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Asymptotic behavior of support points for planar curves

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ABSTRACT

In this paper we prove a universal inequality describing the asymptotic behavior of support points for planar continuous curves. As corollaries we get an analogous result for tangent points of differentiable planar curves and some (partially known) assertions on the asymptotic of the mean-value points for various classical analytic theorems. Some open questions are formulated.

1. Introduction and main results

A starting point of this project is a remarkable property of mean-value points in the first integral mean-value theorem. Let $f:[0,1]\to\mathbb{R}$ be a continuous function. For any $x\in(0,1]$ consider $\xi(x)$ that is the maximum of $\tau\in[0,x]$ such that $x\cdot f(\tau)=\int_0^x f(t)\,dt$. Then the inequality $\overline{\lim}_{x\to 0}\frac{\xi(x)}{x}\geqslant\frac{1}{e}$ holds. This inequality was proposed by Professor V.K. Ionin and was proved at first in the paper [9]. Further, this result was generalized in various ways [10,5,11–14]. But in this paper we suggest another point of view: the main object of our study are not functions, but continuous curves (for example, the graphs of functions).

At first we refine a definition of support points (with respect to a given chord) of a continuous parametric curve in spite of the fact that this notion is quite natural and intuitively clear.

Definition 1. Let $\gamma:(a,b)\to\mathbb{E}^2$ be a continuous parametric curve in the Euclidean plane, $[c,d]\subset(a,b)$. We say that a point $\gamma(\tau_0)$, $\tau_0\in[c,d]$ is a support point for the chord $[\gamma(c),\gamma(d)]$ (if $\gamma(c)\neq\gamma(d)$), if a straight line l passing through $\gamma(\tau_0)$ in parallel $[\gamma(c),\gamma(d)]$ is such that for all τ, rather close to τ_0, the points $\gamma(\tau)$ are in one and the same half-plane determined by l. If $\gamma(c)=\gamma(d)$, then we set that any point $\gamma(\tau_0)$ for $\tau_0\in[c,d]$ is a support point for the (degenerate) chord $[\gamma(c),\gamma(d)]$.

Note that our convention on the set of support points for $\gamma(c)=\gamma(d)$ is stipulated by the universality of an analytic description of such sets under this definition (see below). It is possible to modify this definition, but in any case it is not so important since the case of chords with zero length is trivial (in some sense) for our questions.

Consider a (rectangular) Cartesian coordinate system Oxy in the plane $\mathbb{E}^2$. Then $\gamma(t)=(x(t),y(t))\in\mathbb{R}^2$, $t\in(a,b)$. The fact that a point $\gamma(\tau_0)$, $\tau_0\in[c,d]$, is a support point for the chord $[\gamma(c),\gamma(d)]$ could be expressed in the following form. Consider a function $\Phi:(a,b)\to\mathbb{R}$,

$$\Phi(\tau)=\det\begin{pmatrix} x(d)-x(c) & y(d)-y(c)\\ x(\tau) & y(\tau)\end{pmatrix}.$$

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It is easy to see that a point $\gamma(\tau_0)$, $\tau_0 \in [c,d]$, is a support point for the chord $[\gamma(c),\gamma(d)]$ if and only if τ_0 is a point of local extremum of the function Φ. Since $\Phi(c)=\Phi(d)$, then there is at least one point of local extremum of the function under consideration on the interval (c,d). Moreover, if the curve $\gamma(t)$ is differentiable at the point τ_0, then

$$\Phi'(\tau_0)=\det\begin{pmatrix} x(d)-x(c) & y(d)-y(c) \\ x'(\tau_0) & y'(\tau_0)\end{pmatrix}=0,$$

that means the collinearity of the vectors $\gamma(d)-\gamma(c)$ and $\gamma'(\tau_0)$. Note also that the straight line l in Definition 1 is a tangent line to the curve $\gamma(t)$ at the point $\gamma(\tau_0)$ when $\gamma(c)\neq\gamma(d)$ and $\gamma'(\tau_0)\neq 0$.

Now we can formulate the main results of this paper. Let $\gamma:[a,b)\to\mathbb{E}^2$, where $a,b\in\overline{\mathbb{R}}=\mathbb{R}\cup\{-\infty,\infty\}$, be a continuous parametric curve in the Euclidean plane, that is not a constant in any neighborhood of the point a.

Further, by $D(t)$ we denote a distance between the points $\gamma(a)$ and $\gamma(t)$.

For every $t\in(a,b)$ we denote by $S(t)$ a set of $\tau\in(a,t]$ such that the point $\gamma(\tau)$ is a support point for the chord $[\gamma(a),\gamma(t)]$. Now, consider

$$DS(t)=\sup\bigl\{D(\tau)\bigm|\tau\in S(t)\bigr\}.$$

The main object of our study is the asymptotic of the ratio $DS(t)/D(t)$ when $t\to a$. For a fixed t the set $S(t)\subset(a,t]$ could be organized quite complicated, and this is evident from a geometric interpretation of this set. The case $D(t)=0$ is exceptional. According to the definition, in this case we get $S(t)=(a,t]$. Obviously, there exists $\tau\in S(t)=(a,t]$ with the property $D(\tau)>0$ (otherwise, the curve γ is constant on the interval (a,t)). Hence $DS(t)>0$ for a such t, and we set $DS(t)/D(t)=\infty$ when $D(t)=0$.

For a fixed value t it is possible to choose a curve with the ratio $DS(t)/D(t)$ equal to a given positive number. On the other hand, it is clear that this ratio could not be greater than 1 for all values of a parameter. A rather less evident fact is that the ratio $DS(t)/D(t)$ could not be less than some definite positive number for all values of a parameter. An exact assertion consists in the following theorem that is the main result of this paper.

Theorem 1. *Let $\gamma:[a,b)\to\mathbb{E}^2$ be an arbitrary continuous parametric curve. Then the inequality*

$$\varlimsup_{t\to a}\frac{DS(t)}{D(t)}\geqslant\frac{1}{e}\tag{1}$$

holds, where, as usual, $e=\lim_{n\to\infty}(1+\frac{1}{n})^n$.

For a differentiable curve any support point is a tangent point automatically. Therefore, the above theorem implies some corresponding results for tangent points. Let us clarify the statement of the problem.

Let $\gamma:[a,b)\to\mathbb{E}^2$, where $a,b\in\overline{\mathbb{R}}=\mathbb{R}\cup\{-\infty,\infty\}$, be a continuous parametric curve in the Euclidean plane such that for every $t\in(a,b)$ there exists a non-zero derivative vector $\gamma'(t)$. Note that this vector defines a direction of the tangent line to the considered curve at the point $\gamma(t)$. If the derivative vector is continuous (with respect to a parameter), then such a curve is called smooth regular, but we do not assume the continuity of the derivative vector in what follows (unless otherwise stipulated).

By analogy with general continuous curves, for every $t\in(a,b)$ we denote by $T(t)$ the set of $\tau\in(a,t]$ such that the vector $\gamma'(\tau)$ is collinear to the vector $\overrightarrow{\gamma(a)\gamma(t)}$. It is clear that the set $T(t)$ is non-empty for every $t\in(a,b)$ (since even the set $S(t)\subset T(t)$ is non-empty).

