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# A Theory of Hyperfinite Sets

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**Abstract**

We develop an axiomatic set theory — the Theory of Hyperfinite Sets THS— which is based on the idea of the existence of proper subclasses of large finite sets. We demonstrate how theorems of classical continuous mathematics can be transfered to THS, prove consistency of THS, and present some applications.


*Keywords:* Nonstandard set theory; Hyperfinite set; Thin class; Zermelo universe

## 0. Introduction

Many applications of nonstandard analysis are based on the simulation of infinite structures by hyperfinite structures. When translated into the language of standard mathematics, such simulation means an approximation of infinite structures by finite structures. Thus, nonstandard analysis provides us with a machinery that allows us to obtain new results about infinite structures using such approximations and corresponding results about finite structures. The latter are often much easier to obtain. This approach is realized in the famous monograph [13] for the construction of probability theory on infinite probability spaces. In the monograph [7], it was shown how this approach can be used for the systematic construction of harmonic analysis on locally compact abelian groups, starting from harmonic analysis on finite abelian groups.

The results obtained in this way allow us to look at this approach from another point of view.

According to this point of view, mathematics should be developed on the basis of the hypothesis that all sets are finite (some kind of the ancient Greeks' atomism).

Historically, the first approach, due to Yessenin-Volpin [19,20], to developing this idea on the basis of modern logic is called ultra-intuitionism. It assumes the existence of the maximal natural number. This approach stimulated investigation of the notion of feasible numbers. The first mathematically rigorous formalization of the notion of the feasibility of natural numbers was introduced by Parikh [14]. Many papers develop Parikh's approach, as well as some other approaches, to the notion of feasibility (see, e.g., [8,4,15]). We do not discuss them here. A very interesting discussion of correlation between Real Analysis and Discrete Analysis is contained in [21]. The main idea of this paper

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is as follows: "Continuous analysis and geometry are just degenerate approximations to the discrete world... While discrete analysis is conceptually simpler ... than continuous analysis, technically it is usually much more difficult. Granted, real geometry and analysis were necessary simplifications to enable humans to make progress in science and mathematics....". In some sense, our paper, together with the paper [6], contributes to this idea.

In this paper we develop an axiomatic theory of finite sets, which we call the Theory of Hyperfinite Sets (THS), by the reasoning explained below. Similarly to Kelley-Morse theory or von Neumann–Bernays–Gödel theory (NBG), THS is a theory of classes in the $\in$-language, where sets are defined as elements of classes. The universe of sets satisfies all the axioms of ZF $^{\text{fin}}$ — the theory obtained by replacing in ZF the axiom of infinity by its negation and adding a suitable form of regularity (for instance, an axiom saying that every set has a transitive closure; see [16]).

However, the properties of classes differ essentially from those of NBG. For example, the Separation Axiom fails in THS: there exist sets that contain proper subclasses (subclasses that are not subsets). The reason why we need to include the last statement in our theory is that we want to consider in THS such properties as feasibility discussed above. Indeed, let $F(x)$ be the statement "$x$ is a feasible number" and let $N$ be a non-feasible number. Then the set $A = \{x \leq N \mid F(x)\}$ satisfies the following inconsistent conditions: (1) $0 \in A$, (2) $\forall x(x \in A \longrightarrow x + 1 \in A)$, and (3) $N \notin A$. The only way to avoid this paradox, if one wants to keep the induction principle for sets, is to assume that $A$ is not a set, and thus the separation axiom fails for the finite set $\{0, 1, \ldots, N\}$.

The paradox discussed in the previous paragraph is a version of the well-known paradox about a pile of sand, due to Eubilides in the 4th century B.C.: since one grain of sand is not a pile and if $n$ grains of sand do not form a pile of sand, then $n + 1$ grains do not form a pile of sand also, then how can we get a pile of sand? Paradoxes of this type cannot be considered in the framework of classical set theory, since the objects, like a pile of sand, have a very vague description and, thus, cannot be considered as any objects of classical mathematics, i.e. as sets. On the other hand, there are many examples that show that such notions arise very naturally in mathematics (see, e.g., the example concerning feasibility above). The first mathematician who systematically studied and axiomatized the notions of this type was Vopěnka. In [17], he introduced the first theory of finite sets, the Alternative Set Theory (AST), where the existence of finite sets containing subclasses that are not sets was postulated. Such subclasses of sets are called *semisets*.

In our opinion, the main defect of Vopěnka's approach is the opposition of his theory to classical mathematics. As was mentioned above (see the quotation from [21]), the advantage of continuous mathematics in comparison to discrete one is its simplicity, which often allows us to solve problems concerning discrete objects.

THS, introduced here, is also based on the idea of the existence of proper subclasses of large finite sets. Finite sets that contain proper subclasses are called hyperfinite sets. This term is borrowed from nonstandard analysis. The primary model for THS is the collection of all subclasses of the set of hereditarily finite sets in the Nonstandard Class Theory NCT [1]. The central notion of *thin* class is defined by a formulation equivalent in NCT to the definition of a class of standard size: a class is thin if any subset of it does not contain proper subclasses. Sets that do not contain proper subclasses are called *small*. The class of all small natural numbers is a thin class. It coincides with the set $\omega$ in ZF. Under our approach, the class of all small numbers can be interpreted as the class of feasible numbers.

We prove that all results of classical mathematics that can be formalized in Zermelo set theory can be proved for thin structures in THS. This is a substantial difference between THS and AST. It allows us to formalize within THS those proofs of theorems about finite sets that use continuous mathematics and, hence, it is not necessary to invent any new proofs for such theorems.

In the discrete world, continuous objects have their place as well: they originate from hyperfinite sets or their $\sigma$-subclasses as quotient "sets" by some *indiscernibility* relation. An indiscernibility relation $\rho$ is an equivalence relation that is a $\pi$-class and satisfies some special condition (see Section 7). A class is called an $\sigma$-class ($\pi$-class) if it can be represented by the union (the intersection) of a thin class. We prove that there exists a thin class of representatives of all $\rho$-equivalence classes which represent the quotient "set" (more exactly, the quotient system of classes) by $\rho$.

For example, to obtain the field of reals $\mathbb{R}$ in THS, one should consider a computer arithmetic implemented in an idealized computer with a hyperfinite memory for the simulation of the field of reals. It may be the usual computer arithmetic, based on the representation of reals in the form with floating point. Let $\langle R; \oplus, \odot\rangle$ be this system. It is well known that, because of the rounding off, the operations $\oplus$ and $\odot$ are neither associative nor distributive. Let $R_b = \{x \in R \mid \exists^{\text{small}} n(|x| < n)\}$, where $\exists^{\text{small}} n$ means "there exists a small natural number $n$". Since the class of all small natural numbers is a thin class, it is easy to see that $R_b$ is a $\sigma$-class. We can interpret the elements of $R_b$ as the computer numbers that are far enough from the boundary of the computer's memory, so that, performing computations with these numbers, one can never get overfilling of memory. Indeed, it can be proved that the class $R_b$ is closed under the operations $\oplus$ and $\odot$. The indiscernibility relation $\rho$ is defined by the condition $x \rho y \iff \forall^{\text{small}} n (|x - y| < \frac{1}{n})$. Obviously, $\rho$ is a $\pi$-class. It also has a natural interpretation: we identify those numbers that differ on a number close enough to the computer zero. It can be proved (in THS) that the quotient system $R_b/\rho$ is isomorphic to the field $\mathbb{R}$.

Certainly, there are many other systems from which one can obtain $\mathbb{R}$ in a way similar to that described in the previous paragraph. The system based on representation of reals in floating point form is discussed in details in [5]. In this paper we introduce a hyperfinite system that is a little bit simpler and has some better properties — it is an abelian group for addition. However, we have proved that it is impossible to obtain the field $\mathbb{R}$ from a hyperfinite system that is an associative ring [6]. Similar facts also hold for many locally compact non-commutative groups. It is shown in [6] that we cannot find the hyperfinite groups that have approximate properties of many important Lie groups such as $SO(3)$. The hyperfinite objects with the best properties, from which all unimodular locally compact groups can be constructed, are quasigroups (latin squares) [6]. These facts demonstrate that the continuous world has better properties than the discrete world. The theory THS introduced here allows us to formalize not only all classical mathematics, but also the statements about the connection between the discrete world and its continuous approximation.

In [3], the Non Standard Regular Finite Set Theory was formulated by Baratella and Ferro. Based on a countably saturated universe of hereditarily hyperfinite sets, NRFST contains a rich structure of external sets *over* it. We believe that a theory of classes of higher levels over the universe of hyperfinite sets of THS can be formulated in the pure $\in$-language and simulated within THS.

## 1. Axioms

**Remark 1.1.** In [1, Section 6] we announced a theory of hyperfinite sets THS. The theory presented here is a result of further development of the idea; the implementation is very different though and, we believe, much more interesting than the one described in [1].

**1.2.** THS is a first-order theory. Semantical objects of the theory are ***classes***. Its language contains only one non-logical symbol — the binary predicate symbol $\in$ of membership.

**1.3.** ***Sets*** are *defined* as members of classes:

$$\text{Set}\,(X) \rightleftharpoons \exists Y \; ( X \in Y ).$$

We accept the convention of using lower-case letters for sets and upper-case letters for classes.

