly separated by a hyperplane H and if $a \in {}^{i(V)}A$ and $b \in {}^{i(W)}B$, then $a \neq b$ and the straight line $(a: b)$ meets H in one and only one point necessarily lying in $[a: b]$.*

That a and b are distinct is a direct consequence of Theorem 1.

It is not possible to have $(a: b) \subset H$ and $V \cup W \subset H_0$ since, under these conditions, $A \cup B \subset {}^{l}A \cup {}^{l}B = [a + ({}^{l}A)_0] \cup [b + ({}^{l}B)_0] \subset (a + V + W) \cup (b + V + W) \subset H \cup H = H$ and the subsets A and B are not frankly separated. Thus, if a and b both belong to H, there exists $v \in V \cup W$ (let us suppose that $v \in V$) such that $v \notin H_0$. This situation is also impossible. Indeed, by virtue of the hypothesis $a \in {}^{i(V)}A$, there exists a number $\eta > 0$ such that $x + \alpha v \in A$ $(|\alpha| < \eta)$; due to the fact that $a \in H$ and $v \notin H_0$, these points lie on both sides of H. Thus A and B are not frankly separated by H.

Consequently, $(a: b)$ meets H in one and only one point and the rest of the theorem is a straightforward consequence of the frank separation of A and B.

Remark. In these theorems, the hypothesis $V + W = ({}^{l}A)_0 + ({}^{l}B)_0$ could be replaced by $V + W \supset ({}^{l}A)_0 + ({}^{l}B)_0$. Indeed, $V \subset ({}^{l}A)_0$ and $W \subset ({}^{l}B)_0$ are direct consequence of ${}^{i(V)}A \neq \varnothing$ and ${}^{i(W)}B \neq \varnothing$.

THEOREM 3. *Let A, B be two convex sets and V, W be two linear subspaces of E. If A and B are frankly separated by a hyperplane H and if V (respectively W) is not included in H_0, then ${}^{i(V)}A \cap H = \varnothing$ (respectively ${}^{i(W)}B \cap H = \varnothing$).*

The last step in the proof of Theorem 2 yields this theorem.

COROLLARY 1. *Let A, B be two convex sets and V_1, V_2, W_1, W_2 be four linear subspaces of E such that $V_1 + W_1 = V_2 + W_2 = ({}^{l}A)_0 + ({}^{l}B)_0$. If $V_1 \subset V_2$, $W_1 \subset W_2$, ${}^{i(V_2)}A \neq \varnothing$, and ${}^{i(W_2)}B \neq \varnothing$, then the condition ${}^{i(V_2)}A \cap {}^{i(W_2)}B = \varnothing$ is equivalent to ${}^{i(V_1)}A \cap {}^{i(W_1)}B = \varnothing$.*

This corollary is a consequence of Lemma 1 and Theorem 1.

COROLLARY 2. *Let A and B be two convex sets of E such that $B \neq \varnothing$, ${}^{i}A \neq \varnothing$, and ${}^{l}A = E$. Then A and B are frankly separated if and only if ${}^{i}A \cap B = \varnothing$.*

We just need to take $V = E$ and $W = \{0\}$ in Theorem 1.

COROLLARY 3. *Two convex subsets A and B of E, whose intrinsic cores are nonvoid, are frankly separated if and only if ${}^{i}A \cap {}^{i}B = \varnothing$.*

It suffices to take $V = ({}^lA)_0$ and $W = ({}^lB)_0$ in Theorem 1.

COROLLARY 4. *Let us suppose that A and B are two nonvoid convex sets of E. If the codimension of ${}^lA$ is finite, if ${}^iA \neq \varnothing$, and if ${}^iA \cap B = \varnothing$, then there exists a frank separation of A and B.*

Let $W \subset ({}^lB)_0$ be an algebraic complement of $V = ({}^lA)_0$ in $({}^lA)_0 + ({}^lB)_0$. Since the codimension of A is finite, W is a finite-dimensional linear subspace. Now, the corollary is a simple consequence of Lemma 7, Theorem 1, and the fact that ${}^iA \cap B = \varnothing$ implies ${}^iA \cap {}^{i(W)}B = \varnothing$.

Remark. Corollary 2 is the oldest form of the frank separation. It is mentioned, for instance, in [2, p. 23; 7, 8.10, p. 456; 8, 2.2, p. 253]. Corollary 3 is mentioned in [2, 2.2.1, p. 26; 3, 1.3, p. 475] and admits as a particular case a theorem of [6, p. 541]. Corollary 4 is given in [2, 2.2.3(a), p. 28; 3, 1.4, p. 475].