Let us consider the value

$$DT(t)=\sup\bigl\{D(\tau)\bigm|\tau\in T(t)\bigr\}.$$

We are interested in the asymptotic of the ratio $DT(t)/D(t)$ when $t\to a$. For a fixed t (by analogy with $S(t)$) the set $T(t)\subset(a,t]$ could be rather complicated that follows from the geometric interpretation of this set as a set of points $\tau\in(a,t]$, such that a tangent line to the curve γ at the point $\gamma(\tau)$ is parallel to the chord $[\gamma(a),\gamma(t)]$. For example, for $D(t)=0$ we get $T(t)=(a,t]$, and we set $DT(t)/D(t)=\infty$ in this case. This is motivated by the fact that for some $\tau\in(a,t)\subset T(t)$ the inequality $D(\tau)>0$ holds (otherwise the derivative vector γ' should be trivial on the interval (a,t)) and, therefore, $DT(t)>0$ (compare with analogous convention for $DS(t)/D(t)$ when $D(t)=0$).

It is clear that for some fixed value of t (and for a curve chosen specially) the ratio $DT(t)/D(t)$ could be equal to any positive number. For a differentiable curve γ we obviously get the inclusion $S(t)\subset T(t)$ (but $S(t)\neq\emptyset$ under the condition) and, therefore, the inequality $DS(t)\leqslant DT(t)$ for an arbitrary $t\in(a,b)$. Hence, the following result (obtained at first in [15]) is an immediate consequence of Theorem 1.

Theorem 2. *(See [15].) Let $\gamma:[a,b)\to\mathbb{E}^2$ be an arbitrary continuous parametric curve with non-zero derivative vector $\gamma'(t)$ at every point $t\in(a,b)$. Then the inequality*

$$\varlimsup_{t\to a}\frac{DT(t)}{D(t)}\geqslant\frac{1}{e}\tag{2}$$

holds.

In Section 2 we consider some examples illustrating the assertions of Theorem 1 and Theorem 2. According to these examples, the inequalities (1) and (2) are best possible.

Note also that the inequalities (1) and (2) have local character. Therefore, the domain of definition $[a,b)$ of the curve $\gamma(t)$ in Theorem 1 or in Theorem 2 can be replaced by any interval $[a,b_1)$, where $b_1 \in (a,b)$.

Under conditions of Theorem 1, we can consider curve $\gamma_1 : [a_1,b_1) \to \mathbb{E}^2$ instead of the curve $\gamma : [a,b) \to \mathbb{E}^2$, if $\gamma_1(t) = \gamma(g(t))$ for some continuous bijective function $g : [a_1,b_1) \to [a,b)$. In the case of Theorem 2, it should be required (in addition) the existence of positive derivative for the function g on the interval (a_1,b_1). In other words, the assertions of the above theorems concern the geometry of a nonparametric curve (that could be considered as a class of pairwise equivalent parametric curves). Further (e.g., in the proof of Theorem 1 in Section 3) we will use these properties repeatedly.

On the other hand, the results of Theorem 1 and Theorem 2 may be used to study some special parameterizations of a curve. In such a case we obtain a couple of (partially known) results on the asymptotic of mean-value points in some classical differential and integral theorems (cf. Section 4). In the last section we formulate some unsolved questions that can be used as a basis for further investigations in the designated direction. In particular, it would be desirable to hope that results of this paper initiate more detailed study of the asymptotic behaviour of the ratios $DS(t)/D(t)$ and $DT(t)/D(t)$.

For functions defined on an interval $[\alpha,\beta] \subset \mathbb{R}$, we use limits and derivatives at the point α (β, respectively) having in mind right hand limits and right derivatives (left hand limits and left derivatives, respectively). This natural convention allows us to simplify the presentation.

2. Some examples

We will define curves with using of polar coordinates with a pole in the initial point of the curve. Moreover, a parameter in the following examples is the polar angle t, $t \in [a,b)$. A curve $\gamma(t)$, $t \in [a,b)$ is defined completely by $t \mapsto \rho(t)$ – the distance function from the current point of the curve to the pole ($\rho(a) = 0$). The points $\tau \in T(t) \subset (a,t]$ are determined by the equation

$$\rho(\tau)\cos(t-\tau) - \rho'(\tau)\sin(t-\tau) = 0. \tag{3}$$

Note also that the equality $D(t) = \rho(t)$ holds in all examples below. Moreover, in all these examples, the equality $T(t) = S(t)$ holds for all possible values of the parameter t (it is easy to check), i.e. all tangent points are also support points for the corresponding chords of the curve.

Example 1. Consider a curve defined by the equation $\rho(t) = e^{\alpha t}$, where $\alpha > 0$, the parameter is the polar angle, $t \in [-\infty, b)$. It is clear that $t = \tau + \operatorname{arccot}(\alpha) + \pi n$ ($n \in \mathbb{Z}$, $n \geqslant 0$) for $\tau \in T(t) = S(t)$. Hence,

$$DS(t) = DT(t) = e^{\alpha t - \alpha\cdot\operatorname{arccot}(\alpha)}, \qquad D(t) = \rho(t) = e^{\alpha t}, \qquad \frac{DT(t)}{D(t)} = \frac{DS(t)}{D(t)} = e^{-\alpha\cdot\operatorname{arccot}(\alpha)}.$$

It is necessary to note that $\alpha \cdot \operatorname{arccot}(\alpha) < 1$ for $\alpha > 0$ and $\lim_{\alpha\to\infty}(\alpha \cdot \operatorname{arccot}(\alpha)) = 1$. Therefore, the inequalities in Theorem 1 and Theorem 2 are best possible.

Example 2. Consider a curve defined by the equation $\rho(t) = t^\alpha$, where $\alpha > 0$, the parameter is the polar angle, $t \in [0,b)$. For determining of $\tau \in T(t) = S(t)$ we have the equation

$$\tau\cos(t-\tau) - \alpha\sin(t-\tau) = 0.$$

It is clear that $t = \tau + \operatorname{arccot}(\frac{\alpha}{\tau}) + \pi n$ ($n \in \mathbb{Z}$, $n \geqslant 0$). Therefore,

$$\lim_{t\to 0}\frac{DT(t)}{D(t)} = \lim_{t\to 0}\frac{DS(t)}{D(t)} = \lim_{\tau\to 0}\frac{\tau^\alpha}{(\tau + \operatorname{arccot}(\frac{\alpha}{\tau}))^\alpha} = \lim_{\tau\to 0}\left(1 + \frac{1}{\tau}\operatorname{arccot}\left(\frac{\alpha}{\tau}\right)\right)^{-\alpha} = \left(1 + \frac{1}{\alpha}\right)^{-\alpha} > e^{-1}.$$

Note also that $\lim_{\alpha\to\infty}(1+\alpha^{-1})^{-\alpha} = e^{-1}$.