**1.4.** Formulas where all quantifiers range over set variables are called ***normal formulas***.

**1.5.** ***Set formulas*** are normal formulas where no class constants or variables occur.

Axiom of Extensionality:

**Ext** $$\forall X\, \forall Y \; \Big( \forall x \; ( x \in X \longleftrightarrow x \in Y ) \longrightarrow X = Y \Big).$$

Axioms of class formation (arbitrary formulae are allowed):

**Class** $$\forall X_1, \ldots, X_n \, \exists Y \, \forall y \; ( y \in Y \longleftrightarrow \Phi(y, X_1, \ldots, X_n) ).$$

Axiom of set formation:

**Set** $$\text{Set}\; \varnothing \,\&\, \forall x\, \forall y \;\; \text{Set}\,(x \cup \{y\}).$$

Axioms of induction and regularity (only set formulae are allowed):

**Ind** $$\varphi(\varnothing) \,\&\, \forall x \forall y \; ( \varphi(x) \,\&\, \varphi(y) \longrightarrow \varphi(x \cup \{y\}) ) \longrightarrow \forall x \;\; \varphi(x).$$

**1.6.** The class of all sets is denoted by $\mathbb{H}$.

**1.7.** Subclasses of sets are called ***semisets***.
Sets in THS may contain proper subsemisets, i.e. subclasses which are not sets.

**1.8.** A set is called *small* if it does not contain proper subsemisets:

$$\mathsf{small}\ x \rightleftharpoons \forall Y \subseteq x\ (\ \mathsf{Set}\ Y\ ).$$

**1.9.** A class $X$ which is not small is called *infinitely large* or simply *infinite* ($\mathsf{inf}\ X$).

$$\mathsf{inf}\ X \rightleftharpoons \exists Y \subseteq X\ (\ \neg\,\mathsf{Set}\ Y\ ).$$

**1.10.** A class is called *thin* iff every sub*set* of it is small:

$$\mathsf{thin}\ X \rightleftharpoons \forall a \subseteq X\ \forall C \subseteq a\ (\ \mathsf{Set}\ C\ ).$$

Thus, thin set is the same as small set.

In formulas, we use quantifiers with superscripts thin, small and inf in a natural way.

Axiom of Thin Semisets:

**Thin** $\forall X\ \Big(\mathsf{thin}\ X \longrightarrow \exists x\ (\ X \subseteq x\ )\Big).$

Axiom of Compactness:

**Comp** $\forall^{\mathsf{thin}} X\ \forall u\ \Big(u \subseteq \cup X \longrightarrow \exists x \subseteq X\ (\ u \subseteq \cup x\ )\Big).$

Axiom of Exponentiation:

**Exp** $\forall^{\mathsf{thin}} X\ \exists^{\mathsf{thin}} P\ \forall y\ \exists p \in P\ (\ y \cap X = p \cap X\ ).$

**1.11.** We define the ordered pair $\langle x, y\rangle \stackrel{\text{def}}{=} \{\{x\}, \{x, y\}\}$ and the operations $\times$ (cartesian product), $\mathsf{dom}$ (domain), ” (image: $X\text{”}A = \{b : \exists a \in A\ (\ \langle a, b\rangle \in X\ )\}$) in the usual way. We also use $\mathsf{Fnc}\ F$ as a shorthand for $\forall x\ \forall y\ \forall z\ (\ \langle x, y\rangle \in F\ \&\ \langle x, z\rangle \in F \longrightarrow y = z\ )$.

Axiom of Choice:

**Choice** $\forall X\ \Big(\mathsf{thin}\ \mathsf{dom}(X) \longrightarrow$
$\exists F\ (\ \mathsf{Fnc}\ F\ \&\ \mathsf{dom}(F) = \mathsf{dom}(X)\ \&\ F \subseteq X\ )\Big).$

**1.12.** The class $\mathbb{SN}$ of *small natural numbers* is defined as the smallest class which contains the empty set and is closed under the von Neumann successor operation:

$$\mathbb{SN} = \{x : \forall N\ \Big(\ [\ \varnothing \in N\ \&\ \forall n \in N\ (\ n \cup \{n\} \in N\ )\ ] \longrightarrow x \in N\Big)\}.$$

Axioms of Dependent Choices (arbitrary formulae are allowed):

**DC** $\forall X\ \exists Y\ \ \Phi(X, Y) \longrightarrow$
$\forall X_0\ \exists Z\ \Big(Z\text{”}\{\varnothing\} = X_0\ \&\ \forall n \in \mathbb{SN}\ \ \Phi(Z\text{”}\{n\}, Z\text{”}\{n \cup \{n\}\})\Big).$

**1.13.** Similarly to small natural numbers, the class $\mathbb{S}$ of all *standard sets* is defined as the smallest class containing the empty set and closed under the operation of adjoining one element:

$$\mathbb{S} = \{x : \forall S\ \Big(\ [\ \varnothing \in S\ \&\ \forall a \in S\ \forall b \in S\ (\ a \cup \{b\} \in S\ )\ ] \longrightarrow x \in S\Big)\}.$$

Axioms of Transfer (only set formulae are allowed):

**T** $\forall t_1 \in \mathbb{S} \cdots \forall t_n \in \mathbb{S}\ (\ \exists x\ \ \varphi(x, t_1, \ldots, t_n)$
$\longrightarrow\ \exists x \in \mathbb{S}\ \ \varphi(x, t_1, \ldots, t_n)\ ).$

**1.14.** We denote

$$\mathsf{TFS} = \mathbf{Ext} + \mathbf{Class} + \mathbf{Set} + \mathbf{Ind}$$
$$\mathsf{THS}_0 = \mathsf{TFS} + \mathbf{Thin} + \mathbf{Comp}$$
$$\mathsf{THS} = \mathsf{THS}_0 + \mathbf{Exp} + \mathbf{Choice} + \mathbf{DC} + \mathbf{T}.$$

**Remark 1.15.** Axioms of TFS are borrowed from the Alternative Set Theory AST [17] of Vopěnka. **Thin** and **Comp** are true in AST for countable classes. See also 2.11, 2.14 and 3.4.

**Remark 1.16.** The axioms of transfer are not as important in THS as in other nonstandard frameworks, because standard sets are not the primary object of investigation here. The main reason for including them in the list of axioms is Theorem 3.9.

## 2. Basic facts and notions

**2.1.** *Natural numbers* are defined in the same way that ordinals are defined in ZF: they are transitive sets linearly ordered by the membership relation. The class of all natural numbers is denoted by $\mathbb{N}$. **Ind** implies induction over $\mathbb{N}$ for any set-formula $\varphi$:

$$[\,\varphi(\varnothing)\ \&\ \forall n \in \mathbb{N}\ (\,\varphi(n) \longrightarrow \varphi(n \cup \{n\})\,)\,] \longrightarrow \forall n \in \mathbb{N}\ \varphi(n).$$

**2.2. Ind** also implies that, for any set $x$, its *size* $\sharp(x)$ is uniquely defined as a natural number $k$ such that there is a set-bijection from $x$ onto $k$.

**Theorem 2.3.** (1) *All axioms of* TFS *hold in* $\mathbb{S}$.
(2) *The universe* $\mathbb{H}$ *of all sets and the universe* $\mathbb{S}$ *of standard sets both satisfy the axioms of* $\mathsf{ZF}^{\mathrm{fin}}$.

**Proof.** It follows from the definition of $\mathbb{S}$ that the axioms of TFS are true in $\mathbb{S}$. Sochor [16] proved that $\mathsf{ZF}^{\mathrm{fin}}$ is equivalent to the theory with the axioms **Set**, **Ind** and extensionality for sets. Note that we do not use the axioms of Transfer here. □

**Proposition 2.4** (TFS). *There exists a bijective mapping* ac *from the universe* $\mathbb{H}$ *of all sets onto the class* $\mathbb{N}$ *of natural numbers, definable by a set formula and such that* $x \in y$ *implies* $\mathsf{ac}(x) < \mathsf{ac}(y)$ *for all sets* $x$ *and* $y$.

**Proof.** It can be proved in $\mathsf{ZF}^{\mathrm{fin}}$ that the Ackermann encoding of finite sets defined inductively by the conditions $\mathsf{ac}(\varnothing) = 0$ and $\mathsf{ac}(x) = \sum_{a \in x} 2^{\mathsf{ac}(a)}$ is a total bijection. □

**Proposition 2.5** (TFS). (1) $\mathbb{S}$ *is a thin class;*
(2) $\mathbb{S}$ *coincides with the class of hereditarily small sets;*
(3) $\mathbb{SN} = \{n \in \mathbb{N} : \mathsf{small}\ n\} = \mathbb{S} \cap \mathbb{N}$.

Many properties of small sets and thin classes can be proved in TFS already. We use $X \restriction Y$ for the restriction of $X$ to $Y$, i.e. $X \restriction Y = \{\langle x, y\rangle \in X \mid y \in Y\}$.

