Indag. Mathem., N.S., 10 (1), 101–116 March 29, 1999

Counting conjugacy classes in symmetric spaces over $\mathbf{F}_q$

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Communicated by Prof. T.A. Springer at the meeting of June 23, 1997

ABSTRACT

In this paper we verify a prediction of the Langlands–Lusztig program in the special case of the algebra of double cosets $K(\mathbf{F}_q)\backslash GL_n(\mathbf{F}_q)/K(\mathbf{F}_q)$, where K is the fixed point subgroup of an involution on GL_n (see Conjecture 3.6 and Example 7 of [G]). We calculate the dimension of the algebra by computing the number of $K(\mathbf{F}_q)$-conjugacy classes in the space $GL_n(\mathbf{F}_q)/K(\mathbf{F}_q)$. We compare this dimension with the size of the set parametrizing representations of the double coset algebra, defined in terms of perverse sheaves on the flag variety. These numbers turn out to be the same.

1. INTRODUCTION

Let $\bar{G}$ be a connected reductive group defined over a finite field $\mathbf{F}_q$ of q elements and let $G = \bar{G}(\mathbf{F}_q)$ be the finite group of its rational points. Let $\theta : \bar{G} \to \bar{G}$ be an involutory automorphism, $\bar{G}^\theta$ its fixed point subgroup and $\bar{K}$ a subgroup of $\bar{G}^\theta$ containing the identity component $\bar{G}^\theta_0$. Grojnowski in [G] Conjecture 3.6 (see also [L1] and [L2]) postulated a parametrization of $\bar{K}$-equivariant perverse sheaves on the symmetric variety $\bar{G}/\bar{K}$. Throughout the rest of this paper let G be the general linear group $GL_n(\mathbf{F}_q)$ and $K = \bar{K}(\mathbf{F}_q)$. Consider the algebra of double cosets $K\backslash G/K$. Grojnowski's conjecture for this special case predicts that the dimension of this algebra is equal to the cardinality of the set parametrizing character sheaves on $K\backslash G/K$. In this paper we verify that these numbers are indeed the same.

The dimension of the algebra is calculated by computing the number of K-conjugacy classes in the space G/K. Recall the general strategy for studying the conjugacy classes of a reductive algebraic group (see [SS]) and, in particular, those of G (see Section 2). It is to use the Jordan decomposition. That is, look first at the conjugacy classes of semisimple elements, we next calculate the centralizers of elements in such classes and then find the unipotent conjugacy classes of these centralizers. Likewise. this approach is used here to enumerate the set of K-conjugates of G/K (or equivalently, the set of double cosets $K\backslash G/K$) as a sum over the 'semisimple' K-conjugacy classes of G/K of 'unipotent' K-conjugacy classes taken from centralizers.

The cardinality of the set parametrizing character sheaves on $K\backslash G/K$ is calculated also as a sum over semisimple K-conjugacy classes of G/K (see Theorem 3.4 of [G]). Recall the definition of the coherent continuation representation (see [BV] and [LV]). Let L be a connected reductive linear complex Lie group and $\theta : L \to L$ an involutive automorphism. Let R be a subgroup of L^θ containing L^θ_0. Then R has finitely many orbits on the flag variety of all Borel subgroups of L. Let $\mathcal{T}$ be a set of representatives for these orbits. For $b \in \mathcal{T}$, write R_b for the group of elements of R stabilizing b. The quotient group $R_b/(R_b)_0$ has exponent two. Let $\mathcal{D}$ be the finite set of all pairs (b, τ), with $b \in \mathcal{T}$ and τ a one-dimensional representation of $R_b/(R_b)_0$. Let $\mathcal{F}$ be the $\mathbf{C}$-vector space spanned by $\{e_\gamma : \gamma \in \mathcal{D}\}$. The Weyl group of L has a representation on $\mathcal{F}$ (Proposition 5.5 of [LV] with $u = 1$), the coherent continuous representation of L with respect to R.

Let gK be a semisimple coset of G/K. Working over some extension of $\mathbf{F}_q$ we have a polar decomposition $g = \bar{k}\bar{x}$ where $\theta(\bar{k}) = \bar{k}$ and $\theta(\bar{x}) = \bar{x}^{-1}$. Choose $\bar{x}$ to be as singular as possible. Denote as L_x the centralizer $Z_G(\bar{x})$. The group L_x will be a product of general linear groups over extensions of $\mathbf{F}_q$, that is, $L_x \cong \prod_i GL_{m_i}(\mathbf{F}_{q^{d_i}})$. Let $L_x^{\mathbf{C}} = \prod_i GL_{m_i}(\mathbf{C})$ and $K_x^{\mathbf{C}} = L_x^{\mathbf{C}} \cap K(\mathbf{C})$. Let $\mathcal{F}_{gK}$ be the coherent continuation representation for $L_x^{\mathbf{C}}$ with respect to $K_x^{\mathbf{C}}$. Proposition 2.4 of [BV] computes the coherent continuous representation in this case to be

$$\mathcal{F}_{gK} = \bigoplus_{H \in C_{gK}} \operatorname{Ind}_{W_{K_x^{\mathbf{C}}}(H)}^{W_{L_x^{\mathbf{C}}}(H)} \det_I \tag{1}$$

where C_{gK} is the set of $K_x^{\mathbf{C}}$-conjugacy classes of θ-stable Cartan subgroups of $L_x^{\mathbf{C}}$ and $\det_I$ is the one dimensional representation of $W_{K_x^{\mathbf{C}}}(H)$ given by its action on the subsystem of θ-imaginary roots. Note that if g_1K and g_2K are elements of the same semisimple K-conjugacy class of G/K then $\mathcal{F}_{g_1K}$ is equivalent to $\mathcal{F}_{g_2K}$.

Throughout this paper K will assumed to be $\mathbf{F}_q$-split. Also we will always assume that the involution θ is semisimple, that is, that the characteristic of $\mathbf{F}_q$ is not equal to two.

The following theorem is a special case of a conjecture of Grojnowski, see Conjecture 3.6 of [G]. I would like to thank Ian Grojnowski for the illuminating conversations in which he communicated and explained this problem to me.

Theorem 2. *Let K be as defined above and let $\mathcal{S}$ be a set of representatives for the semisimple K-conjugacy classes of G/K, then*

$$|K\backslash G/K| = \sum_{gK\in\mathcal{S}} (\mathcal{F}_{gK}, \mathcal{F}_{gK}).$$

In fact the number of unipotent conjugacy classes of G/K that commute with a semisimple gK is equal to $(\mathcal{F}_{gK}, \mathcal{F}_{gK})$.

In Section 2 we look at the motivating example, conjugacy classes of G. In Section 3 we expand on the concepts in this introduction and determine the various subgroups K. In the remaining sections we prove Theorem 2 case by case.

2. THE DIAGONAL CASE

Consider the group $G \times G$, with the involution $\theta(g_1, g_2) = (g_2, g_1)$; its fixed point set is ΔG, the diagonal subgroup of $G \times G$. Now $(G \times G)/\Delta G \cong G$ and it follows that finding ΔG-conjugacy classes of $(G \times G)/\Delta G$ is equivalent to finding conjugacy classes of G.