Example 3. Let us set $l > 1$, $[a,b) = [-\infty, 0)$, $\rho(t) = e^{\phi(t)}$, where $\phi(t) = -|t|^l = -(-t)^l$, a parameter is the polar angle again, $t \in [-\infty, 0)$. Obviously, $t = \tau + \operatorname{arccot}(\phi'(\tau)) + \pi n$ ($n \in \mathbb{Z}$, $n \geqslant 0$) for $\tau \in T(t) = S(t)$ (cf. the equality (3)). This implies immediately

$$\lim_{t\to-\infty}\frac{DT(t)}{D(t)} = \lim_{t\to-\infty}\frac{DS(t)}{D(t)} = e^L,$$

where $L = \lim_{s\to-\infty}(\phi(s) - \phi(s+\alpha(s)))$ and $\alpha(s) = \operatorname{arccot}(\phi'(s))$. Since $\phi''(t) \leqslant 0$, then

$$\phi'(s)\cdot\alpha(s) \geqslant \phi\bigl(s+\alpha(s)\bigr) - \phi(s) \geqslant \phi'\bigl(s+\alpha(s)\bigr)\cdot\alpha(s).$$

Since

$$\lim_{\beta\to\infty}\bigl(\beta\cdot\operatorname{arccot}(\beta)\bigr)=1,\qquad \lim_{s\to-\infty}\phi'(s)=\infty,\qquad \lim_{s\to-\infty}\frac{\phi'(s+\alpha(s))}{\phi'(s)}=1,$$

then $L=-1$ and $\lim_{t\to-\infty}\frac{DT(t)}{D(t)}=\lim_{t\to-\infty}\frac{DS(t)}{D(t)}=e^{-1}$. This example helps to understand better some steps in the proof of Theorem 1.

A couple of other examples follows from Theorem 4 and from Corollary 1 due to the asymptotic equation (8), because such examples are considered in various papers, devoted to the asymptotic of mean-value points in classical mean-value theorem [9,10,5,11]. Under some additional restrictions to the asymptotic of the curve $t\to\gamma(t)$ at the point a, there exists a usual limit $\lim_{t\to a}\frac{DT(t)}{D(t)}$ $(\geqslant e^{-1})$. Assertions of such kind for various integral and differential mean-value theorems are obtained in the papers [1–4,6,7,17,19–21]. Moreover, this problematic is adequately depicted in the book [18], where one can find also extensive references.

3. Proof of Theorem 1

Let us consider a Cartesian coordinate system Oxy in $\mathbb{E}^2$ such that $O=\gamma(a)$. Then $\gamma(t)=(x(t),y(t))\in\mathbb{R}^2$, $t\in[a,b)$, and $\gamma(a)=(x(a),y(a))=(0,0)$. The fact, that a point $\tau_0\in(a,t]$ is in the set $S(t)$, can be expressed in the following form. Consider a function $\Phi:[a,b)\to\mathbb{R}$,

$$\Phi(\tau)=\det\begin{pmatrix}x(t) & y(t)\\ x(\tau) & y(\tau)\end{pmatrix}. \tag{4}$$

Then a point $\tau_0\in(a,t]$ is in the set $S(t)$ if and only if τ_0 is a point of local extremum of the function $\tau\mapsto\Phi(\tau)$.

In the rest of this section we prove Theorem 1. *Further, we suppose that the assertion of Theorem* 1 *does not hold, and get the contradiction.*

Without loss of generality we may assume that $\gamma(t)\neq\gamma(a)$ for all $t\in(a,b)$. Indeed, if there is a sequence of points $t_n\in(a,b)$ such that $t_n\to a$ as $n\to\infty$ and $\gamma(t_n)=\gamma(a)$, then $\overline{\lim}_{t\to a}\frac{DT(t)}{D(t)}=\infty>\frac{1}{e}$ (in this case $D(t_n)=0$ and $DS(t_n)/D(t_n)=\infty$ according to our arrangements discussed just before the statement of Theorem 1), that is impossible. Therefore, numbers $t\in(a,b)$ with the property $\gamma(t)=\gamma(a)$ cannot be close to a as much as possible. Therefore, if we decrease (if necessary) the number b, then we get that such points t are absent.

Let us consider functions $\rho,\theta:[a,b)\to\mathbb{R}$, defined in the following way. Put $\rho(t)=D(t)$ – the distance between O and a current point of the curve $\gamma(t)$. As $\theta(t)$ we consider a number satisfied to equations $x(t)=\rho(t)\cos(\theta(t))$ and $y(t)=\rho(t)\sin(\theta(t))$. Such a number (the polar angle) is defined uniquely up to a summand $2\pi n$ $(n\in\mathbb{Z})$. Taking into account the continuity of $\gamma(t)$, it is easy to choose this angle in such a way that the function $t\mapsto\theta(t)$ is continuous for all values of t.

Let us show that we may assume $\theta(t)$ to be strictly increasing. In our new notations, the function Φ (cf. the equality (4)) has the following form:

$$\Phi(\tau)=\rho(t)\rho(\tau)\sin\bigl(\theta(\tau)-\theta(t)\bigr).$$

If t is a point of local maximum (minimum) of the function $\tau\mapsto\theta(\tau)$, then it is also a point of local minimum (maximum, respectively) of the function $\tau\mapsto\Phi(\tau)$, $t\in S(t)$, and $DS(t)\geqslant D(t)$. Therefore (cf. reasonings above), such points cannot be close to a as much as possible. Decreasing (if necessary) the number b, we may assume that the point t is not a point of local extremum of the function $\tau\mapsto\theta(\tau)$ for all $t\in(a,b)$. Taking into account the continuity of this function, we get that it is either strictly decreasing or strictly increasing on the interval (a,b). Replacing (if necessary) the ray Oy with the opposite ray (hence, changing the orientation), we may assume that this function is strictly increasing for $t\in(a,b)$.

Now, we may change (without loss of generality) the parameter t in such a way that $\theta(t)=t$ for all $t\in(a,b)$, i.e. the curve under consideration is parameterized by the polar angle. Further, it will be convenient to consider a function

$$\phi(t)=\ln\bigl(\rho(t)\bigr).$$

Since $\rho(t)=e^{\phi(t)}$, then (according to our conclusions and assumption on $\rho(t)$) the function $\phi:[a,b)\to\overline{\mathbb{R}}$ is continuous, takes finite values for $t\in(a,b)$, and $\phi(a)=-\infty$. Further, we determine some other properties of this function.

Since we supposed Theorem 1 to be false, then we may assume that there is a number $q>1$ such that

$$\frac{D(\tau)}{D(t)}\leqslant e^{-q}$$

for all $t\in(a,b)$ and all $\tau\in S(t)$. In our notations $D(t)=e^{\phi(t)}$, hence this inequality is equivalent to the following one:

$$\phi(t)-\phi(\tau)\geqslant q>1 \tag{5}$$

for all $t\in(a,b)$ and for all $\tau\in S(t)$.

Now, since $x(t) = e^{\phi(t)}\cos(t)$ and $y(t) = e^{\phi(t)}\sin(t)$, then a point $\tau \in (a,t]$ is in the set $S(t)$ if and only if τ is a point of local extremum of the function

$$\tau \mapsto -\big(\rho(t)\big)^{-1}\Phi(\tau) = \rho(\tau)\sin(t-\theta) = e^{\phi(\tau)}\sin(t-\tau).$$

For a fixed t we consider an interval $I(t) = [\max\{t-\pi, a\}, t]$. It is clear that the function $\tau \mapsto e^{\phi(\tau)}\sin(t-\tau)$ is vanished at the endpoints of this interval. Therefore, there is at least one point of extremum of the latter function (i.e. a point in the set $S(t)$) in the interior of this interval.