Thus, Theorem 1 unifies different theorems on frank separation. Furthermore, it introduces a more immediate symmetry between the sets to separate. Finally, it leads to new cases. Here is an example of two sets A and B which satisfy the hypotheses of Theorem 1 but not those of Corollaries 2, 3, and 4.

Let $N = \{1, 2, 3,...\}$ be the set of natural integers and N_1, N_2 be the subsets of N formed by the odd and even integers, respectively. Let us take as a vector space E the space $\mathbb{R}^N$ endowed with the usual laws of composition and let us represent by e_n $(n \in N)$ the point of E whose nth coordinate is 1 and the others are 0. If

$$V = \{(x_i)_{i \in N}: x_i = 0 \text{ when } i \in N_2, \sup_{i \in N_1} |x_i| \text{ exists}\},$$

$$W = \{(x_i)_{i \in N}: x_i = 0 \text{ when } i \in N_1, \sup_{i \in N_2} |x_i| \text{ exists}\},$$

T_1 is the linear subspace spanned by $\{e_n: n \in N_1\}$ and T_2 this spanned by $\{e_n: n \in N_2\}$, then V, W are linear subspaces of E and $T_1 \subset V$, $T_2 \subset W$.

Let us consider $A = \{(x_i)_{i \in N}: x_i > 0$ when $i \in N_1$, $\sup_{i \in N_1} x_i$ exists, $x_i \geqslant 0$ when $i \in N_2$ and $x_i \neq 0$ only for a finite number of $i \in N_2\}$. A is a convex cone with vertex at 0. Then $({}^lA)_0 = {}^lA = A - A$ (general property of nonvoid convex cones).

Let $v = (v_i)_{i \in N}$ be an arbitrary point of V. If $m = (m_i)_{i \in N}$ is such that $m_i = 2v_i$ whenever $v_i > 0$, $m_i = 1$ whenever $i \in N_1$ and $v_i = 0$, $m_i = -v_i$ whenever $v_i < 0$, $m_i = 0$ whenever $i \in N_2$, if $m' = (m_i')_{i \in N}$ is such that $m_i' = v_i$ whenever $v_i > 0$, $m_i' = 1$ whenever $i \in N_1$ and $v_i = 0$, $m_i' = -2v_i$ whenever $v_i < 0$, $m_i' = 0$ whenever $i \in N_2$, then $v = m - m'$. Thus $V \subset A - A$. It is also easy to prove that $T_2 \subset A - A$ and that $({}^lA)_0 = {}^lA = V + T_2$.

If $a = (a_i)_{i \in N}$ is an arbitrary point of A, there exists $k \in N_2$ such that $a_k = 0$. The straight line through a and parallel to $(0: e_k)$ does not insert a into A. Thus ${}^iA \neq \varnothing$.

In other respects, if $u = (u_i)_{i \in N}$ is a point of A for which there exists a real $\tau > 0$ such that $\tau < u_i$ ($i \in N_1$), then $u \in {}^{i(V)}A$. Indeed, for any point $v = (v_i)_{i \in N}$ of V, it is possible to find a real $\rho > 0$ such that $-\rho \leqslant v_i \leqslant \rho$ ($i \in N_1$). Under these conditions, $u + \alpha v \in A$ whenever $|\alpha| < \tau/\rho$.

Similarly, let us consider $B = \{(x_i)_{i \in N}: x_i > 0$ when $i \in N_2$, $\sup_{i \in N_2} x_i$ exists, $x_i \geqslant 0$ when $i \in N_1$ and $x_i \neq 0$ only for a finite number of $i \in N_1\}$. B is a convex cone with its vertex at 0, $({}^lB)_0 = W + T_1$, ${}^iB = \varnothing$, ${}^{i(W)}B \neq \varnothing$.

The hypotheses of Theorem 1 are satisfied since $({}^lA)_0 + ({}^lB)_0 = V + T_2 + W + T_1 = V + W$ and $A \cap B$ (thus a fortiori ${}^{i(V)}A \cap {}^{i(W)}B$) is void. The subsets A and B are frankly separated although the hypotheses of Corollaries 2, 3, and 4 are not satisfied.