For the group G, it is well known that two semisimple elements are in the same conjugacy class if and only if they have the same characteristic polynomial. The centralizer of an element of such a semisimple conjugacy class will be a product of $GL_{m_i}(\mathbf{F}_{q^{d_i}})$'s, the m_i's and $\mathbf{F}_{q^{d_i}}$'s being (respectively) the multiplicities and splitting fields for the $\mathbf{F}_q$-irreducible factors of the characteristic polynomial of elements in the class. The unipotent conjugacy classes which commute with this semisimple conjugacy class are then parametrized by (multi-)partitions of the m_i's (this is just the sizes of the Jordan blocks in the Jordan canonical form of an element of that conjugacy class). Note that the number of (multi-)partitions of the m_i's is also equal to the number of conjugacy classes of the Weyl group of the centralizer (which will be a product of symmetric groups, S_{m_i}'s). We shall put all this information into the language of Theorem 2.

Embed the quotient $(G \times G)/\Delta G$ into $G \times G$ as the set $X = \{(g, g^{-1}) : g \in G\}$. Then $x = (g, g^{-1}) \in X$ is a semisimple element if and only if g is a semisimple matrix. Let L_x be the centralizer of x in $G \times G$ and so

$$L_x = Z_G(g) \times Z_G(g) = \prod_i GL_{m_i}(\mathbf{F}_{q^{d_i}}) \times GL_{m_i}(\mathbf{F}_{q^{d_i}})$$

and $L_x \cap \Delta G = \prod_i \Delta GL_{m_i}(\mathbf{F}_{q^{d_i}})$. The involution acts on L_x by interchanging the components in each factor.

The Weyl group of $GL_m(\mathbf{C})$ is the symmetric group S_m and so the Weyl group of L_x is $\prod_i S_{m_i} \times S_{m_i}$. Recall that the representations of S_m are parametrized by the partitions of m. Let σ_λ be the representation of S_m associated to λ (for example see [FH]). The set of all partitions of m is denoted by $\mathcal{P}_m$ and the set of all partitions by $\mathcal{P}$. We shall denote the diagram of a partition λ by the same symbol λ.

To compute the coherent continuous representation for $L_x^{\mathbf{C}} = \prod_i GL_{m_i}(\mathbf{C}) \times GL_{m_i}(\mathbf{C})$ we first need to compute the coherent continuous representation for each component.

Proposition 3. *The coherent continuous representation for $GL_m(\mathbf{C}) \times GL_m(\mathbf{C})$ with respect to $\Delta GL_m(\mathbf{C})$ is*

$$\mathcal{F}_{\Delta_m} = \bigoplus_{\lambda \in \mathcal{P}_m} \sigma_\lambda \boxtimes \sigma_\lambda.$$

Proof. The group $GL_m(\mathbf{C}) \times GL_m(\mathbf{C})$ has only one $\Delta GL_m(\mathbf{C})$-conjugacy class of θ-stable Cartan subgroups (see [M]). Choose the Cartan subgroup $H = D \times D$, where D is the group of diagonal matrices in $GL_m(\mathbf{C})$. The roots of H will be the disjoint union of the roots of the two components and θ acts by interchanging them. In particular, all the roots are θ-complex and so the one dimensional representation to be induced is the trivial representation. Thus $\mathcal{F}_{\Delta_m} = \mathrm{Ind}_{\Delta S_m}^{S_m \times S_m}$ (trivial) and the result follows since for any finite group W,

$$\mathrm{Ind}_{\Delta W}^{W \times W}(\text{trivial}) = \bigoplus_{\sigma \in \hat{W}} \sigma \boxtimes \sigma^*$$

where $\hat{W}$ is the set of irreducible representations of W and σ^* is the dual representation to σ. □

Denote by $\mathcal{F}_x$ be the coherent continuous representation for $L_x^{\mathbf{C}}$ with respect to $L_x^{\mathbf{C}} \cap \Delta G(\mathbf{C})$ then

$$\mathcal{F}_x = \bigoplus_{\prod_i \lambda^{(i)} \in \prod_i \mathcal{P}_{m_i}} \boxtimes_i (\sigma_{\lambda^{(i)}} \boxtimes \sigma_{\lambda^{(i)}}). \tag{4}$$

Thus $(\mathcal{F}_x, \mathcal{F}_x)$ is the total number of (multi-)partitions of the $m_i's$, that is, the number of unipotent conjugacy classes commuting with g. Hence the total number of conjugacy classes of G is equal to the sum over the semisimple conjugacy classes of G of those $(\mathcal{F}_x, \mathcal{F}_x)$ terms and so Theorem 2 is satisfied in the diagonal case.

3. DEFINITIONS

The map $\Psi : g \mapsto g\theta(g)^{-1}$ is constant on left cosets of K and induces an embedding of G/G^θ into G. Denote the image of Ψ as $X = X(\mathbf{F}_q)$. An elementary calculation gives that X is subset of $\{g \in G : \theta(g) = g^{-1}\}$. Clearly two elements in X are K-conjugate if the cosets they represent are K-conjugate for all $k \in K$ and $g \in G$ (this becomes 'if and only if' when $K = G^\theta$). Thus Ψ induces an endomorphism $G/K \to X$ of K-sets with fibers G^θ/K. The K-action on both sets is given by conjugation.

We call a coset gK *semisimple* (respectively, *unipotent*) if the matrix $\Psi(g)$ is semisimple (respectively, unipotent). Let $x \in G$ have the Jordan decomposition $x = x_s x_u = x_u x_s$, where x_s is semisimple and x_u is unipotent. Then $x \in X$ if and only if both x_s and x_u are in X (see [R], Lemma 6.2). That is, we have a Jordan decomposition for X. Let gK be a semisimple coset with $\Psi(g) = x$. There exists a θ-split torus $T_x = T_x(\mathbf{F}_q)$ containing x (see [R], Proposition 6.3). Thus over the algebraic closure $\bar{\mathbf{F}}_q$ of $\mathbf{F}_q$ there exists a $\bar{x} \in T_x(\bar{\mathbf{F}}_q) \subset X(\bar{\mathbf{F}}_q)$ such that $\bar{x}^2 = x$.

Writing g as $\bar{x}\bar{k}$ we have $\bar{x}\bar{k}\theta(\bar{x}\bar{k})^{-1} = \bar{x}^2$ and it follows that $\bar{k} \in G^\theta$. Hence there is a 'polar decomposition' for elements of G with respect to θ (see for example [Ga] XI, §2) and this gives an alternative definition for $\bar{x}$. The eigenvalues of $\bar{x}$ will be the square roots of the eigenvalues of x and thus determined up to sign. Choose $\bar{x}$ to be as singular as possible, in the sense that the choice of signs is consistent so that the centralizer L_x will be as large as possible. Note that L_x is a subgroup of $Z_G(x)$. Let $l \in L_x$ then $\theta(l\bar{x}l^{-1}) = \theta(\bar{x})$ and so $\theta(l)\bar{x}^{-1}\theta(l)^{-1} = \bar{x}^{-1}$. That is, $\theta(L_x) \subseteq L_x$ and so θ gives an involution of L_x.

When the base field is $\mathbf{C}$ we can use as representatives for the conjugacy classes of involutive automorphisms those that arise from the complexification of a maximal compact subgroup of a real form of $GL_n(\mathbf{C})$. The groups we obtain are $GL_p(\mathbf{C}) \times GL_{n-p}(\mathbf{C})$, the complex orthogonal group $O(n, \mathbf{C})$ and the complex symplectic group $Sp(n, \mathbf{C})$. These groups are all connected except for $O(n, \mathbf{C})$ which has identity component $SO(n, \mathbf{C})$.