Further, we consider the function

$$\tau \mapsto \phi(\tau) + \ln\big(\sin(t-\tau)\big) = \ln\big(e^{\phi(\tau)}\sin(t-\tau)\big) =: F^t(\tau).$$

Claim 1. *For every $t \in (a,b)$, there is $\beta(t) > 0$ such that the function $\tau \mapsto F^t(\tau)$ is strictly decreasing on the interval $[t-\beta(t), t]$.*

Proof. Suppose the contrary. Then there are sequences of numbers $\{\tau_n\}$ and $\{\xi_n\}$ such that $\tau_n < \xi_n < t$ and $F^t(\tau_n) \leqslant F^t(\xi_n)$ for all n, $\tau_n \to t$ as $n \to \infty$. Since $F^t(\tau) \to -\infty$ as $\tau \to t-0$, then there is a point η_n of absolute maximum of the function F^t on the interval $[\tau_n, t)$. Clear, that $\eta_n \in S(t)$ and $\eta_n \to t$ as $n \to \infty$. But according to our assumption, the inequality $\phi(t) - \phi(\eta_n) > q$ (the inequality (5)) holds for all n, that is impossible (it suffices to pass to the limit in this inequality). Therefore, we have proved the existence of the required $\beta(t) > 0$ (it is easy to see also that $\beta(t) < \pi$). □

Remark 1. Recall that every increasing function $f : [\alpha,\beta] \subset \mathbb{R} \to \mathbb{R}$ is differentiable almost everywhere, its derivative is non-negative and summable, and $\int_\alpha^\beta f'(t)\,dt \leqslant f(\beta) - f(\alpha)$. Moreover, if f has derivative *at every point* of the interval $[\alpha,\beta]$, then the above inequality becomes an equality (in this case the function $x \mapsto f(x)$ is absolutely continuous on the interval $[\alpha,\beta]$), cf. [8,16].

Claim 2. *For any $t \in (a,b)$ the function ϕ is differentiable almost everywhere on every interval $[c,d] \subset (t-\beta(t), t)$, its derivative is summable and*

$$\phi(d) - \phi(c) \leqslant \int_c^d \phi'(\tau)\,d\tau.$$

Proof. According to Claim 1 the function $\tau \mapsto F^t(\tau) = \phi(\tau) + \ln(\sin(t-\tau))$ decreases on the interval $[t-\beta(t), t]$. Therefore, the function $-F^t$ increases on this interval. Using properties of increasing functions (cf. Remark 1), differentiability and absolute continuity of the function $\tau \mapsto \ln(\sin(t-\tau))$ on the interval $[c,d] \subset (t-\beta(t), t) \subset (t-\pi, t)$, we easily get the required properties of the function ϕ. □

Claim 3. *The function ϕ is differentiable almost everywhere on the interval (a,b). Its derivative ϕ' is summable on every interval $[c,d] \subset (a,b)$ and satisfies the inequality*

$$\phi(d) - \phi(c) \leqslant \int_c^d \phi'(\tau)\,d\tau.$$

Proof. For every $t \in (a,b)$ we consider the interval $I(t) := (t-\beta(t), t)$ (Claim 1). All these intervals cover jointly the interval $[c,d]$. By compactness, $[c,d]$ is covered also by some finite subset of the intervals $I(t)$, say, by $I(t_0), I(t_1), \dots, I(t_l)$, $t_0 < t_1 < \cdots < t_l$. Now, choose numbers s_i, $i = 0,\dots,l$, such that $c = s_0 < s_1 < \cdots < s_{l-1} = d$ and $[s_i, s_{i+1}] \subset I(t_i)$. According to Claim 2 the function ϕ is differentiable almost everywhere on every interval $[s_i, s_{i+1}]$, and the inequality

$$\phi(s_{i+1}) - \phi(s_i) \leqslant \int_{s_i}^{s_{i+1}} \phi'(\tau)\,d\tau$$

holds. Hence, ϕ is differentiable almost everywhere on the interval $[c,d]$. Summing the obtained inequalities by i from 0 to $s-1$, we get an analogous inequality on the interval $[c,d]$. Since the interval $[c,d] \subset (a,b)$ is arbitrary, the function ϕ is differentiable almost everywhere on the interval (a,b). □

Further, it will be helpful to consider the set

$$Sm = \big\{t \in (a,b) \;\big|\; \text{there exists } \phi'(t) \in \mathbb{R}\big\}.$$

Consider also the function $\alpha : Sm \to \mathbb{R}$, defined by the equation

$$\alpha(t) = \operatorname{arccot}\bigl(\phi'(t)\bigr).$$

It is clear that $\alpha(t) \in (0, \pi)$ for all values of the parameter.

Claim 4. *For every $\tau \in Sm$ either the inequality $\tau + \alpha(\tau) \geqslant b$, or the inequality $\phi(\tau + \alpha(\tau)) - \phi(\tau) \geqslant q$ holds.*

Proof. Let us fix some $\tau_0 \in Sm$ and suppose that $t_0 := \tau_0 + \alpha(\tau_0) < b$. If the point τ_0 is a point of local extremum of the function $\tau \mapsto e^{\phi(\tau)} \sin(t_0 - \tau)$, then $\tau_0 \in S(t_0)$ and (according to our assumptions) the inequality $\phi(t_0) - \phi(\tau_0) > q$ (the inequality (5)) holds, that implies the required result. However, τ_0 should not be a point of local extremum of the above function, but in any case, the point τ_0 is a critical point of the function $\tau \mapsto e^{\phi(\tau)} \sin(t_0 - \tau)$ ($\alpha(\tau_0) = \operatorname{arccot}(\phi'(\tau_0))$ by definition). In other word, the tangent line to the curve $\gamma(t)$ at the point $\gamma(\tau_0)$ is parallel to the chord $[O = \gamma(a), \gamma(t_0)]$.