**Proposition 2.6** (TFS). (1) $\mathsf{small}\ x\ \&\ y \subseteq x \longrightarrow \mathsf{small}\ y$; $\mathsf{thin}\ X\ \&\ Y \subseteq X \longrightarrow \mathsf{thin}\ Y$;
(2) $\mathsf{small}\ x \longleftrightarrow \mathsf{small}\ \sharp(x)$;
(3) $\mathsf{small}\ x \longrightarrow \mathsf{small}\ F \restriction x\ \&\ \mathsf{small}\ F``x$, *for any function* $F$;
(4) $\forall^{\mathrm{inf}} n \in \mathbb{N}\ \exists x\ (\,X \subseteq x\ \&\ \sharp(x) \le n\,) \longrightarrow \mathsf{thin}\ X$;
(5) $\mathsf{small}\ \cup x \longrightarrow \mathsf{small}\ x$; $\mathsf{inf}\ x \longrightarrow \mathsf{inf}\ \cup x$;
(6) $\mathsf{thin}\ X \longrightarrow \mathsf{thin}\ \{y : y \subseteq X\}$;
(7) $[\ \mathsf{thin}\ X\ \&\ \forall x \in X\ \ \mathsf{thin}\ Y``\{x\}\,] \longrightarrow \mathsf{thin}\ Y \restriction X$;
(8) $\mathsf{thin}\ X\ \&\ \mathsf{thin}\ Y \longrightarrow \mathsf{thin}\ X \times Y$.

**Proof.** (2) $\longrightarrow$ Assume $\mathsf{inf}\,\sharp(x)$. Let $f$ be some set-bijection from $\sharp(x)$ onto $x$. Then the class $Y = \{f(n) : n \in \mathbb{SN}\}$ is a proper semiset, since otherwise $\mathbb{SN} = f^{-1}``Y$ would be a set. Thus, $x$ is infinite.

$\longleftarrow$ One should proceed by induction on $\sharp(x)$ over $\mathbb{SN}$. Due to **Set** adjoining one element to a small set gives a small set again.

(3) We use (2) and proceed by induction over $\sharp(x)$.

(4) Let $X$ be not thin. Then, by the definition of thin class, there exists an infinite subset $y \subseteq X$. Therefore, every superset $x \supseteq X$ cannot contain less than $\sharp(y)$ elements.

(5) Let $y = \cup x$ be a small set. Then $\sharp(x) \leq 2^{\sharp(y)}$. Since, by (2), $\sharp(y)$ is small, $\sharp(x)$ is small and $x$ is small.

(6) Denote $Y = \{y : y \subseteq X\}$ and assume that $u \subseteq Y$ is infinite. Then, according to (5), $\cup u$ is infinite as well. But $\cup u \subseteq X$, in contradiction with the fact that $X$ is thin.

(7) Let the left-hand side of the implication hold. Take any set $a \subseteq Y \restriction X$. Then $\mathsf{dom}\ a$ is small, since $X \supseteq \mathsf{dom}\ a$ and $\mathsf{thin}\ X$. By (2), the numbers $p = \sharp(\mathsf{dom}\ a)$ and $q = \max\{\sharp(a``\{u\}) : u \in \mathsf{dom}\ a\}$ are small. Hence, $\sharp(a) \leq p \cdot q$ is also small, and $a$ is small. This proves that $Y \restriction X$ is thin.

(8) Let $X$ and $Y$ be thin classes. Applying (7), we get immediately that $X \times Y$ is thin. □

**Proposition 2.7** ($\mathsf{THS}_0$). *The following statements are equivalent for any class $X$:*

(1) *$X$ is a thin class (all subsets of $X$ are small);*
(2) $\forall^{\mathsf{inf}} n \in \mathbb{N}\ \exists a\ (\ X \subseteq a\ \&\ \sharp(a) = n\ )$*;*
(3) $\forall Y \subseteq X\ \exists y\ (\ Y = y \cap X\ )$.

**Proof.** Implication (3) $\longrightarrow$ (1) is straightforward. Proposition 2.6(4) gives (2) $\longrightarrow$ (1).

Assume that $X$ is thin, $Y \subseteq X$ and $n \in \mathbb{N}$ is infinitely large. By **Thin**, $X \subseteq x$ for some set $x$. To complete the proof, it is enough to show that

$$\exists y \subseteq x\ (\ y \cap X = Y\ \&\ \sharp(y) \leq n). \qquad (*)$$

Denote $s = \{y \subseteq x : \sharp(y) \leq n\}$, $D = \big\{\{y \in s : a \notin y\} : a \in Y\big\}$, and $\bar{D} = \big\{\{y \in s : b \in y\} : b \in X \setminus Y\big\}$. $D$ is thin since, for every set $t \subseteq D$, there is a set $\cup\{x \setminus \cup d : d \in t\} \subseteq Y$ of the same size. Similarly, $\bar{D}$ is thin as well. Suppose that (*) does not hold. Then $\cup(D \cup \bar{D}) \supseteq s$. Hence, by **Comp**, $\cup t \supseteq s$ for some small $t \subseteq D \cup \bar{D}$, which is impossible because $\sharp(t) < n$. □

As an immediate corollary, we get the following proposition.

**Proposition 2.8** ($\mathsf{THS}_0 + \mathbf{Exp}$). $\forall^{\mathsf{thin}} X\ \exists^{\mathsf{thin}} P\ \forall Y \subseteq X\ \exists p \in P\ (\ Y = p \cap X\ )$.

**Proposition 2.9** ($\mathsf{THS}_0$). (1) $\mathsf{thin}\ X \longrightarrow \mathsf{thin}\ (\mathsf{dom}\ X)$*;*
(2) $\mathsf{thin}\ X\ \&\ \mathsf{thin}\ Y \longrightarrow \mathsf{thin}\ X \cup Y$*;*
(3) $\mathsf{thin}\ X \longrightarrow \Big[\mathsf{thin}\ F``X\ \&\ \forall y \subseteq F``X\ \exists x \subseteq X\ (\ y = F``x\ )\Big]$*, for any function $F$;*
(4) $\mathsf{thin}\ X \longrightarrow \exists x\ (\ \cup X \subseteq x\ )$.

**Proof.** (1), (2) It can be easily seen that projections and binary unions preserve the property (2) of Proposition 2.7.

(3) Let $X$ be thin. It follows from Proposition 2.6(7) that $F \restriction X$ is also thin. By (1), $F``X$ is thin. Using induction over $\mathbb{SN}$ on the cardinality of $y \subseteq F``X$, one proves the existence of a set $x$ such that $F``x = y$.

(4) If $X$ is thin, $X \subseteq y$ for some set $y$ by **Thin**. Put $x = \cup y$. □

**2.10.** A class is called <u>*countable*</u> iff it can be bijectively mapped onto the class of small natural numbers.

**2.11.** It follows from the previous proposition that the theory $\mathsf{AST} + \mathbf{Thin} + \mathbf{Comp} +$"there exists an uncountable thin class" is inconsistent. Indeed, in $\mathsf{AST}$, due to the axiom of two cardinalities saying that there is a bijection between any two uncountable classes, an uncountable thin semiset can be bijectively mapped onto an infinite set which is not thin, in contradiction with Proposition 2.9(3).

**2.12.** The property of being a thin infinite class behaves as a cardinality lying between the cardinalities of small sets and those of infinite sets. This fact can be expressed in the following way.

We define *inner cardinality* of a class:

$$\mathsf{ICard}\ X \stackrel{\text{def}}{=} \{n : \exists x\ (\ x \subseteq X\ \&\ \sharp(x) = n + 1\ )\}.$$

Then for sets we have $\mathsf{ICard}\ x = \sharp(x)$, and infinite thin classes are exactly the classes $X$ such that $\mathsf{ICard}\ X = \mathbb{SN}$.

**Proposition 2.13** (*TFS*). *Axioms* **Thin** *and* **Comp** *together are equivalent to the following statement:*

Prolongation principle:

$$\forall^{\text{thin}} X \left[ \forall^{\text{small}} x \subseteq X\, \varphi(x) \longrightarrow \exists y \ (\,\mathsf{Set}\,(y)\,\&\, X \subseteq y \,\&\, \varphi(y)\,) \right]$$

*where $\varphi$ is any set-formula with set-parameters.*

**Remark 2.14.** The prolongation principle formulated here is a generalization of the prolongation axiom of AST, which says that every countable function is a subclass of a set-function.

The next proposition lists counterparts of statements which have become customary tools in nonstandard analysis. All of them are just special cases of the prolongation principle formulated above.

**Proposition 2.15** ($\mathsf{THS}_0$). Saturation: $\forall^{\text{thin}} Y \ (\forall y \subseteq Y \ (\cap y \neq \varnothing) \longrightarrow \cap Y \neq \varnothing)$;
Extension: $\forall F \ \big(\text{thin } F \,\&\, \mathsf{Fnc}\ F \longrightarrow \exists f \ (\mathsf{Fnc}\ f \,\&\, F \subseteq f)\big)$;
Nelson's idealization principle:

$$\forall^{\text{thin}} A \ \Big( \forall a_0 \subseteq A\ \exists x\ \forall a \in a_0\ \varphi(a,x) \longrightarrow \ \exists x\ \forall a \in A\ \varphi(a,x) \Big),$$

*for any set-formula $\varphi$ with set parameters.*

**2.16.** As we have already said in the introduction, the simplest and the most important proper classes are the $\sigma$-classes and $\pi$-classes, which are defined in THS as follows: a *$\sigma$-class* is a union of a thin class, and a *$\pi$-class* is an intersection of a thin class.

Both $\pi$-classes and $\sigma$-classes are semisets (see Proposition 2.9).