Let I_a be the $a \times a$ identity matrix and let $I_{a,b} = \left(\begin{smallmatrix} I_a & 0 \\ 0 & -I_b \end{smallmatrix}\right)$. We prove Theorem 2 case by case. The case of $K = Sp(n, \mathbf{F}_q)$, coming from the real form $GL_{n/2}(\mathbf{H})$, was done by Grojnowski ([G], see also [BKS]). The final two sections deal with the other two possibilities. These two sections are labeled by the associated real form.

4. $U(p, n-p)$

Here the involution θ is given by $g \mapsto I_{p,n-p}gI_{p,n-p}$ (with $p \leq n-p$). Now G^θ is the group $GL_p(\mathbf{F}_q) \times GL_{n-p}(\mathbf{F}_q)$. As $G^\theta = G^\theta_0$, this is the only possibility for the group K. Since $\det\theta(g) = \det g$ we have that $X \subseteq \{g \in SL_n(\mathbf{F}_q) : \theta(g) = g^{-1}\}$. Writing an element of G in the form $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where $A \in M_p(\mathbf{F}_q)$ and $D \in M_{n-p}(\mathbf{F}_q)$, we see that elements of X satisfy

$$\begin{pmatrix} A & -B \\ -C & D \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = I_n$$

and so we must have $AB = BD$, $CA = DC$, $A^2 - BC = I_p$ and $D^2 - CB = I_{n-p}$.

4.1. Counting conjugacy classes

In this subsection we decompose a vector space into two subspaces labeled by different superscripts. In such a situation, we will adopt the convention that $r+1$ is defined to be $r+1$ (mod 2).

We will view the homogeneous space G/K as X', the space of decompositions of $\mathbf{F}_q^n$ into pairs of subspaces $V = (V^0, V^1)$ with $\dim V^0 = p$, $\dim V^1 = n-p$ and $V^0 \cap V^1 = \{0\}$. This isomorphism is induced by $\Psi' : G \to X'$ where $\Psi'(g)^0$ is the span of the first p column vectors of g and $\Psi'(g)^1$ is the span of the last $n-p$ column vectors of g. This is an isomorphism of K-sets, with the K-action on X' being given by $k.V = (k(V^0), k(V^1))$. Let $E = \Psi'(I_n)$ be the standard decomposition, that is, E^0 is the span of the first p standard basis vectors and E^1 is the span of the last $n-p$ standard basis vectors. Denote the standard basis of E and E^r by $\mathcal{E}$ and $\mathcal{E}^r$ (respectively). The projection onto a subspace W will be denoted by π_W. We now give the two invariants which will be used to describe the K-orbits of X'. Since these also describe the K-conjugacy classes of X they are related to the Jordan canonical form of elements of X. The first invariant corresponds to the Jordan blocks that break into smaller orbits when we restrict conjugation to K. These turn out to be the blocks with eigenvalues ± 1. The second invariant corresponds to the blocks for which K-conjugacy is equivalent to G-conjugacy.

The first invariant is used to locate a decomposition V with respect to the standard decomposition E. Let $\lambda \in \{\pm 1\}, s \in \{0, 1\}$ and $t = t(\lambda, s) = s + \delta_{\lambda,-1}$ (the Kronecker delta). Define a *λ_s-eigenvector chain* of V of length l to be a sequence of l linearly indepedent vectors $\mathbf{c} = [\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_l]$ satisfying the following: (1) $\vec{c}_1 \in V^s \cap E^t$; (2) $\vec{c}_i \in V^s$ and $\pi_{E^t}(\vec{c}_i) = \pi_{E^t}(\vec{c}_{i+1})$ for $i > 1$ odd; (3) $\vec{c}_i \in V^{s+1}$ and $\pi_{E^{t+1}}(\vec{c}_i) = \pi_{E^{t+1}}(\vec{c}_{i+1})$ for i even. We call a chain $\mathbf{c}$ a *standard* chain if the nonzero $\pi_{E^r}(\vec{c}_i) \in \mathcal{E}^r$. So $\vec{c}_1$ is a standard basis vector and, for $i > 1$, we have that $\vec{c}_i$ is the sum of two standard basis vectors. Note that if $\mathbf{c}$ is a λ_s-eigenvector chain of V then there exists a $k \in K$ such that $k(\mathbf{c}) = [k(\vec{c}_1), \ldots, k(\vec{c}_l)]$ is a standard λ_s-eigenvector chain of $k.V$ (just use the change of basis matrix between the elements of $\mathbf{c}$, extended to form a basis, and the standard basis). Given a chain $\mathbf{c}$ denote by $\langle \mathbf{c} \rangle$ the span of the elements of the chain.

Let gK correspond to the decomposition V and to $x \in X$, that is, $\Psi(g) = x$ and $\Psi'(g) = V$. Each maximal standard λ_s-eigenvector chain in V corresponds to a Jordan block for the eigenvalue λ in the Jordan canonical form of x. We illustrate this correspondence when $\mathbf{c}$ is a maximal standard 1_0-eigenvector chain of length $2j$ (the other cases being almost identical). Let $\vec{e}_1 = \vec{c}_1$ and for $i > 1$ let $\vec{e}_i$ be the standard basis vector appearing in $\vec{c}_i$ which does not appear in $\vec{c}_{i-1}$ (so that $\vec{c} = \vec{e}_i + \vec{e}_{i-1}$). Form the ordered basis $\mathcal{E}_\mathbf{c}$ by ordering the $\{\vec{e}_i\}_i$ so that the standard basis elements from E^0 come first and otherwise retaining the order of the subscripts. Let N be the $j \times j$ matrix with $N_{k,l} = 1$ for $k = l - 1$ and with $N_{k,l} = 0$ otherwise. Form the matrix

$$g_\mathbf{c} = ([\vec{c}_1]_{\mathcal{E}_\mathbf{c}}[\vec{c}_3]_{\mathcal{E}_\mathbf{c}} \cdots [\vec{c}_{2j-1}]_{\mathcal{E}_\mathbf{c}}[\vec{c}_2]_{\mathcal{E}_\mathbf{c}}[\vec{c}_4]_{\mathcal{E}_\mathbf{c}} \cdots [\vec{c}_{2j}]_{\mathcal{E}_\mathbf{c}}) = \begin{pmatrix} I_j & I_j \\ N & I_j \end{pmatrix}.$$

Let $\theta_\mathbf{c}$ be conjugation by $I_{j,j}$, then

$$(g_\mathbf{c}\theta_\mathbf{c}(g_\mathbf{c})^{-1} - I_{2j})\vec{c}_i = 2\begin{pmatrix} N(I_j - N)^{-1} & (I_j - N)^{-1} \\ N(I_j - N)^{-1} & N(I_j - N)^{-1} \end{pmatrix}\vec{c}_i = 2\sum_{k<i} \vec{c}_k.$$

Thus $g_\mathbf{c}\theta_\mathbf{c}(g_\mathbf{c})^{-1}$ will be a single Jordan block of x for the eigenvalue 1. Hence $\langle \mathbf{c} \rangle \cap V^r$ will have an orthogonal complement in V^r. By induction, each K-orbit of X' contains a decomposition V with maximal standard chains $\mathbf{c}_1^V, \ldots, \mathbf{c}_m^V$ which give orthogonal decompositions

$$V^r = (\oplus_i \langle \mathbf{c}_i^V \rangle \cap V^r) \oplus \hat{V}^r,$$

where $\hat{V}^r \cap E^0 = \hat{V}^r \cap E^1 = \{0\}$. Note that by its definition $\pi_{E^0}\hat{V}^0 = \pi_{E^0}\hat{V}^1$ and

$\pi_{E^1}\hat{V}^0 = \pi_{E^1}\hat{V}^1$. Also the dimensions of $\hat{V}^0$, $\hat{V}^1$ and $\pi_{E^r}\hat{V}^s$ are all equal, denote this dimension by w^V. Thus, for example, the standard decomposition E has p 1_0-eigenvector chains of length 1 and $(n-p)$ 1_1-eigenvector chains of length 1 and so $w^E = 0$.