Now, choose sequences of numbers $\{\tau_n\}$ and $\{t_n\}$ such that $\tau_n \to \tau_0$, $t_n \to t_0$ as $n \to \infty$ and the chord $[O, \gamma(t_n)]$ is parallel to the chord $[\gamma(\tau_0), \gamma(\tau_n)]$ for all n. Since $\tau_0 \in Sm$, then

$$\frac{1}{\tau_n - \tau_0} \overrightarrow{\gamma(\tau_n)\gamma(\tau_0)} \to \gamma'(\tau_0) \quad \text{as } n \to \infty.$$

Let us show that for every n there exists a number $\eta_n \in S(t_n)$ between the numbers τ_0 and τ_n. Such a number should be a point of local extremum of the function $\tau \mapsto e^{\phi(\tau)} \sin(t_n - \tau)$. For this goal we consider the function $\Psi(\tau) = \det\bigl(\begin{smallmatrix} \gamma(\tau_n) - \gamma(\tau_0) \\ \gamma(\tau) \end{smallmatrix}\bigr)$. Since $\Psi(\tau_0) = \Psi(\tau_n)$, then there is a point η_n of local extremum of this function between the points τ_0 and τ_n. But the same point is also a point of local extremum of the function

$$\tau \mapsto \det\begin{pmatrix} \gamma(t_n) \\ \gamma(\tau) \end{pmatrix} = \det\begin{pmatrix} e^{\phi(t_n)} \cos(t_n) & e^{\phi(t_n)} \sin(t_n) \\ e^{\phi(\tau)} \cos(\tau) & e^{\phi(\tau)} \sin(\tau) \end{pmatrix} = \bigl(-e^{\phi(t_n)}\bigr) e^{\phi(\tau)} \sin(t_n - \tau),$$

i.e. $\eta_n \in S(t_n)$. According to the inequality (5) we get $\phi(t_n) - \phi(\eta_n) > q$ for n. Since $\eta_n \to \tau_0$ and $t_n \to t_0$ as $n \to \infty$, then passing to limits in this inequality, we obtain $\phi(t_0) - \phi(\tau_0) \geqslant q$. $\square$

Let us fix a number $b^* \in (a, b)$. Now we obtain one remarkable property of the function $\alpha(t)$ on the interval $(a, b^*]$.

Claim 5. *For every $t \in (a, b^*) \cap Sm$ at least one of the following two assertions holds*:

(1) $t + \alpha(t) \geqslant b^*$;
(2) *there is $\xi = \xi(t) \in (t, t + \alpha(t)) \cap Sm$ such that $\alpha(t) > q \cdot \alpha(\xi)$.*

Proof. Suppose that Assertion (1) does not hold, i.e. $t + \alpha(t) < b^*$. Set $s = t + \alpha(t)$, then $t < s < b^*$. According to Claim 4, $\phi(s) - \phi(t) \geqslant q > 1$. According to Claim 3, the function ϕ is differentiable almost everywhere on the interval $[t, s]$, the derivative ϕ' is summable on this interval, and the inequality

$$1 < q \leqslant \phi(s) - \phi(t) \leqslant \int_t^s \phi'(\tau)\, d\tau$$

holds.

Further, for some number $\xi \in (t, s) \cap Sm$ the inequality $\int_t^s \phi'(\tau)\, d\tau \leqslant (s - t)\phi'(\xi)$ holds. Indeed, the set $(t, s) \cap Sm$ is a set of full measure on the interval (t, s). If for all points ξ of this set we have $\int_t^s \phi'(\tau)\, d\tau > (s - t)\phi'(\xi)$, then we get a contradiction by integrating this inequality with respect to ξ on (t, s). Therefore, the required point $\xi \in (t, s) \cap Sm$ does exist (such points consist of a set with positive measure), hence,

$$1 < q \leqslant \phi(s) - \phi(t) \leqslant \int_t^s \phi'(\tau)\, d\tau \leqslant (s - t)\phi'(\xi).$$

It is clear that $\phi'(\xi) > 0$, therefore, $\alpha(\xi) = \operatorname{arccot}(\phi'(\xi)) \in (0, \pi/2)$. Further,

$$\phi'(\xi) = \cot\bigl(\alpha(\xi)\bigr) = 1/\tan\bigl(\alpha(\xi)\bigr) < 1/\alpha(\xi),$$

because $\tan(x) > x$ for $x \in (0, \pi/2)$. Consequently, $q \leqslant \phi'(\xi) \cdot \alpha(t) < \frac{\alpha(t)}{\alpha(\xi)}$, and Assertion (2) is proved. $\square$

Now, consider the set

$$S^* = Sm \cap \big(a, b^*\big].$$

It has full measure on the interval $(a, b^*]$. Later on we will need some properties of the function $t \mapsto \alpha(t)$ on the set S^*.

Claim 6. *At least one of the following assertions holds*:

(1) *there are a point $t^* \in (a, b^*]$ and a sequence $\{t_n\}$, $t_n \in S^*$, such that $\alpha(t_n) \to 0$ and $t_n \to t^*$ as $n \to \infty$;*
(2) *$a > -\infty$ and there is $c > 0$ such that $\alpha(t) \geqslant c$ for all $t \in S^*$.*

Proof. Suppose that Assertion (1) does not hold and prove Assertion (2).

Consider any $b_0 \in (a, b)$ and let $c_1 \geqslant 0$ be the greatest lower bound of the function $t \mapsto \alpha(t)$ on the set $S^* \cap [b_0, b]$. If $c_1 = 0$, then using the compactness of the interval $[b_0, b]$, it is easy to find a sequence $\{t_n\}$, $t_n \in S^* \cap [b_0, b) \subset (a, b)$, that tends to some $t^* \in S^* \cap [b_0, b]$ and such that $\alpha(t_n) \to 0$ as $n \to \infty$. But Assertion (1) does not hold and, consequently, we get the inequality $c_1 > 0$.

Now, set $c = \min\{c_1, b^* - b_0\} > 0$. Let us show that $\alpha(t) \geqslant c$ for all $t \in S^*$. Suppose that the set

$$S = \big\{t \in S^* \,\big|\, \alpha(t) < c\big\}$$

is non-empty. Obviously, $S \subset (a, b_0]$. Note that for all $t \in S$ the inequality $t + \alpha(t) < b^*$ holds (otherwise $\alpha(t) \geqslant b^* - t \geqslant b^* - b_0 \geqslant c$), hence, by Claim 5 there exists $\xi = \xi(t) \in (t, t + \alpha(t)) \cap Sm$ such that $\alpha(\xi) < \alpha(t)/q < \alpha(t) < c$, in particular, $\xi \in S$.

Now, choose some $t_1 \in S$ and construct a sequence of points $\{t_n\}$ from S by the following method: if t_i is defined, then put $t_{i+1} = \xi(t_i)$. By construction $t_i < t_{i+1}$, and, since $c > \alpha(t_i) \geqslant q \cdot \alpha(\xi(t_i)) = q \cdot \alpha(t_{i+1}) > \alpha(t_{i+1})$, then $t_{i+1} \in S$. Since the constructed sequence increases and is bounded from above by the number b_0 ($S \subset (a, b_0]$), it has a finite limit $t^* \in (a, b_0]$, and the inequality $\alpha(t_i) \geqslant q \cdot \alpha(t_{i+1})$ implies $\alpha(t_n) \to 0$ as $n \to \infty$. Therefore, Assertion (1) holds that is impossible by our assumptions. Therefore, $S = \emptyset$, i.e. $\alpha(t) \geqslant c$ for all $t \in (a, b]$.

If $a > -\infty$, then we get Assertion (2) from statement of the claim. Hence, we consider now the case $a = -\infty$.

For all $i \geqslant 1$ define the numbers b_i by the recurrent formula $b_i = b_{i-1} - q^i \cdot c$ (b_0 has been chosen earlier). Let us prove by induction that

$$\alpha(t) \geqslant c \cdot q^i$$

for all $t \in (-\infty, b_i] \cap Sm$. We have proved this inequality for $i = 0$. Assume that it is proved for all $i < k$ and prove it for $i = k$.