**2.17.** Together with sets and classes, one can also consider in THS *systems of classes* defined as collections of classes satisfying a certain formula and written as terms of the form

$$\{X : \Phi(X)\},$$

where $\Phi$ is an arbitrary formula with some class- or set-parameters.

**2.18.** A system of classes $\mathcal{X} = \{X : \Phi(X)\}$ is called *codable* if there is a class $C$ such that

$$\mathcal{X} = \{C''\{d\} : d \in \mathsf{dom}(C)\}.$$

Such coding by a class $C$ is called *extensional* iff $C''\{d\} \neq C''\{d'\}$ for distinct elements $d, d' \in \mathsf{dom}(C)$.

If a system $\mathcal{X}$ is coded by a class $C$, one can use quantification over subsystems of $\mathcal{X}$. For instance, $\forall \mathcal{Y} \subseteq \mathcal{X}\ \ \Phi(\mathcal{Y})$ can be interpreted as $\forall D \subseteq \mathsf{dom}\ C\ \ \Phi(\{C''\{d\} : d \in D\})$.

**2.19.** If there exists a coding $C$ such that $\mathsf{dom}(C)$ is a thin class, the system $\mathcal{X}$ is called a *thin system of classes*.

If $\mathcal{X}$ is a thin system, we can always assume, without loss of generality, due to an axiom of Choice, that a given coding of $\mathcal{X}$ is extensional.

Let $\mathcal{X}$ be a thin system of classes and $C$ be a thin encoding of $\mathcal{X}$. By the axiom **Exp**, there exists a thin class $P$ such that $\{P''\{p\} : p \in \mathsf{dom}(P)\}$ is the family of all subclasses of $\mathsf{dom}(C)$. Thus, we can consider the system $\{\{C''\{d\} : d \in P''\{p\}\} : p \in \mathsf{dom}(P)\}$ of *systems of classes*. Certainly the variable $p$ here can be restricted to an arbitrary subclass of $\mathsf{dom}(P)$. In the same way, we can speak about thin systems of higher-level systems using an appropriate encoding.

**Proposition 2.20** (THS). (1) *For every thin class $X$, there exists a strong well-ordering of $X$ (a well-ordering is strong iff every sub*class *of $X$ has a least element).*
(2) *Every infinite class contains a countable infinite subclass.*

**Proof.** If an infinite class $X$ is thin, then one can build a strong well-ordering of $X$, applying Proposition 2.8 and **Choice**, very much like the way that one gets a well-ordering of a set using the axiom of choice. The least infinite initial segment of $X$ under that ordering will be countable.

If $X$ is not thin, it contains an infinite subset $x$. Hence, there is a bijective mapping $h$ from an infinite natural number onto $x$. The class $h''\mathbb{SN}$ will be countable and infinite. □

**Remark 2.21.** The converse statement to Proposition 2.20(1) is true if we accept an additional axiom analogous to the axiom of chromatic classes of NCT.

The following proposition describes small sets in a way similar to the classical Dedekind's characterization of finite sets (a set is finite iff it is not of equal cardinality with any of its proper subset).

**Proposition 2.22** (THS). $\forall X \ \Big(\text{small } X \longleftrightarrow \forall Y \subseteq X \ \Big[ Y \neq X \longrightarrow \forall F : X \to Y \ \ (\text{“}F \text{ is not injective”}) \Big]\Big).$

### 3. ∈-structures and Zermelo universes

It is known (see [10,9]) that saturation principles allow to simulate structures satisfying the axioms of Zermelo set theory or even ZFC within nonstandard models of arithmetic. On the other hand, in “fully saturated” nonstandard set theories (such as E. Nelson's Internal Set theory, NCT or Hrbacek Set Theory of V. Kanovei and M. Reeken), every ∈-structure of standard size is isomorphic to an ∈-substructure of some hereditarily hyperfinite set.

In accordance with the above-mentioned facts, the main result of this section states that every thin semiset can be embedded in a thin subuniverse that satisfies axioms of Zermelo set theory with choice, subclasses in the sense of THS corresponding to subsets in the sense of the Zermelo subuniverse.

**3.1.** Under Zermelo Set Theory with Choice ZC, we understand the theory with the axioms of Extensionality, Union, Powerset, Separation, Infinity, Choice and Regularity. Thus, $\mathsf{ZFC} = \mathsf{ZC} +$ Replacement. We denote by $\mathsf{ZC}^-$ the theory ZC without the axiom of Regularity.

**Theorem 3.2** (THS). *For any thin class $X$, there exists a thin class $Z$ such that*

(1) $X = Z \cap x$ *for some* $x \in Z$;
(2) $\forall x \in Z \ \forall C \subseteq x \cap Z \ \exists q \in Z \ \ (q \cap Z = C)$;
(3) $\forall x \ \ (x \subseteq Z \longrightarrow x \in Z)$;
(4) *All axioms of* $\mathsf{ZC}^-$ *are true in* $Z$.

**3.3.** A class satisfying the conditions (2), (3) and (4) of Theorem 3.2 is called a *Zermelo universe.*

**Remark 3.4.** The definition of Zermelo universe and Theorem 3.2 are very close in formulation to ZF -classes and Cantorian axioms in AST(given in Chapter 12 of [18]). But the important condition (2) does not hold in AST.

This theorem becomes a theorem of THS if we give a formal meaning to (4) using encoding of formulas.

We fix some explicit coding, by standard sets, for symbols of logical connectives, quantifiers, membership relation, punctuation signs and a countable set of variables, and a coding for sets as parameters. *Formal formulas* are naturally defined within THS by induction as special (well-formed) sequences of codes. Every formula $\varphi$ of THS, with parameters, gets its formal counterpart $\ulcorner\varphi\urcorner$ — the code of $\varphi$. Any formal formula of small length with standard parameters is standard (as a set).

The language $\mathsf{SL}(P)$ is defined as the class of all formal formulas of small length with parameters from the class $P$.

Evidently, the language $\mathsf{SL}(P)$ is thin for any thin $P$.

**Proposition 3.5** (TFS). *For any class $X$, there exists a unique class $T$ which consists of closed formulas of* $\mathsf{SL}(X)$ *and satisfies the following properties:*

(1) $\ulcorner x_1 = x_2 \urcorner \in T \longleftrightarrow x_1, x_2 \in X \,\&\, x_1 = x_2$;
(2) $\ulcorner x_1 \in x_2 \urcorner \in T \longleftrightarrow x_1, x_2 \in X \,\&\, x_1 \in x_2$;
(3) $\theta_1 \cdot {'\vee'} \cdot \theta_2 \in T \longleftrightarrow \theta_1 \in T \vee \theta_2 \in T$;
(4) ${'\neg'} \cdot \theta_1 \in T \longleftrightarrow \theta_1 \notin T$;
(5) ${'\exists'} \cdot v \cdot \theta \in T \longleftrightarrow \exists x \in X \ \ (\theta_{v \to x} \in T)$,

*where $\theta_1, \theta_2$ are closed formulas of $\mathsf{SL}(X)$; $\theta$ is a formula of $\mathsf{SL}(X)$ with the only free (symbol of) variable $v$; $\theta_{v\to x}$ is obtained from $\theta$ by replacing every occurance of $v$ with the code of the set $x$; $'\vee'$, $'\neg'$, and $'\exists'$ are the codes of the corresponding symbols; and the dot $\cdot$ denotes concatenation of sequences and codes.*

We denote as $\mathsf{True}(X)$ the class, the existence of which is stated by Proposition 3.5.

For any closed formula $\varphi$ with parameters from some class $X$, it is provable in THS that

$$\varphi^X \longleftrightarrow \ulcorner\varphi\urcorner \in \mathsf{True}(X),$$

where $\varphi^X$ is a relativization of $\varphi$ to the class $X$.

We denote, for any $\theta \in \mathsf{SL}(X)$,

$$X \models_f \theta \rightleftharpoons \theta \in \mathsf{True}(X).$$

Now we formalize (4) from Theorem 3.2:

(4) $\quad Z \models_f \theta$ for each $\theta$ such that "$\theta$ is an axiom of $\mathsf{ZC}^-$",

where the phrase in quotes is appropriately expressed as a formula of THS.

We define the class $\mathsf{Def}(X)$ of sets definable with a formula from $\mathsf{SL}(X)$ as follows:

$$\mathsf{Def}(X) \stackrel{\text{def}}{=} \big\{x : x = \{y : \mathbb{H} \models \theta(y)\} : \theta \in \mathsf{SL}(X) \text{ has exactly one free variable}\big\}.$$

Obviously, $\mathsf{Def}(X)$ is a class. If $X$ is thin, $\mathsf{Def}(X)$ is also thin.

We will say that a class $C$ is an *f-elementary submodel* of a class $M \supseteq C$ (notation: $C \preccurlyeq_f M$) iff

$$C \models_f \varphi \longleftrightarrow M \models_f \varphi$$

for any $\varphi \in \mathsf{SL}(C)$.

**Theorem 3.6** (TFS). *For any class X, the class* $\mathsf{Def}(X)$ *is an f-elementary submodel of* $\mathbb{H}$.

**Proof.** Note that $\mathsf{ac}^{-1}(\min\{\mathsf{ac}\ (a) : \theta(a)\}) \in \mathsf{Def}(X)$ for $\theta \in \mathsf{SL}(X)$. □

**Corollary 3.7.** *In the theory* THS *without the transfer axioms, the transfer axioms follow from the statement* $\mathsf{Def}(\varnothing) = \mathbb{S}$.