Note that other decompositions in the same K-orbit have similar decompositions but these need not necessarily be orthogonal. However the number and lenghts of each type of chain will be an invariant of the K-orbit and independent of any choice. Thus given a decomposition V, we associate a partition $\mu^V \in \mathcal{P}_{n-2w^V}$ by taking the parts of μ to be the lengths of the chains constructed above. So, for example, with the standard decomposition we have $\mu^E = (1^n)$. Further, to keep record of what type of chain corresponds to a part, we construct a function f^V from the diagram of μ^V to $\{\pm 1, \pm\sqrt{-1}\}$ by: for a row μ_i^V corresponding to an λ_r-eigenvector chain let $f^V(i,j) = (-1)^{j+r}(\sqrt{-1})^{\delta_{\lambda,-1}}$. To continue our example, for the standard decomposition we get a function that takes on the value -1 on p points and $+1$ on $n-p$ points (this function will depend on some ordering of the rows).

For $\mu \in \mathcal{P}_{n-2w}$ let A_μ be the set of functions f from the diagram of μ to $\{\pm 1, \pm\sqrt{-1}\}$ such that $\sum_{(i,j)\in\mu} f(i,j) = n-2p$ and $f(i,j+1) = -f(i,j)$. Denote by $A_\mu^{\mathbf{R}}$ the subset of A_μ consisting of real valued functions. Let $A\mu/\mathrm{Row}_\mu$ be the set of equivalence classes of A_μ under the relation whereby two functions are equivalent if they only differ by a permutation of rows of equal length. For the example $U(2, 2)$, representatives of the ten elements of $A_{(2^2)}/\mathrm{Row}_{(2^2)}$ are given by

$$\begin{array}{|c|c|}\hline 1 & -1 \\ \hline 1 & -1 \\ \hline\end{array}\qquad \begin{array}{|c|c|}\hline 1 & -1 \\ \hline -1 & 1 \\ \hline\end{array}\qquad \begin{array}{|c|c|}\hline -1 & 1 \\ \hline -1 & 1 \\ \hline\end{array}\qquad \begin{array}{|c|c|}\hline i & -i \\ \hline i & -i \\ \hline\end{array}\qquad \begin{array}{|c|c|}\hline i & -i \\ \hline -i & i \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline -i & i \\ \hline -i & i \\ \hline\end{array}\qquad \begin{array}{|c|c|}\hline 1 & -1 \\ \hline i & -i \\ \hline\end{array}\qquad \begin{array}{|c|c|}\hline 1 & -1 \\ \hline -i & i \\ \hline\end{array}\qquad \begin{array}{|c|c|}\hline -1 & 1 \\ \hline i & -i \\ \hline\end{array}\qquad \begin{array}{|c|c|}\hline -1 & 1 \\ \hline -i & i \\ \hline\end{array}$$

The first three representatives displayed are those in $A^{\mathbf{R}}_{(2^2)}/\mathrm{Row}_{(2^2)}$.

Let $\mathbf{c}$ be an eigenvector chain V. If $\mathbf{c}$ has even length then the dimensions of the projections of $\langle\mathbf{c}\rangle$ onto the subspaces E^0, E^1, V^0 and V^1 are all equal. If $\mathbf{c}$ is a λ_r-eigenvector chain of odd length then

$$\dim\pi_{V^r}(\langle\mathbf{c}\rangle) = \dim\pi_{E^r}(\langle\mathbf{c}\rangle) = \dim\pi_{V^{r+1}}(\langle\mathbf{c}\rangle) + 1 = \dim\pi_{E^{r+1}}(\langle\mathbf{c}\rangle) + 1.$$

Let V have a_r 1_r-eigenvector chains of odd length and b_r -1_r-eigenvector chains of odd length. Denote by $[a]$ the integer part of a. Since $\dim V^r = \dim E^r$,

$$w^V + \sum_i [\mu_i^V/2] + a_1 + b_1 = w^V + \sum_i [\mu_i^V/2] + a_1 + b_2 = p \quad \text{and}$$
$$w^V + \sum_i [\mu_i^V/2] + a_2 + b_2 = n - p.$$

Hence $b_1 = b_2$ and $a_2 = n - 2p + a_1$. Thus the function f^V constructed above from the chains of V will be in A_{μ^V} and decompositions in the same K-orbit of X' will have the same element of A_μ/Row_μ. This function f^V is the first invariant.

To construct the second invariant choose a basis $\{\vec{e}_i^{\,0}\}$ of $\pi_{E^0}(\hat{V}^0) = \pi_{E^0}(\hat{V}^1)$. There is then a basis $\{\vec{e}_i^{\,1}\}$ of $\pi_{E^1}(\hat{V}^0) = \pi_{E^1}(\hat{V}^1)$ such that $\{\vec{e}_i^{\,0} + \vec{e}_i^{\,1}\}$ is a basis of $\hat{V}^0$. This choice gives coefficients $c_{i,j}^V \in \mathbf{F}_q$ such that a basis for $\hat{V}^1$ is

$$(5) \qquad \{\vec{e}_i^{\,0} + \sum_j c_{i,j}^V \vec{e}_j^{\,1}\}.$$

The matrix $(c_{i,j}^V) \in M_{w^V}(\mathbf{F}_q)$ can have no 0-eigenvector as that would yield an element of an eigenvector chain and it can have no 1-eigenvector as that would give an element of $V^0 \cap V^1$. This matrix is defined up to the choice of a basis for $\pi_{E^0}(\hat{V}^0)$, that is, by conjugation by an element of $GL_{w^V}(\mathbf{F}_q)$. Denote the set of conjugacy classes of elements of $GL_w(\mathbf{F}_q)$ without 1 as an eigenvalue by $\mathcal{C}_w^{\neq 1}(\mathbf{F}_q)$. An element of this set is the second invariant.