Consider any $t \in (-\infty, b_k] \cap Sm$. If $t + \alpha(t) \geqslant b_{k-1}$, then $\alpha(t) > b_{k-1} - t \geqslant b_{k-1} - b_k = c \cdot q^k$. If $t + \alpha(t) \leqslant b_{k-1}$ (that contradicts to the inequality $t + \alpha(t) > b > b_{k-1}$), then by Claim 5 there is $\xi = \xi(t) \in (t, t + \alpha(t)) \cap Sm \subset (t, b_{k-1})$ such that $\alpha(t) \geqslant q \cdot \alpha(\xi)$. Since $\xi < b_{k-1}$, then $\alpha(\xi) \geqslant c \cdot q^{k-1}$ by the inductive assumption. Therefore, $\alpha(t) \geqslant c \cdot q^k$ in this case too.

Now, it suffices to note that the just proved inequality $\alpha(t) \geqslant c \cdot q^i$ contradicts to the inequality $\alpha(t) = \operatorname{arccot}(\phi'(t)) < \pi$. Actually, for rather large i the inequality $c \cdot q^i > \pi$ holds. This contradiction completes the proof of the claim. □

Now, we are ready to finish the proof of Theorem 1. As we have proved, either Assertion (1), or Assertion (2) from the statement of Claim 6 holds, therefore, it suffices to get a contradiction in both these cases.

Suppose that Assertion (1) holds, i.e. there are a point $t^* \in (a, b^*]$ and a sequence $\{t_n\}$, $t_n \in S^* \subset Sm$ such that $\alpha(t_n) \to 0$ and $t_n \to t^*$ as $n \to \infty$. Put $s_n = t_n + \alpha(t_n)$. By Claim 4 for rather large n the inequality $\phi(s_n) - \phi(t_n) \geqslant q$ holds (since $s_n \to t^* < b$ as $n \to \infty$). But it is impossible, since $t_n \to t^*$, $s_n = t_n + \alpha(t_n) \to t^*$, and the function $t \mapsto \phi(t)$ is continuous at the point t^*. Therefore, we have proved the theorem in this case.

Now, suppose that Assertion (2) holds, i.e. $a > -\infty$ and there is $c > 0$ such that $\alpha(t) \geqslant c$ for all $t \in S^*$. Since $\alpha(t) = \operatorname{arccot}(\phi'(t))$, we get the inequality $\phi'(t) \leqslant \cot(c) \in \mathbb{R}$, $t \in S^*$. According to Claim 3 for every $\eta \in (a, b^*)$ the derivative ϕ' is summable on the interval $[\eta, b^*] \subset (a, b)$ and satisfies the inequality

$$\phi\big(b^*\big) - \phi(\eta) \leqslant \int_\eta^{b^*} \phi'(\tau)\, d\tau \leqslant \cot(c)\big(b^* - \eta\big).$$

Tending η to a, we get $\phi(b^*) - \phi(a) \leqslant \cot(c)(b^* - a) \in \mathbb{R}$, but the latter is impossible because of $\phi(a) = -\infty$. Consequently, we have proved the theorem in this case too.

4. Various consequences and connections with other results

At first we will use the assertion of Theorem 1 for parameterizations of some special type. Let us consider two continuous functions $h, g : [a, b) \subset \mathbb{R} \to \mathbb{R}$ and suppose that the function h is increasing and is not a constant in any neighborhood of the point a. For any $x \in (a, b)$ we consider the set of numbers $\tau \in (a, x]$, that are points of local extremum of the function

$$t \mapsto \bigl(g(x) - g(a)\bigr)h(t) - \bigl(h(x) - h(a)\bigr)g(t). \tag{6}$$

Let $\mu(x)$ be the supremum of such τ. The following theorem gives a non-trivial information on a behavior of $\mu(x)$ as $x \to a$.

Theorem 3. *Suppose in addition that there exists a finite limit* $\lim_{x\to a} \frac{g(x)-g(a)}{h(x)-h(a)}$, *then the following inequality holds*

$$\varlimsup_{x\to a} \frac{h(\mu(x)) - h(a)}{h(x) - h(a)} \geqslant \frac{1}{e}. \tag{7}$$

Proof. We use the assertion of Theorem 1 for the curve $\gamma(t) = (g(t), h(t)) \in \mathbb{R}^2$. For this curve, it is clear that $\tau \in S(x)$ if and only if τ is a point of extremum of the function (6), $D(x) = \sqrt{(g(x) - g(a))^2 + (h(x) - h(a))^2}$. Since the limit $P := \lim_{x\to a} \frac{g(x)-g(a)}{h(x)-h(a)}$ exists and is finite, then it is easy to see that

$$DS(x) = \sup\bigl\{D(\tau) \bigm| \tau \in S(x)\bigr\} \sim D\bigl(\mu(x)\bigr) = \sqrt{\bigl(g\bigl(\mu(x)\bigr) - g(a)\bigr)^2 + \bigl(h\bigl(\mu(x)\bigr) - h(a)\bigr)^2} \quad \text{as } x \to a.$$

Set $L(x) = \frac{h(\mu(x))-h(a)}{h(x)-h(a)}$, then taking into account the above asymptotic equality, we get

$$\left(\frac{DS(x)}{D(x)L(x)}\right)^2 \sim \frac{1 + (\frac{g(\mu(x))-g(a)}{h(\mu(x))-h(a)})^2}{1 + (\frac{g(x)-g(a)}{h(x)-h(a)})^2} \to \frac{1 + P^2}{1 + P^2} = 1 \tag{8}$$

as $x \to a$. Therefore, by Theorem 1

$$\varlimsup_{x\to a} \frac{h(\mu(x)) - h(a)}{h(x) - h(a)} = \varlimsup_{x\to a} \frac{DS(x)}{D(x)} \geqslant \frac{1}{e}. \qquad \square$$

Now, suppose in addition that the function $h, g : [a, b) \subset \mathbb{R} \to \mathbb{R}$ have derivatives and $h'(t) > 0$ on the interval (a, b). Then by Cauchy's mean-value theorem, for any $x \in (a, b)$ there is $\tau \in (a, x)$ with the property:

$$\frac{g(x) - g(a)}{h(x) - h(a)} = \frac{g'(\tau)}{h'(\tau)}. \tag{9}$$

Let $\xi(x)$ be the supremum of such τ. Obviously, any point τ of extremum of the function (6) satisfies the equality (9). Hence, $\xi(x) \geqslant \mu(x)$ for all x, and Theorem 3 implies a non-trivial information on a behavior of $\xi(x)$ as $x \to a$.