**Proof of Theorem 3.2.** Using the axiom of dependent choices, we construct the sequence of structures $S_n$ as follows.

We start from some thin class $S_0 \supseteq X$ such that $S_0 \preccurlyeq \mathbb{H}$. Such a class does exist by Theorem 3.6.

Given a class $S_n$, using the axioms of exponentiation and choice, we can choose a thin class $S_{n+1}$ to satisfy the following properties:

(1) $\forall C \subseteq S_n\ \exists y \in S_{n+1}\ \Big(\ y \cap S_n = C\ \&\ \ (\,\mathsf{Set}\ (C) \longrightarrow y = C\ )\Big)$;
(2) $\forall x, y \in S_{n+1}\ \ (\ x \neq y \longrightarrow x \cap S_n \neq y \cap S_n\ )$;
(3) $\forall x \in S_{n+1} \setminus S_n\ \forall y \in S_n\ \ (\ x \notin y\ )$.

We put

$$Z = \bigcup_{n \in \mathbb{SN}} S_n.$$

It is easy to see that the axioms of extensionality, union, power set, separation, infinity and choice hold in $Z$. □

**Corollary 3.8.** (1) THS *is not a conservative extension of* $\mathsf{ZF}^{\text{fin}}$;
(2) THS *is strictly stronger than* $\mathsf{ZC}^-$.

If we take $X = \mathbb{S}$ in the conditions of the previous theorem and apply transfer, we immediately get the following theorem.

**Theorem 3.9.** *Every statement about finite sets provable in* $\mathsf{ZC}^-$ *holds in* THS *as well.*

**Remark 3.10.** Nonstandard extensions of superstructures over a thin class can be constructed easily in THS as thin $\in$-structures, which allows us to use "essentially external" methods of nonstandard analysis such as nonstandard hulls of Banach spaces or Loeb measures.

## 4. Real numbers

Real numbers can be introduced in THS in a quite usual and straightforward way — as elements of a complete linearly ordered field.

We introduce explicitly the rational numbers first.

**4.1.** First of all, define operations $+$ and $\cdot$ on $\mathbb{N}$ by the formulas

$$x + y = \sharp(x \cup \{0\} \times y); \; x \cdot y = \sharp(x \times y).$$

Obviously, the operations that have been introduced satisfy the classical recursive definitions of addition and multiplication of natural numbers. It is easy to see that the subclass $\mathbb{SN}$ of $\mathbb{N}$ is closed under these operations.

**4.2.** Usually, the ring of integers is defined as a quotient set of $\mathbb{N} \times \mathbb{N}$ under the equivalence relation

$$\langle a, b\rangle \sim \langle a_1, b_1\rangle \Longleftrightarrow a + b_1 = a_1 + b.$$

Since, in our case, $\mathbb{N}$ is a class, we must define the quotient class by a system of representatives.

Thus, the class $\mathbb{Z}$ of integers can be defined, for example, by the formula

$$\mathbb{Z} = (\{0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{0\})$$

with obviously defined addition, multiplication and linear order relations. It is also easy to prove that the thin class $\mathbb{SZ}$ of standard elements of $\mathbb{Z}$ is a subring of $\mathbb{Z}$.

**4.3.** The field $\mathbb{Q}$ is defined as the quotient field of the integral domain $\mathbb{Z}$. As before, we must define this quotient field by a system of representatives, e.g. by the formula

$$\mathbb{Q} = \{\langle a, b\rangle \in \mathbb{Z} \times \mathbb{Z} \mid b > 0, \; \mathsf{gcd}\,(a, b) = 1\}.$$

Once again, it is easy to prove that the thin class $\mathbb{SQ}$ of standard elements of $\mathbb{Q}$ is a subfield of $\mathbb{Q}$.

**4.4.** A class $\langle R; +, \cdot, \le\rangle$ is called a *field of real numbers* iff it satisfies the axioms of linearly ordered fields and the following completeness property:

every bounded above *subclass* of $R$ has a supremum.

**Theorem 4.5** (THS). (1) *There exists a thin class that is a field of real numbers.*
(2) *The field $\mathbb{SQ}$ is dense in a field of real numbers.*
(3) *Any two fields of real numbers are isomorphic.*

**Proof sketch.** The proof quite repeats the classical proof. We can choose any usual way of constructing real numbers. Take, for example, Dedekind cuts. Due to the axiom of exponentiation, $\{C : C \subseteq \mathbb{SQ}\} = \{P\text{''}\{c\} : c \in \mathsf{dom}(P)\}$ for some thin class $P$. Every Dedekind cut can be identified then with an element of $\mathsf{dom}(P)$, and we build the field of real numbers as a subclass of $\mathsf{dom}(P)$.

The classical proofs of (2) and (3) can also be transfered easily to THS. $\square$

**Remark 4.6.** In every Zermelo subuniverse $Z$, the field of reals in the sense of $Z$ is a field of reals in the global sense.

**4.7.** In what follows, we fix some field of real numbers $\langle \mathcal{R}; +, \cdot, \le\rangle$, and call it *the* field of reals.

**Remark 4.8.** There is no definable field of real numbers in THS (see Proposition 6.3).

As in nonstandard analysis, every bounded rational number has a standard part in $\mathcal{R}$.

Put $\mathbb{Q}_b = \{x \in \mathbb{Q} : \exists r \in \mathbb{SQ}\,(|x| < r)\}$. We call elements of $\mathbb{Q}_b$ bounded rational numbers. Obviously, $\mathbb{Q}_b$ is a subring of $\mathbb{Q}$ and $\mathbb{SQ} \subseteq \mathbb{Q}_b$.

Let $\mu(0) = \{\alpha \in \mathbb{Q}_b : \forall r \in \mathbb{SQ} \;\, (r > 0 \longrightarrow |\alpha| < r)\}$. Then $\mu(0) \subseteq \mathbb{Q}_b$ is an ideal in $\mathbb{Q}_b$. The following theorem shows that, actually, $\mu(0)$ is a maximal ideal in $\mathbb{Q}_b$.

**Theorem 4.9.** *There exists a unique surjective homomorphism* $\mathsf{st} : \mathbb{Q}_b \to \mathcal{R}$. *The kernel* $\ker(\mathsf{st}) = \mu(0)$.

The real number $\mathsf{st}(x)$ is called *the standard part of a bounded rational $x$*.

## 5. Ordinary mathematics in THS

Intuitively, thin classes behave exactly as usual infinite sets. We would like to transfer notions and results of ordinary mathematics to systems of thin classes. The informal principle is:

*Everything that is true in ordinary mathematics about sets, their subsets, powersets and so on is true in* THS *about thin classes, their subclasses, systems of their subclasses and so on.*

In what follows, we will give a formal account of the formulated principle.

**5.1.** As an example, we would like to say whether a system $\mathcal{T}$ of subclasses of a thin class $X$ is a topology on $X$. If $\mathcal{T} = \{T"\{d\} : d \in D\}$, this can be expressed in the following way:

$$T"D = X \,\&\, \forall d_1, d_2 \in D \exists d \;\; ( T"\{d_1\} \cap T"\{d_2\} = T"\{d\} ) \,\&\, \forall D' \subseteq D \exists d \;\; \left( \bigcup_{e \in D'} T"\{e\} = T"\{d\} \right). \qquad (1)$$

If $X$ is represented in a Zermelo universe $Z$ by an element $x \in Z$ ($Z \cap x = X$), then $\mathcal{T}$ is also represented in $Z$ by some $t \in Z$:

$$\forall Y \;\; \Big( Y \in \mathcal{T} \longleftrightarrow \exists y \in t \cap Z \;\; ( y \cap Z = Y ) \Big),$$

and (1) is true iff $Z \models$ "$t$ is a topology on $x$".

**5.2.** In a more general setting, we may need to refer to higher levels of cumulative hierarchy over some thin class. Some encoding is necessary for that. To describe a general situation and to abstract from a particular encoding of systems of classes (as we did in 5.1), we consider extensional systems over thin classes.

**5.3.** Consider a system of classes (see 2.17) $\mathcal{X}$ and a system of pairs of classes $\mathcal{E} \subseteq \mathcal{X}^2$. In what follows we use the following notation:

$$X \,\mathcal{E}\, Y \iff \langle X, Y \rangle \in \mathcal{E};$$
$$Y_{\mathcal{E}} \stackrel{\text{def}}{=} \{Z : Z \,\mathcal{E}\, Y\};$$
$$\cup_{\mathcal{E}} \mathcal{S} \stackrel{\text{def}}{=} \{Y : \; \exists Z \;\; ( Z \in \mathcal{S} \,\&\, Y \,\mathcal{E}\, Z )\}.$$

It is easy to prove the existence and the uniqueness of the class $TC(X)$ for any class $X \in \mathcal{X}$ such that

(1) $\mathsf{dom}(TC(X)) = \mathbb{SN}$;
(2) $\forall n \in \mathbb{SN} \; TC(X)"\{n+1\} = \cup_{\mathcal{E}} TC(X)"\{n\}$.