The above discussion gives for each decomposition a pair consisting of an element of A_μ/Row_μ and an element of $\mathcal{C}_{(n-|\mu|)/2}^{\neq 1}(\mathbf{F}_q)$, where μ is a partition with $|\mu| \le n$ and $|\mu| \equiv n \pmod 2$. Now any two decompositions with the same pair of invariants lie in the same K-orbit (as we can write out bases for V^0 and V^1 from these invariants). Thus

$$(6) \qquad |K\backslash G/K| = \sum_{w=0}^{p} \left(\sum_{\mu \in \mathcal{P}_{n-2w}} |A_\mu/\mathrm{Row}_\mu| \right) . \, |\mathcal{C}_w^{\neq 1}(\mathbf{F}_q)|.$$

4.2. The coherent continuous representation

Remark 7. A torus T of G is called θ-split if $\theta(t) = t^{-1}$ for all $t \in T$. Every semisimple $x \in X$ is contained in a maximal θ-split torus of $\bar{G}$ (see [R], Proposition 6.3). If T and T' are maximal θ-split tori of $\bar{G}$ then there exists an element k of $\bar{G}_0^\theta$ such that $kTk^{-1} = T'$ (see [V], §1). Hence all semisimple elements of X are $\bar{K}$-conjugate to a matrix of the form

$$\begin{pmatrix} A & B & \\ -B & A & \\ & & I_{n-2p} \end{pmatrix}$$

where A and B are diagonal $p \times p$ matrices satisfying $A_{i,i}^2 + B_{i,i}^2 = 1$. Since the $\bar{K}$-conjugacy classes of these elements are determined by the diagonal entries of the submatrix A the semisimple conjugacy classes of X is determined by the characteristic polynomial of this submatrix. This gives a one to one correspondence between semisimple conjugacy classes of X and monic polynomials in $\mathbf{F}_q[y]$ of degree p. Thus there are q^p semisimple conjugacy classes of X.

Let gK correspond to V and x, that is, $\Psi(g) = x$ and $\Psi'(g) = V$. Let $C = (c_{i,j}^V)$ (see Equation 5), then

$$(8)\qquad \begin{pmatrix} I & I \\ I & C \end{pmatrix}\begin{pmatrix} I & -I \\ -I & C \end{pmatrix}^{-1} = \begin{pmatrix} (C+I)(C-I)^{-1} & 2(C-I)^{-1} \\ 2C(C-I)^{-1} & (C+I)(C-I)^{-1} \end{pmatrix}.$$

After conjugating by an element of $\Delta GL_{w,V}(\bar{\mathbf{F}}_q)$ to put C into Jordan canonical form we see that the right hand side Equation 8 is semisimple if and only if C is semisimple. Thus semisimple cosets correspond to the decompositions V which have all their eigenvalue chains of lenght one and whose matrix $(c^V_{i,j})$ is semisimple. Using this, the fact that there are q^p semisimple conjugacy classes can now be verified from Equation 6. Now let V be semisimple with a_r 1_r-eigenvector chains and b_r -1_r-eigenvector chains. Recall that $b_1 = b_2$ (which we will write now simply as b) and $a_2 = n - 2p + a_1$. The $(a_1 + a_2)$ 1-eigenvector chains to correspond to $(a_1 + a_2)$ 1-eigenvalues of $\bar{x}$. Each -1_0-eigenvector chain is paired with a -1_1-eigenvector chain and each of these pairs corresponds to a pair of -1-eigenvalues of x. These b pairs of chains corresponds to b pairs of $\sqrt{-1}$-eigenvalues and $-\sqrt{-1}$-eigenvalues of $\bar{x}$.

Let C have characteristic polynomial f. Write $f = \prod_i f_i^{m_i}$ for distinct $\mathbf{F}_q$-irreducible factors f_i with multiplicities m_i. Let $\{\lambda_{i,j}\}_j$ be the roots of the polynomial f_i. Thus $\mathbf{F}_{q^{d_i}} = \mathbf{F}_q(\{\lambda_{i,j}\}_j)$ is the splitting field for f_i. Clearly if for some j we have $\sqrt{\lambda_{i,j}} \in \mathbf{F}_{q^{d_i}}$ then $\sqrt{\lambda_{i,k}} \in \mathbf{F}_{q^{d_i}}$ for all k. Calculating the eigenvalues of the right hand side of Equation 8 yields that a $\lambda_{i,j}$-eigenvalue of C corresponds to a $\beta_{i,j}$-eigenvalue and a $\beta_{i,j}^{-1}$-eigenvalue for x, where $\beta_{i,j} = (\sqrt{\lambda_{i,j}} + 1)/(\sqrt{\lambda_{i,j}} - 1)$. Let $\hat{f}_i(y) = \prod_j y^2 - 2((\lambda_{i,j} + 1)/(\lambda_{i,j} - 1))y + 1$. The characteristic polynomial of x is

$$(y-1)^{a_1+a_2}(y+1)^{2b}\prod_i (\hat{f}_i(y))^{m_i}.$$

Now $\mathbf{F}_{q^{d_i}}(\{\sqrt{\lambda_{i,j}}\}_j) = \mathbf{F}_{q^{d_i}}(\{\beta_{i,j}\}_j)$ and the polynomial $\hat{f}_i$ is irreducible over $\mathbf{F}_{q^{d_i}}$ if and only if $\sqrt{\lambda_{i,j}} \notin \mathbf{F}_{q^{d_i}}$. Thus $\hat{f}_i$ is $\mathbf{F}_q$-irreducible when $\sqrt{\lambda_{i,j}} \in \mathbf{F}_{q^{d_i}}$ and is the product of two distinct $\mathbf{F}_q$-irreducible polynomials of equal degree when $\sqrt{\lambda_{i,j}} \notin \mathbf{F}_{q^{d_i}}$. The group $Z_G(x)$ is then given from this data as a product of general linear groups.

Since $\bar{x}$ and x lie in the same torus, comparing $Z_G(\bar{x})$ with $Z_G(x)$ is equivalent to comparing the respective sets of eigenvalues (with multiplicities). Note we cannot compute $Z_G(\bar{x})$ directly from the characteristic polynomial of $\bar{x}$ as this may not lie in $\mathbf{F}_q[y]$. We have seen that pairs of -1-eigenvalues of x split into pairs of distinct eigenvalues for $\bar{x}$. Now pairs of $\beta_{i,j}$ and $\beta_{i,j}^{-1}$ eigenvalues of x correspond to pairs of $\sqrt{\beta_{i,j}}$ and $\sqrt{\beta_{i,j}^{-1}}$ eigenvalues for $\bar{x}$. Thus there can only be a change in multiplicities here if $\beta_{i,j} = \beta_{i,j}^{-1}$, forcing an invalid choice of $\lambda_{i,j}$. Hence L_x is the product of the same general linear groups that occur in $Z_G(x)$ with the exception of the factor that comes from the -1-eigenvector chains. This factor is the centralizer in $GL_{2b}(\mathbf{F}_q)$ of $\begin{pmatrix} 0 & I_b \\ -I_b & 0 \end{pmatrix}$ and so is the subgroup of matrices of the form $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ with A, $B \in M_b(\mathbf{F}_q)$. When $\sqrt{-1} \in \mathbf{F}_q$ it is isomorphic to $GL_b(\mathbf{F}_q) \times GL_b(\mathbf{F}_q)$ by $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto (A + \sqrt{-1}B, A - \sqrt{-1}B)$. Thus the involution θ acts here by swapping components. When $\sqrt{-1} \notin \mathbf{F}_q$ this factor is isomorphic to $GL_b(\mathbf{F}_{q^2})$ by $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \sqrt{-1}B$. Thus the involution θ

acts here by conjugation by $\sqrt{-1}$. Similar calculations show that θ acts by swapping components on the $GL_{m_i}(\mathbf{F}_{q^{d_i}}) \times GL_{m_i}(\mathbf{F}_{q^{d_i}})$ factors and acts by conjugation on the $GL_{m_i}(\mathbf{F}_{q^{2d_i}})$ factors. (Note that the centralizer of the right hand side of Equation 8 is the subgroup of matrices of the form $\left(\begin{smallmatrix} A & B \\ CB & A \end{smallmatrix}\right)$ where $A, B \in M_{w^\nu}(\mathbf{F}_q)$ commuting with C.) The involution θ acts on the $GL_{a_1+a_2}(\mathbf{F}_q)$ component by conjugation by I_{a_1,a_2}.