Theorem 4. *Suppose in addition that there exists a finite limit* $\lim_{x\to a} \frac{g(x)-g(a)}{h(x)-h(a)}$, *then the following inequality holds*

$$\varlimsup_{x\to a} \frac{h(\xi(x)) - h(a)}{h(x) - h(a)} \geqslant \frac{1}{e}. \tag{10}$$

Remark 2. Note that for any fixed function h the inequality (10) (as well as the inequality (7)) is best possible in general. To show this, set $g(x) = (h(x) - h(a))^{1+\alpha}$, where $\alpha > 0$. Since h is monotone, for all $x \in (a, b)$ there is a unique $\tau = \xi(x) \in (a, x)$ that satisfies Eq. (9). Simple calculations imply

$$\frac{h(\xi(x)) - h(a)}{h(x) - h(a)} = \left(\frac{1}{1 + \alpha}\right)^{1/\alpha}$$

for all $x \in (a, b)$. Therefore, $\lim_{x\to a} \frac{h(\xi(x))-h(a)}{h(x)-h(a)} = (\frac{1}{1+\alpha})^{1/\alpha}$. Note also that $\lim_{\alpha\to 0}(\frac{1}{1+\alpha})^{1/\alpha} = e^{-1}$. The same example and the last formula in the proof of Theorem 3 imply also the unimprovability of the inequality (2).

In the case $h(x) = x$ Theorem 4 implies an assertion on the asymptotic of mean-value points in Lagrange's theorem.

Corollary 1. *Let $g : [a,b) \subset \mathbb{R} \to \mathbb{R}$ be a continuous function that is differentiable on the interval (a,b). For every $x \in (a,b)$ denote by $\xi(x)$ the supremum of numbers $\tau \in (a,x]$ such that $g'(\tau)\cdot(x-a) = g(x) - g(a)$. If the function g has (right hand) derivative at the point a, then the inequality*

$$\overline{\lim_{x\to a}}\, \frac{\xi(x)-a}{x-a} \geqslant \frac{1}{e} \tag{11}$$

holds.

The inequality (11) becomes an equality, for example, for the function $g : [0,1) \to \mathbb{R}$, defined by the equality $g(x) = -\int_0^x \frac{dt}{\ln t}$. The conjecture of validity of the above corollary has been stated (as well as some other conjectures) by Professor V.K. Ionin. In the case, when the derivative $g' =: f$ is continuous, this conjecture could be reformulated in the integral form. Consider a continuous function $f : [a,b] \to \mathbb{R}$. For any $x \in (a,b]$ there exists $\tau \in [a,x]$ such that

$$\int_a^x f(t)\,dt = (x-a) f(\tau)$$

(this is a partial case of the integral mean-value theorem). Such τ is unique if f is strictly decreases or strictly increases. In general case we set

$$\eta(x) := \max\left\{\tau \in [0,x] \;\Big|\; \int_a^x f(t)\,dt = x f(\tau)\right\}.$$

Then (this is equivalent to Corollary 1) the inequality

$$\overline{\lim_{x\to a}}\, \frac{\eta(x)-a}{x-a} \geqslant \frac{1}{e}$$

holds. The latter inequality was proved at first in the paper [9], one can find various generalisations of this result in more recent papers [5,10,12–14].

Theorems 3 and 4 give us a non-trivial information on a behavior of the functions $\mu(x)$ and $\xi(x)$ by estimating the asymptotic of $\frac{h(\mu(x))-h(a)}{h(x)-h(a)}$ and $\frac{h(\xi(x))-h(a)}{h(x)-h(a)}$ respectively.

However, it would be desirable to get analogues assertions for the values $\frac{\mu(x)-a}{x-a}$ and $\frac{\xi(x)-a}{x-a}$ (in the case of Lagrange's theorem $h(x) = x$ we have got the required results, of course). The following results imply some results of this kind.

Definition 2. For a function $f : [a,b) \to \mathbb{R}$ we denote by $\overline{\lim}_{x\to a} \operatorname{ess} f(x)$ the greatest lower bound of numbers $t \in \mathbb{R}$ such that $f(x) \leqslant t$ almost everywhere on some interval $[a,\delta] \subset [a,b)$ (essential upper limit). By analogy, $\underline{\lim}_{x\to a} \operatorname{ess} f(x)$ means the least upper bound of numbers $t \in \mathbb{R}$ such that $f(x) \geqslant t$ almost everywhere on some interval $[a,\delta] \subset [a,b)$ (essential lower limit).

Lemma 1. *Let $h : [a,b) \to \mathbb{R}$ be an increasing function and suppose that*

$$C := \overline{\lim_{x\to a}} \operatorname{ess} \frac{h(x)-h(a)}{(x-a)h'(x)} < \infty.$$

Then for any number $q \in (0,1)$ we get the inequality

$$\overline{\lim_{x\to a}}\, \frac{h(a+q(x-a))-h(a)}{h(x)-h(a)} \leqslant q^{1/C}.$$

Proof. Let us fix some number $\varepsilon > 0$. Decreasing (if necessary) the number b, we may assume that for almost all $t \in (a,b]$ the inequality

$$\frac{h(t)-h(a)}{(t-a)h'(t)} < C + \varepsilon$$

holds, or, equivalently

$$\frac{h'(t)}{h(t)-h(a)} > \frac{1}{C+\varepsilon}\cdot\frac{1}{t-a}.$$

Integrating the latter inequality from $t = x_q := a + q(x-a)$ to $t = x$, we get

$$\frac{1}{C+\varepsilon} \cdot \ln\frac{1}{q} < \int\limits_{x_q}^{x} \frac{h'(t)\,dt}{h(t)-h(a)} \leqslant \ln\bigl(h(t)-h(a)\bigr)\Big|_{t=x_q}^{t=x} = \ln\frac{h(x)-h(a)}{h(x_q)-h(a)}.$$

Here we used properties of the increasing function $x \mapsto \ln(h(x)-h(a))$ (its increment at the interval is not less than the integral of its derivative on the same interval, cf. Remark 1). After simple transformations we get

$$\frac{h(x_q)-h(a)}{h(x)-h(a)} \leqslant q^{\frac{1}{C+\varepsilon}} \quad \text{and} \quad \varlimsup_{x\to a} \frac{h(x_q)-h(a)}{h(x)-h(a)} \leqslant q^{\frac{1}{C+\varepsilon}}.$$

Since $\varepsilon > 0$ is arbitrary, then the lemma is proved. □

Theorem 5. *Let $C = \varlimsup_{x\to a} \operatorname{ess} \frac{h(x)-h(a)}{(x-a)h'(x)}$ in the assumptions and notations of Theorem 3. Then the inequality*

$$\varlimsup_{x\to a} \frac{\mu(x)-a}{x-a} \geqslant e^{-C}$$

holds. If, in addition, the assumptions of Theorem 4 are fulfilled, then the inequality

$$\varlimsup_{x\to a} \frac{\xi(x)-a}{x-a} \geqslant e^{-C}$$

holds too.

Proof. Let us prove the first inequality. For $C = \infty$ all is clear. Further consider the case $C < \infty$. Suppose that the theorem is false. Choose some number q between $\varlimsup_{x\to a} \frac{\mu(x)-a}{x-a}$ and e^{-C} ($0 < q < e^{-C} < 1$, in particular). Without loss of generality we may assume that the inequality

$$\frac{\mu(x)-a}{x-a} < q$$

(or, equivalently, the inequality $\mu(x) < x_q = a + q(x-a)$) holds for all $x \in (a,b)$. Since the function h increases, then $h(\mu(x)) < h(a + q(x-a))$. By Theorem 3 we get

$$\varlimsup_{x\to a} \frac{h(a+q(x-a))-h(a)}{h(x)-h(a)} \geqslant \varlimsup_{x\to a} \frac{h(\mu(x))-h(a)}{h(x)-h(a)} \geqslant \frac{1}{e}.$$

Now, Lemma 1 implies $e^{-1} \leqslant q^{1/C}$, i.e. $e^{-C} \leqslant q$, that contradicts to the choice of the number q. This contradiction proves the first inequality of the theorem.