For $k \in \mathbb{SN}$ put

$$\underbrace{\cup_{\mathcal{E}} \cdots \cup_{\mathcal{E}}}_{k \text{ times}} X_{\mathcal{E}} \stackrel{\text{def}}{=} TC(X)"\{n+1\}.$$

The system of classes $X$ equipped with the system of pairs of classes $\mathcal{E}$ is called a (thin) <u>*extensional system*</u> over a thin class $A$ iff the following conditions hold:

(1) $X \,\mathcal{E}\, Y \longrightarrow X \in \mathcal{X} \,\&\, Y \in \mathcal{X}$;
(2) $A \in \mathcal{X}$, $A_{\mathcal{E}} = A$, $\forall a \in A \;\; ( a_{\mathcal{E}} = a \cap A )$;
(3) $\forall X, Y \in \mathcal{X} \setminus A \; \Big( \forall Z \;\; ( Z \,\mathcal{E}\, X \longleftrightarrow Z \,\mathcal{E}\, Y ) \longrightarrow X = Y \Big)$;
(4) $\forall X \in \mathcal{X} \; \exists^{\mathsf{small}} k \;\; ( \underbrace{\cup_{\mathcal{E}} \cdots \cup_{\mathcal{E}}}_{k \text{ times}} X_{\mathcal{E}} \subseteq A )$.

We put

$$\mathcal{X}_0 = A; \;\; \mathcal{X}_1 = \{X : X_{\mathcal{E}} \subseteq A\}; \;\; \mathcal{X}_k = \{X \in \mathcal{X} : \underbrace{\cup_{\mathcal{E}} \cdots \cup_{\mathcal{E}}}_{k-1 \text{times}} X_{\mathcal{E}} \subseteq A\}, k > 1.$$

A system $\mathcal{X}$ is called <u>*k-full*</u> iff

$$\forall \mathcal{Y} \subseteq \mathcal{X}_k \; \exists Y \in \mathcal{X} \; ( Y_{\mathcal{E}} = \mathcal{Y} ).$$

**5.4.** We define a <u>*formula of ordinary mathematics*</u> (o.m.-formula) to be an $\in$-formula $\varphi(A, X_1, \ldots, X_n)$ where all quantifiers have the form $\exists x \in \mathcal{P}^k(A)$ or $\forall x \in \mathcal{P}^k(A)$, where each quantifier has its own natural number $k$ of iterations of the powerset operation $\mathcal{P}$. (Formally, "$\forall x \in \mathcal{P}(A) \ldots$" is to be read as $\forall x \; \Big( \forall z \; ( z \in x \longrightarrow z \in A ) \longrightarrow \ldots \Big)$, and so on). The greatest $k$ is called the <u>*height*</u> of the formula $\varphi$.

For any o.m.-formula $\varphi$ of height $k$, <u>the truth</u> of $\varphi(A, X_1, \ldots, X_n)$ in a $k$-full extensional system $\mathcal{X}$ over $A$ ($X_i \in \mathcal{X}$) is defined in a straightforward way (we omit the obvious details). We write $\mathcal{X} \models \varphi$ if $\varphi$ is true in $\mathcal{X}$.

**Theorem 5.5.** *Let $\mathcal{X}$ be an extensional system over a thin class A. Suppose A is represented in a Zermelo universe Z: $A = Z \cap a$ for some $a \in Z$. Then there exists a* unique *embedding $\mathcal{J} : \mathcal{X} \to Z$ such that*

$$\mathcal{J}(A) = a \;\&\; \forall X, Y \in \mathcal{X} \; ( X \, \mathcal{E} \, Y \longleftrightarrow \mathcal{J}(X) \in \mathcal{J}(Y) ).$$

*Moreover, if $\mathcal{X}$ is k-full then, for any $X_1, \ldots, X_n \in \mathcal{X}$ and any o.m.-formula $\varphi(A, X_1, \ldots, X_n)$ of height $\leq k$, we have*

$$\mathcal{X} \models \varphi(A, X_1, \ldots, X_n) \longleftrightarrow Z \models \varphi(\mathcal{J}(A), \mathcal{J}(X_1), \ldots, \mathcal{J}(X_n)).$$

The proof is quite straightforward.

**5.6.** Theorem 5.5 allows us to extend the definition of truth of o.m.-formulas so that it can be applied to extensional systems that are not necessarily $k$-full, where $k$ is the height of the formula.

Let $\mathcal{X}$ be an extensional system over a thin class $A$, $X_1, \ldots, X_n \in \mathcal{X}$ and $\varphi(A, X_1, \ldots, X_n)$ be an o.m.-formula of height $k$. Then we say that $\varphi(A, X_1, \ldots, X_n)$ is true for $\mathcal{X}$ iff $\mathcal{X}' \models \varphi(A, X_1, \ldots, X_n)$ for some (and then for any) $k$-full extensional system $\mathcal{X}' \supseteq \mathcal{X}$ over $A$.

**Theorem 5.7.** *Suppose that $\forall A \;\; \forall X_1 \in \mathcal{P}^{k_1}(A) \ldots \; \forall X_n \in \mathcal{P}^{k_n}(A) \;\; \varphi(A, X_1, \ldots, X_n)$ is a theorem of ordinary mathematics provable in* $\mathsf{ZC}^-$. *Let $X_1 \in \mathcal{X}_{k_1}, \ldots, X_n \in \mathcal{X}_{k_n}$ in an extensional system $\mathcal{X}$ over a thin class A. Then $\varphi(A, X_1, \ldots, X_n)$ is true for $\mathcal{X}$.*

## 6. Interpretations of THS

**Theorem 6.1.** *The collection of all subclasses of the set $V_\omega$ of hereditarily finite sets, together with the original membership relation, gives an interpretation of* THS *in* NCT. *Moreover, under this interpretation:*

(1) *sets are exactly finite subsets of $V_\omega$;*
(2) *thin classes are exactly subsemisets of $V_\omega$ of standard size.*

**Proof.** By Theorem 3.12 of [1], a set is S-finite in NCT (that is, having a standard finite cardinality) iff every subclass of it is a set. Corollary 4.12 of [1] states that a semiset $X$ has a standard size iff every subset of $X$ is S-finite. So we have that small sets are interpreted as S-finite, and thin classes are interpreted as semisets of standard size. **Ext** holds obviously. The axiom **Class** follows from Corollary 4.17 of [1], stating that any formula in which only semisets are quantified is equivalent to a normal formula. The axioms **Set** and **Ind** can be derived from the theorem of Sochor mentioned in the proof of Theorem 2.3. Proposition 4.13 of [1] says that any semiset of standard size can be embedded into a set of any given infinitely large cardinality. The axiom of thin semisets follows. The truth of **Comp** can be derived easily from the Saturation Theorem 4.7 of [1].

The Choice Theorem 4.20 of [1] implies **Choice**. The axiom of exponentiation **Exp** also follows from the Choice theorem because, if $\kappa$ is the "standard size" of a semiset $X$, then $2^\kappa$ is the "standard size" of the class of its subclasses.

Theorem 3.2.11 of Kanovei and Reeken [11] shows that the scheme of Inner Dependent Choice holds in **BST** (Bounded Set Theory). Since NCT is a conservative extension of **BST** [1, Theorem 5.1] and semisets are uniformly parameterized by sets [1, Theorem 4.16], the scheme **DC** is also true. □

**Remark 6.2.** In a model of Nelson's **IST** [12], the collection of all subclasses of $V_\omega$ gives a model of $\mathsf{THS}_0$ but does not give a model of the full THS. The reason is that there exists a subclass $O$ of $V_\omega$ that can be one-to-one mapped onto the class of all standard sets and therefore will be thin, but neither **Choice** nor **Exp** can be proved for $O$.

**Proposition 6.3.** *There is no formula $\Phi$ with one free variable such that $\exists! X\,\Phi(X)\,\&\,\forall X\quad(\Phi(X)\longrightarrow$ "$X$ is an uncountable thin class") would be a theorem of* THS.

**Proof.** Under the interpretation of THS in NCT described above, every formula of THS gets translated to a normal formula of NCT. By Proposition 7 from [2], any class of standard size defined by a formula without parameters consists of standard elements.[1] Therefore, any thin class definable by a formula in such a model has to consist of hereditarily small sets, and hence cannot be uncountable. □

For any class $X$, denote

$$\mathsf{ADef}(X) = \bigcap_{n\in \mathsf{Def}(X)\setminus\mathbb{S}} \{x : \mathsf{ac}(x) < n\}.$$

Due to compactness, $\mathsf{ADef}(X)$ is nonempty for any thin class $X \neq \mathbb{S}$.

**Proposition 6.4.** *For any thin class $X \neq \mathbb{S}$, the collection of all subclasses of the class $\mathsf{ADef}(X)$, together with the original membership relation, forms an interpretation of* THS *in* THS.

Thus, there are interpretations of THS in THS of any size: $\mathsf{ADef}(l)$ is a subclass of a set having less than $l$ elements. Moreover, the universe $\mathbb{H}$ of all sets can be thought of as a subclass of some highly unfeasibly large hyperfinite set.

## 7. Indiscernibilty equivalences and locally compact topological spaces

As was mentioned in the Introduction, continuous objects in the discrete world originate from "accessible" subclasses of hyperfinite sets by identifying "indiscernible" elements. In this section, we discuss how this approach can be formalized in THS.