Hahn–Banach Theorem, Extension of the Analytic Form

Recall that a ***real-valued function*** f on D (where D is a subset of E) is a mapping of D into $\mathbb{R}$. This function is called *convex* whenever D is convex and, whatever $x, y \in D$ and the real λ of $]0{:}\, 1[$ may be, $f[\lambda x + (1 - \lambda) y] \leqslant \lambda f(x) + (1 - \lambda) f(y)$. A *concave* function is a real-valued function whose opposite is convex.

A real-valued function f on D is called ***sublinear*** if D is a convex cone with its vertex at 0 (i.e., λ, $\mu > 0$ and $x, y \in D$ imply $\lambda x + \mu y \in D$), if f is positively homogeneous ($f(\lambda x) = \lambda f(x)$ when $\lambda > 0$ and $x \in D$) and subadditive ($f(x + y) \leqslant f(x) + f(y)$ whatever $x, y \in D$ may be).

The sum of a constant and a linear form on a linear subspace D is called an *affine function* on D.

An *extended function* f' on D' is a mapping of D' into $\mathbb{R} \cup \{\infty\}$.

Let D be a subset of D' and f be a real-valued function on D. Then the mapping f' of D' into $\mathbb{R} \cup \{\infty\}$ defined by $f'(x) = f(x)$ whenever $x \in D$ and $f'(x) = \infty$ whenever $x \in D' \backslash D$ is an extended function on D'. Any extended function on D' is of this form. In this paper, an extended convex function on a convex set D' gives a finite convex function f on a convex subset $D \subset D'$ [9, p. 10] and a hypolinear function on a cone D' gives a sublinear function on a subcone D of D' [1, p. 129].

These two points of view are thus closely related. According to the problem under consideration, each of them is more or less advantageous. In the sequel, we essentially employ the real-valued functions.

Note that a sublinear function can be defined here on an unpointed cone D (i.e., $0 \notin D$) and that, in this case, the associated hypolinear function is such that $f(0) = \infty$ when D' is pointed (i.e., $0 \in D'$). In other respects, for a sublinear function f defined on a pointed cone D, we always have $f(0) = 0$ (this is a consequence of the definitions).

Let f be a real-valued function on D. The *epigraph* (respectively, the *strict* *epigraph*) of f is the subset epi(f) (respectively, stepi(f)) of $E \times \mathbb{R}$ defined by $\{(x, \lambda): x \in D$ and $f(x) \leqslant \lambda\}$ (respectively, $f(x) < \lambda$). Similarly, the ***hypograph*** (respectively, the ***strict hypograph***) of f is the subset hyp(f) (respectively, sthyp(f)) of $E \times \mathbb{R}$ defined by $\{(x, \lambda): x \in D$ and $f(x) \geqslant \lambda\}$ (respectively $f(x) > \lambda$).

We say that a function f (defined on F) ***dominates*** a function g (defined on G) on a subset $A \subset F \cap G$ when, for all $x \in A$, $g(x) \leqslant f(x)$.

LEMMA 8. *If f is a real-valued function on D and if V is a linear subspace of E, then:*

(a) ${}^{l}[\mathrm{epi}(f)] = {}^{l}[\mathrm{stepi}(f)] = {}^{l}D \times \mathbb{R}$;

(b) ${}^{l}[\mathrm{hyp}(f)] = {}^{l}[\mathrm{sthyp}(f)] = {}^{l}D \times \mathbb{R}$;

(c) ${}^{i(V\times\mathbb{R})}[\mathrm{epi}(f)] = {}^{i(V\times\mathbb{R})}[\mathrm{stepi}(f)] \subset \{(x, \lambda): x \in {}^{i(V)}D$ *and* $f(x) < \lambda\}$;

(d) ${}^{i}[\mathrm{epi}(f)] = {}^{i}[\mathrm{stepi}(f)] \subset \{(x, \lambda): x \in {}^{i}D$ *and* $f(x) < \lambda\}$;

(e) ${}^{i(V\times\mathbb{R})}[\mathrm{hyp}(f)] = {}^{i(V\times\mathbb{R})}[\mathrm{sthyp}(f)] \subset \{(x, \lambda): x \in {}^{i(V)}D$ *and* $f(x) > \lambda\}$;

(f) ${}^{i}[\mathrm{hyp}(f)] = {}^{i}[\mathrm{sthyp}(f)] \subset \{(x, \lambda): x \in {}^{i}D$ *and* $f(x) > \lambda\}$.