The Weyl group of L_x is the appropriate products of symmetric groups. To compute $\mathcal{F}_x$ we need first to compute the coherent continuous representation for each component, there being three cases to consider. The one for $GL_m(\mathbf{C}) \times GL_m(\mathbf{C})$ with respect to $\Delta GL_m(\mathbf{C})$ was handled in Proposition 3. For the other two cases we have the following Proposition.

Proposition 9. *The coherent continuous representation for:*
(i) $GL_{a_1+a_2}(\mathbf{C})$ *with respect to* $GL_{a_1}(\mathbf{C}) \times GL_{a_2}(\mathbf{C})$ *is*

$$\bigoplus_{\lambda \in \mathcal{P}_{a_1+a_2}} |A^{\mathbf{R}}_\lambda(a_2 - a_1)/\mathrm{Row}_\lambda| \sigma_\lambda.$$

(ii) $GL_l(\mathbf{C})$ *with respect to* $GL_l(\mathbf{R})$ *is*

$$\mathcal{F}_{\mathbf{R},l} = \bigoplus_{\lambda \in \mathcal{P}_l} \sigma_\lambda.$$

Proof. Part (i) is calculated in [BV] Section 4.

Part (ii). Let τ be the involution given by complex conjugation. There are $[\frac{l}{2}] + 1$ $GL_l(\mathbf{R})$-conjugacy classes of τ-stable Cartan subalgebras of $\mathfrak{gl}_l(\mathbf{C})$. Representatives of these classes is given by the set of Cartan subalgebras $\{\mathfrak{h}_i\}_{0 \le i \le \frac{l}{2}}$, where $\mathfrak{h}_i$ is the Lie algebra of matrices of the form

$$h = \begin{pmatrix} A(\alpha_1, \beta_1) & & & & & \\ & \ddots & & & & \\ & & A(\alpha_i, \beta_i) & & & \\ & & & \gamma_1 & & \\ & & & & \ddots & \\ & & & & & \gamma_{l-2i} \end{pmatrix} \tag{10}$$

with $A(\alpha, \beta)$ being the matrix $\left(\begin{smallmatrix} \alpha & \beta \\ -\beta & \alpha \end{smallmatrix}\right)$ and α_j, β_j and $\gamma_j \in \mathbf{C}$. Now $W_{GL_l(\mathbf{C})}(\mathfrak{h}_i) = S_l$ while the little Weyl group $W_{GL_l(\mathbf{R})}(\mathfrak{h}_i) = W(B_i) \times S_{l-2i}$, (where $W(B_i) = (\mathbf{Z}/2\mathbf{Z})^i \ltimes S_i$ is the Weyl group of type B_i). Now $\mathfrak{h}_i$ has linear functionals α_j^*, β_j^* and γ_j^* given by $\nu_j^*(h) = \nu_j$. The τ-imaginary roots of $\mathfrak{h}_i$ are then $\{\gamma_j^* - \gamma_k^* : 1 \le j \ne k \le l - 2i\}$. Thus

$$\begin{aligned} \mathcal{F}_{\mathbf{R},l} &= \bigoplus_i \mathrm{Ind}^{S_l}_{W(B_i)\times S_{l-2i}}(\text{trivial}) \boxtimes (\text{sign}) \\ &= \bigoplus_i \mathrm{Ind}^{S_l}_{S_{2i}\times S_{l-2i}} \mathrm{Ind}^{S_{2i}\times S_{l-2i}}_{W(B_i)\times S_{l-2i}}(\text{trivial}) \boxtimes (\text{sign}) \end{aligned}$$

Now by [BV], Lemma 4.1.b we have

$$(11)\qquad \mathrm{Ind}_{W(B_i)}^{S_{2i}}(\text{trivial}) = \bigoplus_{\substack{\lambda \in \mathcal{P}_{2i} \\ \lambda \text{ has only even rows}}} \sigma_\lambda.$$

If σ_λ is a representation of S_j then, by the Littlewood–Richardson rule (see [FH], §A.1), we have

$$(12)\qquad \mathrm{Ind}_{S_j \times S_k}^{S_{j+k}} \sigma_\lambda \boxtimes (\text{sign}) = \bigoplus_{\lambda * \in \mathcal{P}^*_{j+k}} \sigma_\lambda.$$

where $\mathcal{P}^*_{j+k}$ is the set of partitions that can be obtained by adding k squares to λ so that no two are added to the same row. Using induction and applying Equations 11 and 12 we see that each partition occurs exactly once in the above expression for $\mathcal{F}_{\mathbf{R},l}$. □

Since, in this case, Ψ induces an isomorphism, we will refer to $\mathcal{F}_{\Psi(gK)}$ instead of $\mathcal{F}_{gK}$. Piecing together these three different types of coherent continuous representations yields

$$(13)\qquad \mathcal{F}_x = \bigoplus_{\lambda \times \mu \times \prod_i \nu^{(i)} \in \mathcal{P}_x} |A_\lambda^{\mathbf{R}}/\mathrm{Row}_\lambda| \cdot \sigma_\lambda \boxtimes \sigma_\mu^{e_1} \boxtimes (\boxtimes_i \sigma_{\nu^{(i)}}^{e_i})$$

where $\mathcal{P}_x = \mathcal{P}_{a_1+a_2} \times \mathcal{P}_b \times \prod_i \mathcal{P}_{m_i}$ and the e_j's are either 1 or 2. Thus the right hand side of the equation in Theorem 2 becomes

$$(14)\qquad \sum_{x \in \mathcal{S}} (\mathcal{F}_x, \mathcal{F}_x) = \sum_{x \in \mathcal{S}} \sum_{\lambda \times \mu \times \prod_i \nu^{(i)} \in \mathcal{P}_x} |A_\lambda^{\mathbf{R}}/\mathrm{Row}_\lambda|^2.$$

Let $\mathcal{U}(c_{i,j}^V)$ be the set of unipotent conjugacy classes in $GL_{w^V}(\mathbf{F}_q)$ that commute with $(c_{i,j}^V)$. By the discussion in Section 2 we have that the right hand side of the above equation is equal to

$$\sum_{w=0}^{l} \sum_{\substack{x \in \mathcal{S} \\ w^1 = w}} \left(\sum_{(\lambda, \mu) \in \mathcal{P}_{a_1+a_2} \times \mathcal{P}_b} |A_\lambda^{\mathbf{R}}/\mathrm{Row}_\lambda|^2 \right) \cdot |\mathcal{U}(c_{i,j}^V)| = \sum_{w=0}^{l}$$
$$\times \left(\sum_{\mu \in \mathcal{P}_{n-2w}} |A_\lambda^{\mathbf{R}}/\mathrm{Row}_\lambda|^2 \right) \cdot |\mathcal{C}_w^{\neq 1}(\mathbf{F}_q)|.$$

The following is combinatorial lemma which completes the proof of Theorem 2 for this case.