**7.1.** Let $x$ be a set, let $X \subseteq x$ be a $\sigma$-subset of $X$, and let $\overset{\mathrm{E}}{\approx}$ be an $\pi$-equivalence relation on $x$ (see Section 2.16). The relation $\overset{\mathrm{E}}{\approx}$ is called an <u>*indiscernibility equivalence*</u> on $X$ iff

$$\forall^{\mathsf{inf}} u \subseteq X\ \exists a, b \in u\ (a \neq b\,\&\, a \overset{\mathrm{E}}{\approx} b).$$

The following example shows that it is natural to assume that accessible elements of a hyperfinite set form a $\sigma$-subset of this set, while the relation of indiscernibility is a $\pi$-relation that satisfies the above definition. Indeed, accessible elements should be bounded in some sense by small natural numbers, while indiscernible elements should be, in some sense, infinitesimally close.

**Example 7.2.** To consider an example of an indiscernibility relation, we introduce the following notation.

Let $\alpha, \beta \in \mathbb{Q}$. Then

(1) $\alpha \approx \beta \rightleftharpoons \alpha - \beta \in \mu(0)$ (cf. Theorem 4.9);
(2) $\alpha \sim \infty \rightleftharpoons \alpha \in \mathbb{Q} \setminus \mathbb{Q}_b$.

Note that $\mathbb{N} \ni n \sim \infty \Longleftrightarrow n \in \mathbb{N} \setminus \mathbb{SN}$.

Fix $n, m \in \mathbb{N} \setminus \mathbb{SN}$ such that $\frac{n}{m} \sim \infty$. Let $x = \{\pm\frac{k}{l}\ |0 \le k \le n,\ 0 < l \le m\}$. Consider the restriction of the relation $\approx$ to $x$. We denote this restriction also by $\approx$ in this example. Let $X = x \cap \mathbb{Q}_b$. Obviously, $X$ is a $\sigma$-class. Let us show that $\approx$ is an indiscernibility relation on $X$.[2] Indeed, let $u \subseteq X$. Put $v = \{|a - b| \mid a, b \in u,\ a \neq b\}$. If $\min v \approx 0$, then — by definition — there exist $a \neq b$ such that $a \approx b$. Thus, for every $n \in \mathbb{SN}$, the set $u_n = \{a, b \in u \mid a \neq b,\ |a - b| < \frac{1}{n}\} \neq \emptyset$. Since the decreasing countable sequence $\{u_n \mid n \in \mathbb{SN}\}$ consists

[1] The proof is given in [2] for **BST**, but can be transfered literally to NCT.
[2] This is obvious for those who are familiar with nonstandard analysis.

of nonempty sets, its intersection is also nonempty by Proposition 2.15. Thus there exist $a, b \in u$ such that $\alpha \neq b$ but $a \approx b$. Therefore, in this case our statement is proved.

Now let $\min v > \delta > 0$, where $\delta \in \mathbb{SQ}$. This implies that the map $\mathsf{st} : \mathbb{Q}_b \longrightarrow {}^{\circ}\mathbb{R}$ defined in Theorem 4.9 is injective on $u$ and $\inf\{|\,\mathsf{st}(a) - \mathsf{st}(b)| \mid a \neq b,\ a, b \in U\} \geq \delta$.

On the other hand, the set $\mathsf{st}''u$ is bounded. Indeed, since $u \subseteq \mathbb{Q}_b$, we have $st''u \subseteq [\mathsf{st}(\min u) - 1, \mathsf{st}(\max u) + 1]$. Applying the theorem of ordinary mathematics, which states that every infinite bounded set of reals has an accumulation point, we obtain that $st''u$ is finite and, thus, $u$ is small.

Using Theorem 5.7, we can easily formalize this consideration in THS.

**Theorem 7.3.** *Let $X \subseteq x$ be a $\sigma$-subset of a set $x$ and $\overset{\mathrm{E}}{\approx}$ be a $\pi$-relation on $x$. The following conditions are equivalent:*

(1) $\overset{\mathrm{E}}{\approx}$ *is an indiscernibility equivalence on $X$;*
(2) $\forall e \in E \forall y \subseteq X \exists^{\mathsf{small}}\, v \subseteq X (y \subseteq e''v)$*;*
(3) $\exists^{\mathsf{thin}}\, N \subseteq X (\forall a \in X\, \exists n \in N (a \overset{\mathrm{E}}{\approx} n)\, \&\, \forall n, m \in N (n \overset{\mathrm{E}}{\approx} m \longrightarrow n = m))$.

**Proof.** [3] (1) $\longrightarrow$ (2). If (2) does not hold, then there exist $e \in E$ and $y \subseteq X$ such that $y \setminus (e''v) \neq \varnothing$ for any small $v \subseteq y$. Using induction for $\mathbb{SN}$, we will find, for any small $n$, an injective function from $n$ into $y$ such that its range consists of pairwise $e$-nonequivalent elements. Applying prolongation, we get an infinite set of pairwise $e$-nonequivalent elements.

(2) $\longrightarrow$ (3). Assume that $X = \cup D$, where $D$ is a thin class (recall that $X$ is an $\sigma$-set). Using **Choice**, we assign to any $e \in E$ and $d \in D$ a set $v_{ed}$ such that $d \subseteq e''v_{ed}$.

We say that a class $C$ is an f.i.p. class if it has the finite intersection property: $\forall^{\mathsf{small}}\, c \subseteq C\ \cap c \neq \emptyset$.

Consider the thin class $B = \{e''\{z\} : z \in v_{ed}\, \&\, e \in E\, \&\, d \in D\}$. Using **Exp**, Proposition 2.15 and **Choice**, we prove the existence of a class $N \subseteq X$ that contains exactly one element in the intersection of each f.i.p. subclass of $B$. Let us prove that $N$ satisfies the condition (3).

Fix an arbitrary $a \in X$ and consider a centered class $C_a = \{e''\{z\} \in B \mid a \in e''\{z\}\}$. By the construction of $N$, there exists $n \in N$ such that $n \in \bigcap C_a$. Let us show that $a \overset{\mathrm{E}}{\approx} n$. We have to prove that, for every $e \in E$, there holds $\langle a, n\rangle \in e$. Since $E$ is an equivalence relation, there is an $e_1 \in E$ such that $e_1 \circ e_1^{-1} \subseteq e$. Take an arbitrary $d \in D$ such that $a \in d$. Then $a \in e_1''v_{e_1 d}$. Thus, there exists $z \in ve_1 d$ such that $a \in e_1''\{z\}$. Since $e_1''\{z\} \in B$, we have $n \in e_1''\{z\}$. Thus, $\langle a, n\rangle \in e$.

Let $n, m \in N$ and $n \overset{\mathrm{E}}{\approx} m$. Thus, for every $e \in E$, there holds $\langle n, m\rangle \in E$, i.e. $m \in e''\{n\}$. Notice that $e''\{n\} \in B$, since there exists $d \in D$ such that $n \in D$. Thus, the class $B_n = \{e''\{n\} \mid e \in E\}$ is an f.i.p. subclass of $B$, which contains both elements $m, n$. Since $m, n \in N$, we have $m = n$.

The implication (3) $\longrightarrow$ (1) is straightforward. □

Obviously, a class $N$ satisfying the condition (3) of Theorem 7.3, generally speaking, is not unique. Fix any such $N$. For $u \subseteq X$, define $\overset{\circ}{u} = \{a \in u : \forall b \in X (a \overset{\mathrm{E}}{\approx} b \longrightarrow b \in u)\}$ and $\overset{\circ}{u}{}^{\#} = \overset{\circ}{u} \cap N$.

**Proposition 7.4** (THS).

$$\left[\forall n \in N\ \exists u \subseteq X (n \in \overset{\circ}{u}{}^{\#})\right] \&\, \forall n \in N\ \forall u_1, u_2 \subseteq X \left(n \in \overset{\circ}{u}_1{}^{\#} \cap \overset{\circ}{u}_2{}^{\#} \longrightarrow \exists u \subseteq X \ (n \in \overset{\circ}{u}{}^{\#} \subseteq \overset{\circ}{u}_1{}^{\#} \cap \overset{\circ}{u}_2{}^{\#})\right). \quad (2)$$

Semantically, this proposition means that the system of classes $\mathcal{T} = \{\overset{\circ}{u}{}^{\#} \mid u \subseteq X\}$ forms a base of topology on $N$.

It can be proved that this topology is locally compact. Let us show how this statement can be formulated explicitly in THS. Let $C \subseteq N$. We say that $F$ is an open covering of $C$ if $F$ is a function, $\mathsf{dom}\ F = I$ is a thin class, and $\forall i \in I F(i) \subseteq X$ and $N \subseteq \bigcup_{i \in I} \overset{\circ}{F}(i)^{\#}$. A class $C \subseteq N$ is compact iff $\forall F (F$ is an open covering of $C) \longrightarrow \exists^{\mathsf{small}}\, p \subseteq \mathsf{dom}\ F (C \subseteq \bigcup_{i \in p} \overset{\circ}{F}(i)^{\#})$ (cf. Proposition 2.22). Now, "$(N, \mathcal{T})$ is a locally compact space" is equivalent

[3] See [11, Theorem 1.4.11] for a generalization of the equivalence (1) $\longleftrightarrow$ (3); a corresponding statement can be proved in THS as well.

to the following THS-formula:

$$\forall n \in N \exists u \subseteq X \left( n \in \overset{\circ}{u}^{\#} \;\&\; \exists C \subseteq N((C \text{ is compact}) \;\&\; \left( \overset{\circ}{u}^{\#} \subseteq C \right) \right). \qquad \text{(LC)}$$

A similar approach to the construction of locally compact spaces was developed in [7] in terms of nonstandard analysis for the case of locally compact abelian groups. The proof of Theorem 2.2.4 of [7] can be easily transformed to a proof of (LC) in THS.