It is possible to prove these properties by classical arguments.

LEMMA 9. *If f is a convex function on D (respectively, concave on D) and if V is a linear subspace of E, then ${}^{i(V\times\mathbb{R})}[\mathrm{epi}(f)] = \{(x, \lambda): x \in {}^{i(V)}D$ and $f(x) < \lambda\}$ (respectively, ${}^{i(V\times\mathbb{R})}[\mathrm{hyp}(f)] = \{(x, \lambda): x \in {}^{i(V)}D$ and $f(x) > \lambda\}$).*

Let us give the proof for a convex function.

Let (x_0, λ_0) be an arbitrary point of $\{(x, \lambda): x \in {}^{i(V)}D$ and $f(x) < \lambda\}$. By virtue of Lemma 8, it is sufficient to prove that $(x_0, \lambda_0) \in {}^{i(V\times\mathbb{R})}[\mathrm{epi}(f)]$. Thus, for a point (v, r) of $V \times \mathbb{R}$, we just have to find a real $\eta > 0$ such that $(x_0, \lambda_0) + \alpha(v, r) \in \mathrm{epi}(f)$ whenever $|\alpha| < \eta$. Since $x_0 \in {}^{i(V)}D$, there exists $\eta_1 > 0$ such that $x_0 + \beta v \in D$ if $|\beta| \leqslant \eta_1$. So, if ρ is a real such that $\rho > \sup\{0, f(x_0 + \eta_1 v) - f(x_0) - \eta_1 r, f(x_0 - \eta_1 v) - f(x_0) + \eta_1 r\}$, it is possible to take $\eta = (\eta_1/\rho)(\lambda_0 - f(x_0))$. Indeed, when $\alpha \geqslant 0$, $f(x_0 + \alpha v) = f[(\alpha/\eta_1)(x_0 + \eta_1 v) + (1 - (\alpha/\eta_1))\, x_0] \leqslant (\alpha/\eta_1) f(x_0 + \eta_1 v) + (1 - (\alpha/\eta_1)) f(x_0) \leqslant \lambda_0 + \alpha r$; when $\alpha < 0$, $f(x_0 + \alpha v) = f[-(\alpha/\eta_1)(x_0 - \eta_1 v) + (1 + (\alpha/\eta_1))\, x_0] \leqslant -(\alpha/\eta_1) f(x_0 - \eta_1 v) + (1 + (\alpha/\eta_1)) f(x_0) \leqslant \lambda_0 + \alpha r$.

THEOREM 4. *Let f be a convex function on F, g be a concave function on G, and V, W be two linear subspaces of E such that $V + W = ({}^{l}F)_0 + ({}^{l}G)_0$. When ${}^{i(V)}F \cap {}^{i(W)}G \neq \varnothing$, f dominates g on ${}^{i(V)}F \cap {}^{i(W)}G$ if and only if there exists an affine function φ on E such that f dominates φ on F and φ dominates g on G.*

It is trivial that the existence of φ implies the domination of g by f on ${}^{i(V)}F \cap {}^{i(W)}G$.

Conversely, let $\tilde{F}$ be the epigraph of f and $\tilde{G}$ the hypograph of g. The subset ${}^{i(V\times\mathbb{R})}\tilde{F} \cap {}^{i(W\times\mathbb{R})}\tilde{G}$ is void. Indeed, if $(x_0, \lambda_0) \in {}^{i(V\times\mathbb{R})}\tilde{F} \cap {}^{i(W\times\mathbb{R})}\tilde{G}$, then, by virtue of Lemma 9, $x_0 \in {}^{i(V)}F \cap {}^{i(W)}G$, $f(x_0) < \lambda_0$ and $g(x_0) > \lambda_0$. This is not possible since f dominates g on ${}^{i(V)}F \cap {}^{i(W)}G$.