Lemma 15. *For $\lambda \in \mathcal{P}_{n-2w}$ we have*

$$|A_\lambda/\mathrm{Row}_\lambda| = \sum_{(\mu, \nu) \in \mathcal{P} \times \mathcal{P}: \mu \cup \nu \cup \nu = \lambda} |A_\mu^{\mathbf{R}}/\mathrm{Row}_\mu|^2.$$

Proof. Consider the map $\Phi : A_\lambda^{\mathbf{R}} \times A_\lambda^{\mathbf{R}} \to A_\lambda$ induced by

$$\Phi(f_1, f_2)(r, 1) = f_1(r, 1)\sqrt{f_1(r, 1) f_2(r, 1)}.$$

To calculate the sum of an element of A_λ over λ we need only to sum over the first columns of rows of odd length. Let λ have k odd rows, on these k first squares an element of $A_\lambda^{\mathbf{R}}$ takes $\frac{n+k}{2}-p$ times the value $+1$ and takes $\frac{n-k}{2}+p$ times the value -1. It is then easy to check that Φ is well-defined. Conversely, we have a well-defined inverse Φ^{-1}, induced by

$$\Phi^{-1}(f)((r,1),(s,1)) = (\mathrm{sign}(f(r,1)), \Delta(f(s,1)))$$

where $\Delta(x)$ is 1 if $x \in \{+1, -\sqrt{-1}\}$ and is -1 if $x \in \{-1, +\sqrt{-1}\}$. Thus the map Φ is a bijection, however it doesn't remain a bijection under the equivalence relation given by permuting rows. This is since restricting to two rows of λ of equal even (resp. odd) length yields nine (resp. two) possibilities for $A_\lambda^{\mathbf{R}}/\mathrm{Row}_\lambda \times A_\lambda^{\mathbf{R}}/\mathrm{Row}_\lambda$ whereas there are ten (resp. one) possibilities for $A_\lambda/\mathrm{Row}_\lambda$. Hence to induce a bijection onto $A_\lambda/\mathrm{Row}_\lambda$ we need also to consider partitions μ which we can obtain from λ by deleting pairs of equal rows. $\square$

5. $GL_n(\mathbf{R})$

Let w_0 be the element in S_n given by $(1\,n)(2n-1)\cdots([\frac{n+1}{2}][\frac{n+2}{2}])$. We will also write w_0 for the corresponding permutation matrix. In this case we will use for θ the involution given by $\theta : g \mapsto w_0(g^{-1})^t w_0$. Thus G^θ is the orthogonal group $O_n(\mathbf{F}_q, Q)$ where Q is the nondegenerate quadratic form given by $Q(A,B) = \sum_{i=1}^{n} A_i B_{w_0(i)}$. This quadratic form has zero Witt defect and so G^θ is $\mathbf{F}_q$-split (for example, see [C]). The group G_0^θ is the special orthogonal group $SO_n(\mathbf{F}_q, Q)$. Now $|G^\theta : G_0^\theta| = 2$ so these are the only two possibilities for K. For $\theta(g) = g^{-1}$ we must have that g is an element of the set $Y = \{g \in G : g_{i,j} = g_{w_0(j), w_0(i)}\}$. The set X is the set of elements of Y whose determinant is a square. Let x be a semisimple element in X. We can choose the element $\bar{x}$ to have the same set of multiplicities of eigenvalues as x and so L_x will be isomorphic to $Z_G(x)$. Hence we need only calculate with x.

Let x have characteristic polynomial f and let $f = \prod_i f_i^{m_i}$ be its factorization into $\mathbf{F}_q$-irreducibles with the degree of f_i equaling d_i. Then

$$L_x \cong \prod_i GL_{m_i}(\mathbf{F}_{q^{d_i}})$$

and its Weyl group is $\prod_i S_{m_i}$. Let $\mathcal{P}_x = \prod_i \mathcal{P}_{m_i}$. Since L_x is θ-stable the above isomorphism preserves Q, that is, $L_x \cap K$ is isomorphic to $\prod_i GL_{m_i}(\mathbf{F}_{q^{d_i}}) \cap \bar{K}$. Thus $L_x \cap G_0^\theta$ is isomorphic to $(\prod_i O_{m_i}(\mathbf{F}_{q^{d_i}}, Q)) \cap \bar{G}_0^\theta$. Whilst $L_x \cap G^\theta$ is isomorphic to $\prod_i O_{m_i}(\mathbf{F}_{q^{d_i}}, Q)$.

5.1. Counting conjugacy classes

The strategy here is to first examine $\bar{K}$-conjugacy classes. If any two elements of $Y(\bar{\mathbf{F}}_q)$ are similar then they are orthogonally similar (see, for example, [Ga] XI Theorem 4 and using the fact that all quadratic forms are similar over an algebraically closed field of odd characteristic). Thus any two elements of $X(\bar{\mathbf{F}}_q)$ with the same Jordan canonical form are conjugate by an element of $\bar{G}^\theta$. For n odd we have $\bar{G}^\theta = \langle \pm 1 \rangle \times \bar{G}_0^\theta$ so in this case any two elements of $X(\bar{\mathbf{F}}_q)$ with the same Jordan canonical form are conjugate by an element of $\bar{G}_0^\theta$. To examine the $\bar{G}_0^\theta$-orbits of a single Jordan block of even length we use the following sharpening of Witt's theorem for the orthogonal case found in E. Artin's book [A], Theorem 3.16.

Theorem 16. *Let V be an n-dimensional vector space with an non-singular quadratic form Q. Let g_U be an isometry of a subspace U of V into V. It is possible to prescribe the value ± 1 for the determinant of an extension g_V of g_U to all of V if and only if* $\dim U + \dim \operatorname{rad} U < n$.

Let $J_{2m}(\lambda)$ be the set of elements of $X(\bar{\mathbf{F}}_q)$ whose Jordan canonical form consists of a single $2m \times 2m$ Jordan block with eigenvalue λ. Since $|\bar{G}^\theta : \bar{G}_0^\theta| = 2$ we have that $J_{2m}(\lambda)$ is either the union of one or two $G_0^\theta(\bar{\mathbf{F}}_q)$-conjugacy classes of X.

Lemma 17. *There are two $\bar{G}_0^\theta$-orbits in $J_{2m}(\lambda)$.*

Proof. We show that there has to be more than one conjugacy class. Let

$$J = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}.$$

Note that $J \in Y$ and so is in $J_{2m}(\lambda)$. For an arbitrary $L \in J_{2m}(\lambda)$ let $V_i(L)$ be the nullspace of $(L - \lambda I_{2m})^i$. We know that $L = kJk^{-1}$ for some $k \in \bar{G}^\theta$ and so $k(V_m(J)) = V_m(L)$. Note that $V_m(J) = \langle e_1 \cdots e_m \rangle$ and so is a Q-isotropic subspace. That is, $\dim V_m(J) + \dim \operatorname{rad} V_m(J) = 2m$. Theorem 16 then shows we cannot prescribe the determinant of k. $\square$

Using induction we get that there are two $(\prod_i O_{m_i}(\bar{\mathbf{F}}_q)) \cap \bar{G}_0^\theta$-orbits on $\bigoplus_i J_{m_i}(\lambda_i)$ if all the m_i are even and one orbit otherwise. Now we descend from $\bar{\mathbf{F}}_q$ down to $\mathbf{F}_q$. To do this we using the following extension of Lang's theorem ([SS], Theorem 2.7).