The second inequality obviously follows from the first one and the fact that $\xi(x) \geqslant \mu(x)$ for all x in the conditions of Theorem 4. □

Let $f : [0,1] \to \mathbb{R}$ be a continuous function, and let $\varphi : [0,1] \to \mathbb{R}$ be a summable and non-negative. Let us define the function $\eta : [0,1] \to \mathbb{R}$ in the following way: for any $x \in (0,1]$, $\eta(x)$ is the maximum of numbers $\tau \in (0,x]$ satisfied the equation

$$\int\limits_0^x \varphi(t) f(t)\,dt = f(\tau) \int\limits_0^x \varphi(t)\,dt$$

(such numbers $\tau \in (0,x]$ do exist because of the integral mean-value theorem). Theorem 5 implies

Theorem 6. *(See [10,11].) Let $C = \varlimsup_{x\to 0} \operatorname{ess}(\int_0^x \varphi(t)\,dt \cdot (x\varphi(x))^{-1})$, then in the notations as above the inequality*

$$\varlimsup_{x\to 0} \frac{\eta(x)}{x} \geqslant e^{-C} \tag{12}$$

holds.

Proof. We may assume that the function $t \mapsto \varphi(t)$ is not zero almost everywhere in any neighborhood of the point 0 (otherwise, in such neighborhood the equality $\eta(x) = x$ holds, and all is clear). By the same manner we may assume that for some $\varepsilon > 0$ the value $\varphi(t)$ is not zero almost everywhere on any interval $[c,d] \subset (0,\varepsilon)$ of non-zero length (otherwise, $C = \infty$, and nothing to prove).

Now, we define two functions $g, h : [0, 1] \to \mathbb{R}$ by the formulas

$$g(x) = \int_0^x \varphi(t) f(t)\, dt, \qquad h(x) = \int_0^x \varphi(t)\, dt.$$

It is clear that the function $x \mapsto h(x)$ increases and is not a constant in any neighborhood of 0, $g(0) = h(0) = 0$, $\lim_{x\to 0} \frac{g(x)-g(0)}{h(x)-h(0)} = f(0) \in \mathbb{R}$. Therefore, we can apply the part of Theorem 5, dealing with the function $x \mapsto \mu(x)$, to these two functions (in this case $a = 0$, $b = 1$). Now, it suffices to verify the inequality

$$\eta(x) \geqslant \mu(x)$$

for all x (sufficiently close to 0). In turn, for this goal it is enough to prove the following assertion for all x sufficiently close to 0: Every $\tau \in (0, x]$ provided a local extremum to the function

$$t \mapsto \big(g(x) - g(0)\big)h(t) - \big(h(x) - h(0)\big)g(t) = \int_0^x \varphi(s) f(s)\, ds \int_0^t \varphi(s)\, ds - \int_0^x \varphi(s)\, ds \int_0^t \varphi(s) f(s)\, ds =: \Psi(t),$$

satisfies the equality $\int_0^x \varphi(s) f(s)\, ds = f(\tau) \int_0^x \varphi(s)\, ds$.

Let us suppose the contrary. Then by the integral mean-value theorem we get

$$\Psi(\tau + \Delta) - \Psi(\tau) = \int_\tau^{\tau+\Delta} \varphi(s)\, ds \left(\int_0^x \varphi(s) f(s)\, ds - f(\nu) \int_0^x \varphi(s)\, ds \right),$$

where ν is some number between $\tau + \Delta$ and τ. For sufficiently small Δ the sign of the expression in the brackets coincides with the sign of the (non-zero by our assumption!) expression $\int_0^x \varphi(s) f(s)\, ds - f(\tau) \int_0^x \varphi(s)\, ds$. At the same time, the sign of $\int_\tau^{\tau+\Delta} \varphi(s)\, ds$ coincides with the sign of Δ (at least for all $\tau \in (0, \varepsilon)$, where ε is the number discussed in the beginning of the proof). This means that the point τ could not be a point of local extremum of the function $t \mapsto \Psi(t)$. The theorem is proved. □

Notice that one can find various generalizations and refinements of the proved theorem in the papers [10,11].

5. Open questions

Note that there is another (but quite natural from a geometric point of view) proximity estimation of points $\gamma(\tau)$, $\tau \in T(t)(S(t))$, to a point $\gamma(t)$. Suppose that the curve $\gamma : [a, b] \to \mathbb{E}^2$ is continuous and rectifiable. Of course, this assumption essentially narrows a class of curves under investigation. Let $L : [a, b] \to \mathbb{R}$ be such that $L(t)$ is the length of an arc (of the curve γ), corresponding to values of the parameter from the interval $[a, t]$. It is clear that for every $t \in (a, b)$ the inequality $0 \leqslant L(\tau) \leqslant L(t)$ holds for all $\tau \in S(t)$ (for all $\tau \in T(t)$ in the case of differentiable curve), therefore, $0 \leqslant \sup\{L(\tau) \mid \tau \in S(t)\ (\tau \in T(t))\} \leqslant L(t)$. It is quite possible that the following conjecture is true.

Conjecture 1. *Let $\gamma : [a, b] \to \mathbb{E}^2$ be an arbitrary continuous rectifiable parametric curve. Then the inequality*

$$\varlimsup_{t\to a} \frac{\sup\{L(\tau) \mid \tau \in S(t)\}}{L(t)} \geqslant \frac{1}{e}$$

holds.

A version of this conjecture for differentiable curves also has doubtless interest.

Conjecture 2. *Let $\gamma : [a, b] \to \mathbb{E}^2$ be an arbitrary continuous rectifiable parametric curve such that for every $t \in (a, b)$ there is a non-zero derivative vector $\gamma'(t)$, then the inequality*

$$\varlimsup_{t\to a} \frac{\sup\{L(\tau) \mid \tau \in T(t)\}}{L(t)} \geqslant \frac{1}{e}$$

holds.

Obviously, the latter conjecture is true for all curves with the property

$$L(t) \sim D(t) \quad \text{as } t \to a. \tag{13}$$

However, this asymptotic equality is not universal, hence, the result of Theorem 2 is not sufficient for studying of Conjecture 2, it demands some special approach. Nevertheless, it is well known that in the case of a smooth curve $\gamma : [a, b] \to \mathbb{E}^2$ (when a curve has a continuous derivative vector $\gamma'(t)$, $t \in [a, b]$), the relation (13) is fulfilled. Hence, we get the following corollary from Theorem 2.

Corollary 2. *Let $\gamma : [a, b] \to \mathbb{E}^2$ be an arbitrary regular smooth parametric curve. Then the inequality*

$$\varlimsup_{t\to a} \frac{\sup\{L(\tau) \mid \tau \in T(t)\}}{L(t)} \geqslant \frac{1}{e}$$

holds.

Finally, we note the following independent interesting problem: to find a convenient criterion for fulfillment of the asymptotic equality (13).

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