Moreover, the existence of a point u_0 in ${}^{i(V)}F \cap {}^{i(W)}G$ implies the nonemptiness of ${}^{i(V\times\mathbb{R})}\tilde{F}$ and ${}^{i(W\times\mathbb{R})}\tilde{G}$ (for instance, $(u_0, \lambda) \in {}^{i(V\times\mathbb{R})}\tilde{F}$ when $\lambda > f(u_0)$). Taking account of Theorem 1, $\tilde{F}$ and $\tilde{G}$ can be frankly separated by a hyperplane $\tilde{H}$. There exist a nonnull linear form $\tilde{\varphi}$ on $E \times \mathbb{R}$ and a real h such that

$$\tilde{H} = \{(x, \lambda) \in E \times \mathbb{R}: \tilde{\varphi}(x, \lambda) = h\},$$
$$\tilde{F} \subset \tilde{H}_1 = \{(x, \lambda) \in E \times \mathbb{R}: \tilde{\varphi}(x, \lambda) \geqslant h\},$$

and

$$\tilde{G} \subset \tilde{H}_2 = \{(x, \lambda) \in E \times \mathbb{R}: \tilde{\varphi}(x, \lambda) \leqslant h\}.$$

Furthermore, by virtue of Theorem 2, the "vertical" through the point u_0 of ${}^{i(V)}F \cap {}^{i(W)}G$ (i.e., $\{(u_0, \alpha): \alpha \in \mathbb{R}\}$) meets H in one and only one point. Thus, when $\alpha > f(u_0) \geqslant g(u_0)$, $\tilde{\varphi}(u_0, g(u_0)) < \tilde{\varphi}(u_0, \alpha)$. So $\tilde{\varphi}(0, 1) > 0$. Then it is possible to deduce from $\tilde{\varphi}$ a linear form φ' on E such that $\tilde{\varphi}(x, \varphi'(x)) = 0$ for all $x \in E$ [4, Lemma].

If k is the real defined by $k\tilde{\varphi}(0, 1) = h$, the affine function $\varphi' + k$ on E satisfies the conditions of the function φ of this theorem. Indeed, whatever $y \in F$ and $\epsilon \geqslant 0$ may be, $\tilde{\varphi}(y, \varphi'(y)) = 0$ and $\tilde{\varphi}(y, f(y) + \epsilon) \geqslant h = k\tilde{\varphi}(0, 1)$. Thus $f(y) + \epsilon \geqslant \varphi'(y) + k$ for all $y \in F$ and $\epsilon \geqslant 0$. This result proves that f dominates φ on F. It is also easy to verify that φ dominates g on G.

This theorem admits as particular cases different theorems, for instance, [4, Theorems 2 and 3] and [12, Theorem V.1.4].

We also give the following consequences.

COROLLARY 5. *Let f be a convex function on F, τ be a linear form on a linear subspace T of E, and V be a linear subspace of E such that $V + T = ({}^lF)_0 + T$. When $T \cap {}^{i(V)}F \neq \varnothing$, f dominates τ on $T \cap {}^{i(V)}F$ if and only if there exists a linear form ψ on E such that f dominates ψ on F and the restriction of ψ to T is τ.*

Since a linear form is concave and since ${}^{i(T)}T = {}^{i}T = T$, there exists, by virtue of Theorem 4, an affine function φ on E such that $\tau(x) \leqslant \varphi(x)$ for all $x \in T$ and $\varphi(x) \leqslant f(x)$ for all $x \in F$. But $\varphi = \varphi' + k$, where k is a real and where φ' is a linear form on E. In these conditions, the restriction of φ' to T is τ. Indeed, if t is an arbitrary point of T, $\alpha t \in T$ for all $\alpha \in \mathbb{R}$ and $\tau(\alpha t) \leqslant \varphi'(\alpha t) + k$ (or $\alpha[\tau(t) - \varphi'(t)] \leqslant k$), which is only possible when $\tau(t) = \varphi'(t)$.

Furthermore, $0 \leqslant k$ ($0 \in T$ and $\tau(0) \leqslant \varphi'(0) + k$) and f dominates $\varphi' + k$ on F. Thus f dominates φ' on F and φ' is the linear form ψ of this corollary.

COROLLARY 6. *Let f be a sublinear function on F* (thus F is a convex, pointed, or unpointed cone with vertex at 0), *g be a concave function on the convex G, and*

V, W be two linear subspaces such that $V + W = ({}^lF_0) + ({}^lG)_0$. When ${}^{i(V)}F \cap {}^{i(W)}G \neq \varnothing$, f dominates g on ${}^{i(V)}F \cap {}^{i(W)}G$ if and only if there exists a linear form ψ on E such that f dominates ψ on F and ψ dominates g on G.

The necessary condition is evident. Concerning the sufficient condition, there exists an affine function $\psi + k$ on E (ψ and k being, respectively, a linear form on E and a real) such that $g(x) \leqslant \psi(x) + k$ for all $x \in G$ and $\psi(x) + k \leqslant f(x)$ for all $x \in F$.