Theorem 18. *Let L be a connected linear algebraic group and σ an endomorphism of L such that the fixed point subgroup L^σ is finite. Let M be a nonempty L-homogeneous space on which σ acts.*
(i) *M contains a point fixed by σ.*
(ii) *Fix $m_0 \in M^\sigma$ and set $A = Z_L(m_0)$. Assume that A is a closed subgroup of L. Then the elements of the orbit space L^σ / M^σ are in one to one correspondence with those of $H^1(\sigma, A/A_0)$.*

Here $H^1(\sigma, A)$ denotes A modulo the equivalence relation: $a \sim b$ if $a = cb\sigma(c)^{-1}$ for some $c \in A$.

Now $Z_{G_0^\theta}(J) = \pm I_n$ and σ acts trivially on this. Thus the $\mathbf{F}_q$-points of the

$\bar{G}_0^\theta$-orbit containing J consists of two $\bar{G}_0^\theta$-orbits. By Witt's theorem these two $\bar{G}_0^\theta$-orbits form one G^θ-orbit. So by induction there are four $(\prod_i O_{m_i}(\bar{\mathbf{F}}_{q^{d_i}})) \cap \bar{G}_0^\theta$-orbits on the $\mathbf{F}_q$-points of $\bigoplus_i J_{m_i}(\lambda_i)$ if all the m_i are even and there is just one orbit otherwise.

Recall that $\mathcal{S}$ is the set of semisimple elements of G/K. Let $\mathcal{S}_{\text{even}}$ be the subset consisting of those elements with all their associated m_i's even. Since there exists elements of $Y(\bar{\mathbf{F}}_q)$ with arbitrary preassigned Jordan canonical form (for example, see [Ga] XI, Theorem 5) we have that the left hand side of Theorem 2, for this case, is then

$$\sum_{x \in \mathcal{S}-\mathcal{S}_{\text{even}}} |\mathcal{P}_x| + \sum_{x \in \mathcal{S}_{\text{even}}} c_K|\mathcal{P}_x| \tag{19}$$

where $c_K = 1$ when $K = G^\theta$ and $c_K = 4$ when $K = G_0^\theta$.

5.2. The coherent continuous representation

The following proposition gives the coherent continuous representation for the case $i = 1$.

Proposition 20. *The coherent continuous representation for:*
(i) $GL_n(\mathbf{C})$ *with respect to* $O_n(\mathbf{C})$ *and* (ii) $GL_n(\mathbf{C})$ *with respect to* $SO_n(\mathbf{C})$ *when* $n = 2m+1$ *is*

$$\bigoplus_{\lambda \in \mathcal{P}_n} \sigma_\lambda.$$

(iii) $GL_n(\mathbf{C})$ *with respect to* $SO_n(\mathbf{C})$ *when* $n = 2m$ *is*

$$\mathcal{F}_n = \bigoplus_{\lambda \in \mathcal{P}_n} a_\lambda \sigma_\lambda$$

where a_λ *equals two when all the columns of* λ *are even and is equal to one otherwise.*

Proof. Case (i) follows from Proposition 9 (ii). For cases (ii) and (iii) there are $m\, SO_n(\mathbf{C})$-conjugacy classes of θ-stable Cartan subalgebras of $\mathfrak{gl}_n(\mathbf{C})$. Representatives of these classes is given by the set of Cartan subalgebras $\{\mathfrak{h}_i\}_{0 \le i \le m}$ where $\mathfrak{h}_i$ is the Lie algebra given in Proposition 9 by Equation 10. The little Weyl groups are $W_{SO_n(\mathbf{C})}(\mathfrak{h}_i) = W(B_i) \times S_{n-2i}$ unless $n = 2m$ and $i = m$, in which case $W_{SO_{2m}(\mathbf{C})}(\mathfrak{h}_m) = W(D_m)$ where $W(D_m) = (\mathbf{Z}/2\mathbf{Z})^{m-1} \ltimes S_m$ the Weyl group of type D_m. Part (ii) now follows the same proof as for Proposition 9 (ii). For part (iii) we have:

$$\mathcal{F}_{2m} = \left(\bigoplus_{1 \le i < m} \operatorname{Ind}_{W(B_i) \times S_{2(m-i)}}^{S_{2m}} (\text{trivial}) \boxtimes (\text{sign}) \right) \bigoplus \operatorname{Ind}_{W(D_m)}^{S_{2m}} (\text{trivial}).$$

Now $\operatorname{Ind}_{W(D_m)}^{S_{2m}}$ (trivial) $= \operatorname{Ind}_{W(B_m)}^{S_{2m}} \operatorname{Ind}_{W(D_m)}^{W(B_m)}$ (trivial) $= \operatorname{Ind}_{W(B_m)}^{S_{2m}}$ (trivial) $\bigoplus$ (sign). Thus from the proof of Proposition 9 (ii)

$$\mathcal{F}_{2m} = \left(\bigoplus_{\lambda \in \mathcal{P}_{2m}} \sigma_\lambda \right) \bigoplus \mathrm{Ind}_{W(B_m)}^{S_{2m}}(\mathrm{sign}).$$

But $\mathrm{Ind}_{W(B_m)}^{S_{2m}}(\mathrm{sign})$ is the sum of the representations of S_{2m} which are parametrized by diagrams which are the transpose of the diagrams in $\mathrm{Ind}_{W(B_m)}^{S_{2m}}(\mathrm{trivial})$. The lemma now follows from Equation 11. □

By induction, as a corollary to the above Proposition 20 we have the following.

Corollary 21. *The coherent continuous representation for*
(i) $\prod_i GL_{m_i}(\mathbf{C})$ *with respect to* $(\prod_i O_{m_i}(\mathbf{C})) \cap SO_n(\mathbf{C})$ *is*

$$\begin{cases} \bigoplus_{\prod \lambda(i) \in \mathcal{P}_x} 2 \boxtimes_i \sigma_{\lambda(i)} & \text{if } x \in \mathcal{S}_{\text{even}}, \\ \bigoplus_{\prod \lambda(i) \in \mathcal{P}_x} \boxtimes_i \sigma_{\lambda(i)} & \text{otherwise.} \end{cases}$$

(ii) $\prod_i GL_{m_i}(\mathbf{C})$ *with respect to* $(\prod_i O_{m_i}(\mathbf{C}))$ *is*

$$\bigoplus_{\prod \lambda(i) \in \mathcal{P}_x} \boxtimes_i \sigma_{\lambda(i)}.$$

Comparing this with Equation 19 we see that Theorem 2 is proved for this case.

Remark 22. Note that Theorem 2 does not hold if we use a K which is not a split form. For example, let Q be a nondegenerate quadratic form with a Witt defect of one, then the orthogonal groups $O_n(\mathbf{F}_q, Q)$ and $SO_n(\mathbf{F}_q, Q)$ are non-split forms (note that this case can only occur for n even). In this case the right hand side of Theorem 2 and the number of semisimple conjugacy classes are the same as in the corresponding split form case just discussed but that the number of unipotent conjugacy classes decreases. For example, when $n = 2$, there are no unipotent elements in G/K as there are no isotropic vectors for Q in $\mathbf{F}_q^2$ and thus all elements of X are diagonalizable (using the same proof for the diagonalizability of real symmetric matrices). In general, there cannot exist elements of X whose Jordan canonical form consists entirely of even dimensional blocks.

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Received January 1997