It is easy to see that any locally compact space can be represented as a quotient class $N$ constructed by an appropriate triple $\langle x, X, \overset{\mathrm{E}}{\approx} \rangle$, where $X$ is a thin subclass of a set $x$ and $\overset{\mathrm{E}}{\approx}$ is an indiscernibility relation on $X$.

Let us consider such representations of the field $\mathbb{R}$. As was mentioned in the Introduction, they can be considered as numerical systems that simulate reals in an idealized computer of infinite (i.e. hyperfinite nonsmall) memory. This point of view gives a motivation for the following definition.

**7.5.** Consider a tuple $R = (\langle r; \oplus, \odot \rangle; R_b, \rho)$, where $r$ is a hyperfinite set with binary operations $\oplus, \odot$ on it. We assume that $R_b \subseteq r$ is a $\sigma$-class and $\rho$ is a $\pi$-equivalence relation on $r$. Define the topology $\mathcal{T}_R$ on $R_b/\rho$ in the same way as is done in Proposition 7.4. Take $r$, $R_b$, $\rho$, $R_b/\rho$ for $x$, $X$, $\overset{\mathrm{E}}{\approx}$ and $N$, respectively.

We say that $R$ is a *hyperfinite computer arithmetic* if the following conditions hold:

(1) $\rho$ is an indiscernibility relation on $R_b$;
(2) $R_b$ is closed under the operations $\oplus$ and $\odot$;
(3) $\rho$ is a congruence relation on $R_b$;
(4) the quotient algebra $R_b/\rho$ is topologically isomorphic to the field $\mathbb{R}$.

The previous considerations show that this definition can be formalized in THS.

We interpret the elements of $R_b$ the same way as was discussed in the Introduction. Elements of $R_b$ are computer reals that are not too big, i.e. not too close to the boundary of the computer memory. Obviously, this is not a definition in the framework of the classical mathematics. That is why $R_b$ is a proper semiset. Operating with these numbers does not imply overfilling of the memory. Thus the computer operations restricted to $R_b$ approximate the corresponding operations on reals. This fact is formalized in statement (3) of definition 7.5.

**Example 2.** Consider a rational number $e$, $0 < e \in \mu(0)$, and a natural number $\omega$, $\omega \in \mathbb{N} \setminus \mathbb{SN}$, such that $\omega e \sim \infty$. Consider a tuple $R(\omega, e) = (\langle r_\omega; \oplus, \odot \rangle; R_b, \rho)$, where

(1) $r_\omega = \{-\omega, \dots, \omega\}$;
(2) the operation $\oplus$ is the addition modulo $2\omega + 1$;
(3) the operation $\odot$ is defined by the formula

$$k \odot m = [km\, e](\mathrm{mod}(2\omega + 1)),$$

where $k, m \in r_\omega$ and $[\alpha]$ is the integral part of a real number $\alpha$;
(4) $R_b = \{k \in r_\omega \mid ke \in \mathbb{Q}_b\}$;
(5) an equivalence relation $\rho \subseteq r_\omega \times r_\omega$ is such that

$$k \rho m \iff ke \approx me.$$

**Proposition 7.6.** *The tuple $R(\omega, e)$ is a nonstandard computer arithmetic.*

**Proof.** It is easy to see that if $k, m \in \mathbb{R}_b$ then $k \oplus m = k + m$ and $k \odot m = n$, where $ne \le kme^2 < (n+1)e$.

Define the map $F : R_b \longrightarrow \mathbb{R}$ by the formula $F(k) = \mathsf{st}(ke)$. It is easy to see that, for all $k, m \in R_b$, there holds

(1) $F(k) = F(m) \iff k\rho m$;
(2) $\forall q \in \mathbb{Q}_b \exists k \in R_b (q \approx ke)$;
(3) $(k \odot m)e \approx ke \cdot me$.

These properties prove that $F$ is a surjective homomorphism and that $R_b/\rho$ is isomorphic to $\mathbb{R}$. □

The nonstandard computer arithmetic $R(\omega, e)$ discussed in Example 2 is not a hyperfinite version of the computer arithmetic, which is implemented in existing computers. The last one is based on the floating point representation of reals. We will call it FP-arithmetic. Its hyperfinite version was discussed in [5] in terms of nonstandard analysis. The computer arithmetic $R(\omega, e)$ (no matter whether standard or nonstandard $\omega$ and $e$ are considered) has some better than FP-arithmetic algebraic properties. Indeed, it is well known that addition and multiplication in FP-arithmetic are neither associative nor distributive, while $\langle r_\omega, \oplus\rangle$ is an abelian group. However, multiplication in $R(\omega, e)$ is not associative and the law of distributivity also fails [5].

It is not quite clear how the good algebraic properties of numerical systems would affect the quality of numerical computations. It was shown in [7] that the convergence properties of approximation of the Fourier Transformation on $\mathbb{R}$ by sampling its kernel are better when the result of this sampling is the matrix of Finite Fourier Transformation, i.e. when we approximate the additive group $\mathbb{R}$ is by finite abelian groups. The theory of approximation of locally compact groups by finite abelian groups was developed in [7]. It can be proved (cf. [5], where similar questions were discussed in terms of nonstandard analysis) that the problem of approximation of locally compact algebraic systems by finite systems can be reduced to a problem of representation of locally compact systems by the quotients of $\sigma$-subsystems of hyperfinite systems under indiscernibility relations.

The following theorem shows that it is impossible to construct a computer arithmetics that is an associative ring.

**Theorem 7.7.** *There does not exist a hyperfinite computer arithmetic* $R = (\langle r; \oplus, \odot\rangle; R_b, \rho)$ *such that* $\langle r; \oplus, \odot\rangle$ *is an associative ring (even noncommutative).*

A similar theorem about finite approximations of locally compact fields was proved in [5] (see also [6]). Theorem 7.7 is a little bit more general than those of [5]. However, the proof presented in [5] can be easily adjusted to Theorem 7.7. This proof can be formalized in THS.

Let $L$ be the first-order language in the signature $\sigma_1 = \langle +, \cdot\rangle$ and $L_h$ the first-order language in the signature $\sigma_2 = \langle \oplus, \odot; R_b, \rho\rangle$. Here, $\oplus$ and $\odot$ are symbols of binary operations, $R_b$ is a symbol of a unary predicate, and $\rho$ is a symbol of binary predicate.

Formulas of $L_h$ have the natural interpretation in any hyperfinite computer arithmetic $R = (\langle r; \oplus, \odot\rangle; R_b, \rho)$. We use notations $\forall^b$ and $\exists^b$ for the universal and existential quantifiers restricted to $R_b$.

Let $t$ be a term in the signature $\sigma_1$. Replace each occurrence of the operation $+$ $(\cdot)$ in $t$ by $\oplus$ $(\odot)$. The obtained term in the signature $\sigma_2$ will be denoted by $t_h$.

Let $\varphi(x_1, \ldots, x_n)$ be an $L$-formula. Denote by $\varphi_h$ the $L_h$-formula obtained from $\varphi$ by the replacement of each atomic formula $t = s$ by $t_h\, \rho\, s_h$ and each quantifier $Qx$ by $Q^b x$. Here $t, s$ are $\sigma_1$-terms. We call $\varphi_h$ *the hyperfinite analog* of $\varphi$.

Let $R = (\langle r; \oplus, \odot\rangle; R_b, \rho)$ be a hyperfinite computer arithmetic. By Definition 7.5(4), there exists a a homomorphism $\psi : R_b \longrightarrow {}^{\circ}\mathbb{R}$ such that $\psi(a) = \psi(b) \iff a\, \rho\, b$. This homomorphism may not be unique. We call $\psi$ a canonical homomorphism. The following theorem follows immediately from the definitions.

**Theorem 7.8.** *For any $L$-formula $\varphi(x_1, \ldots, x_n)$, the following statement holds.*

*If* $R = (\langle r; \oplus, \odot\rangle; R_b, \rho)$ *is a hyperfinite computer arithmetic,* $\psi : R_b \longrightarrow \mathbb{R}$ *is the canonical homomorphism and* $a_1, \ldots a_n \in R_b$*, then*

$$R \models \varphi_h(a_1, \ldots, a_n) \Longleftrightarrow \mathbb{R} \models \varphi((\psi(a_1), \ldots, \psi(a_n)).$$

This theorem gives some qualitative formalization of the fact that, if we operate with relatively small numbers so that the memory overfilling cannot occur during the computations, we obtain results that are approximately true, at least when we deal with algebraic statements that can be formalized in the language $L$.

A version of Theorem 7.8 can be formulated in the language of classical mathematics only for some specific $L$-formulas — the *positive bounded formulas* [5].

The investigation of the correlation between continuous and computer mathematics in terms of THS (i.e. on the qualitative level) for higher-order properties (e.g. formulated in the language of type theory) is an interesting problem. It can help to discover some new phenomena concerning the numerical investigation of some more complicated structures.

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