Let u be an arbitrary point of F. For all $\alpha > 0$, $\alpha u \in F$ (F is a cone) and $\alpha\psi(u) + k \leqslant \alpha f(u)$. Consequently, $\psi(u) \leqslant f(u)$ (letting $\alpha \to \infty$) for all u of F and $k \leqslant 0$ (letting $\alpha \to 0$).

Thus, $g(x) \leqslant \psi(x) + k \leqslant \psi(x)$ for all $x \in G$, $\psi(x) \leqslant f(x)$ for all $x \in F$ and the proof is complete.

COROLLARY 7. *Let f be a convex function on F, g be a concave function on G, and V, W be two linear subspaces of E such that $V + W = ({}^lF)_0 + ({}^lG)_0$. If ${}^{i(V)}F \cap {}^{i(W)}G \neq \varnothing$ and if f dominates g on ${}^{i(V)}F \cap {}^{i(W)}G$, then f dominates g on $F \cap G$.*

This corollary is a direct consequence of Theorem 4.

Remarks. 1. If, in Corollary 5, f is defined on the whole vector space ($F = E$), then we obtain [5, Theorem 2, of p. 358].

If, in Corollary 6, f is defined on the whole vector space ($F = E$) and if g is the opposite of a sublinear function, then we obtain [7, Theorem 12.2, p. 462] and [10, Corollary 6, p. 116]. Indeed, it is sufficient to take $V = E$ and $W = \{0\}$.

2. Let E be a topologized vector space and let f be a convex function on an *open* convex set F. It is known that the following four conditions are equivalent:

(a) f is upper semicontinuous at a point x_0 of F;

(b) f is upper semicontinuous on F;

(c) f is continuous on F;

(d) there exists a nonvoid open set (included in F) in which f is bounded above.

A proof of this theorem (this proof is valid for the β-topologies which are introduced by Klee in [7] and which are more general than the classical vectorial topologies) is given in [12, III.3.1, p. 39].

By virtue of this property, the continuity of the affine function of Theorem 4 is ensured when the interior of F is nonvoid and f is upper semicontinuous at an interior point of F or when the interior of G is nonvoid and g is lower semicontinuous at an interior point of G (an affine function is also concave).

So [1, Corollary 1.9, p. 133] is a particular case of Theorem 4.

We give the following corollaries.

COROLLARY 8. *Let E be a locally convex vector space, f be a sublinear function on F* (thus F is a convex cone of E with vertex at 0), *and u be an arbitrary point of F. The function f is lower semicontinuous at u if and only if $f(u) = \sup\{\gamma(u): \gamma \in \mathscr{L}\}$ where $\mathscr{L}$ denotes the set of all continuous linear forms on E which are dominated by f on F.*

The necessary condition is obvious. About the sufficient condition, let k be an arbitrary real such that $k < f(u)$. Since f is lower semicontinuous at u and since E is a locally convex space, there exists a convex open neighbourhood U of u such that $k < f(x)$ for all $x \in U \cap F$. Let g be the concave function on U defined by $g(x) = k$ for all $x \in U$.

${}^{l}U = ({}^{l}U)_0 = E$ and $u \in {}^{i(E)}U$ (U is a neighbourhood of u). Thus the hypotheses of Corollary 6 are satisfied (with $W = E$ and $V = \{0\}$) and there exists a linear form φ on E such that $\varphi(x) \leqslant f(x)$ for all $x \in F$ and $k \leqslant \varphi(x)$ for all $x \in U$.

It follows from the fact that φ is bounded below in the open U that φ is continuous on E.

Thus for every real k such that $k < f(u)$, there exists $\varphi \in \mathscr{L}$ such that $k \leqslant \varphi(u)$. This proves the corollary.

COROLLARY 9. *Let E be a locally convex vector space, F be a convex function on F, and u be an arbitrary point of F. The function f is lower semicontinuous at u if and only if $f(u) = \sup\{\gamma(u): y \in \mathscr{A}\}$, where $\mathscr{A}$ denotes the set of all continuous affine functions on E which are dominated by f on F.*

The proof of Corollary 9 is similar to that of Corollary 8: It suffices to employ Theorem 4 instead of Corollary 6.

Remark. Corollaries 8 and 9 correspond to [1, Theorem 3.4, p. 141] and [1, 4.7, p. 146], respectively.

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