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An Oscillating Hydromagnetic Non-Newtonian Flow in a Rotating System

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Abstract—An exact solution of an oscillatory boundary layer flow bounded by two horizontal flat plates, one of which is oscillating in its own plane and the other at rest, is developed. The fluid and the plates are in a state of solid body rotation with constant angular velocity about the z-axis normal to the plates. The fluid is assumed to be second grade, incompressible, and electrically conducting. A uniform transverse magnetic field is applied. During the mathematical analysis, it is found that the steady part of the solution is identical to that of viscous fluid. The structure of the boundary layers is also discussed. Several known results of interest are found to follow as particular cases of the solution of the problem considered. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

The analysis of the effects of rotation and magnetic field in fluid flows has been an active area of research because of its geophysical and technological importance. Interest in MHD flow began in 1918, when Hartmann [1] invented the electromagnetic pump. The study of magnetic field effects on the laminar flow of an incompressible electrically conducting fluid is an important problem that is related to many practical applications, such as the MHD power generator and boundary layer flow control. Historically, Rossow [2] was the first to study the hydrodynamic behavior of the

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boundary layer on a semi-infinite flat plate in the presence of a uniform transverse magnetic field. Since then, a large amount of literature has been developed on this subject. A review of this topic today can be found in [3]. The effects of a uniform transverse magnetic field were investigated by Gupta [4], Debnath [5], Soundalgekar and Pop [6], Mazumder *et al.* [7]. In another paper, Mazumder [8] investigated the unsteady oscillatory Couette flow of a viscous incompressible fluid between two parallel plates, one of which is oscillating and the other at rest. Later, Ganapthy [9] proposed an alternative solution for the problem in [8]. More recently, Singh [10] extended this analysis by including the magnetic field effects. Despite the above studies, attention has hardly been given to the study of hydromagnetic flows of non-Newtonian fluids. The study of non-Newtonian fluid dynamics is important in connection with plastics manufacture, performance of lubricants, applications of paints, processing of food, and movement of biological fluids. Most biologically important fluids contain higher molecular weight components and are, therefore, non-Newtonian.

Keeping in view the importance of non-Newtonian fluids, the main objective of this communication is to present an exact solution to the study of oscillatory flow between two parallel plates. The fluid considered is electrically conducting and second grade. The entire system rotates about an axis perpendicular to the planes of plates. The study of hydromagnetic flow of second grade fluid between two horizontal plates responds to oscillations in one plate in a rotating system has remained attented. For $\beta_1 = 0$, the problem reduces to the one discussed by Singh [10]. Moreover, for $\beta_1 = 0 = M$, the problem reduces to that of Ganapathy [9]. Similar to Singh [10] it is noted that the claim of Ganapathy [9] that the solution of Mazumder [8] explains resonance phenomenon in rotating system is incorrect.

2. GOVERNING EQUATIONS

The physical situation considered is that of the unsteady hydromagnetic flow of a second grade, incompressible, and electrically conducting fluid bounded by infinite-parallel plates, distant d apart, when both the fluid and plates rotate with a constant angular velocity Ω about the z-axis taken normal to the plates. It is assumed that the plates are electrically nonconducting and an applied uniform magnetic field $\mathbf{B}_0$ is acting parallel to the z-axis. The lower plate is at rest and the upper plate oscillating in its own plane with a velocity $U(t) = U_0(1+ \in \cos\omega t)$ about a nonzero constant mean velocity U_0. The origin is taken on the lower plate and the x-axis parallel to the direction of motion of the upper plate. Since the plates are infinite in extent, all the physical quantities, except the pressure, depend on z and t only. In a coordinate system rotating with the fluid, the governing equations of continuity and motion and Maxwell's equations are

$$\boldsymbol{\nabla}.\mathbf{V} = 0, \tag{1}$$

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}.\boldsymbol{\nabla})\,\mathbf{V} + 2\boldsymbol{\Omega}\times\mathbf{V} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})\right] = \boldsymbol{\nabla}.\mathbf{T} + \mathbf{J}\times\mathbf{B}, \tag{2}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_e\mathbf{J}, \tag{3}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{4}$$

$$\boldsymbol{\nabla}.\mathbf{B} = 0, \tag{5}$$

$$\mathbf{J} = \sigma\left(\mathbf{E} + \mathbf{V}\times\mathbf{B}\right), \tag{6}$$

where $\mathbf{V} = (u, v, w)$ is the velocity vector, $\Omega = \Omega\mathbf{k}$, $\mathbf{k}$, the unit vector in the z-direction, t, the time, ρ, the density, $\mathbf{J}$, the current density, $\mathbf{B}$, the magnetic induction, $\mathbf{E}$, the electric field, μ_e, the magnetic permeability, σ, the electrical conductivity, $\mathbf{T}$, the Cauchy stress, and $\mathbf{r}$, the radial coordinate given by

$$r^2 = x^2 + y^2. \tag{7}$$

The constitutive equation for Cauchy stress tensor $\mathbf{T}$ is [11,12]

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2. \tag{8}$$

Here p is the dynamic pressure function, $\mathbf{I}$ the unit tensor, μ the constant dynamic viscosity, and α_1, α_2 the normal stress moduli, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the Rivlin-Eriksen tensors [13] and are given by

$$\mathbf{A}_1 = (\operatorname{grad}\mathbf{V}) + (\operatorname{grad}\mathbf{V})^{\top}, \tag{9}$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + (\operatorname{grad}\mathbf{V})^{\top}\mathbf{A}_1 + \mathbf{A}_1(\operatorname{grad}\mathbf{V}), \tag{10}$$

where $\frac{d}{dt}$ is the material time derivative. According to Dunn and Fosdick [14], the second grade fluid model is compatible with thermodynamics when the Helmholtz free energy of the fluid is a minimum for the fluid in equilibrium. The fluid model then has general and pleasant boundedness and stability properties. The aforementioned and the Clausius-Duhem inequality imply that the coefficients μ, α_1, and α_2 must satisfy

$$\mu \geq 0, \quad \alpha_1 \geq 0, \qquad \alpha_1 + \alpha_2 = 0. \tag{11}$$

Since we are dealing with second grade fluid flow, the strict inequality holds true. Fosdick and Rajagopal [15] showed that when $\alpha_1 < 0$, the fluid exhibits anomalous behavior that is incompatible with any fluid of rheological interest, and so results in a fluid that is unstable. Further, we assume that the magnetic Reynolds number is so small that the induced magnetic field can be neglected in comparison with the applied one [16], so that

$$\mathbf{B} = (0, 0, B_0), \tag{12}$$

where B_0 is a constant. It is assumed that no applied and polarization voltage exists (i.e., $\mathbf{E} = 0$). This then corresponds to the case when no energy is added to or extracted from the fluid by the electric field. Since the plates are infinite in extent, all physical variables (except pressure) are functions of z and t only. Thus, equation (1) becomes $\frac{\partial w}{\partial z} = 0$ which, because $w = 0$ on the boundaries, implies that $w = 0$ every where in the fluid. Now, the equation for the conservation of electric charge, $\mathbf{\nabla}.\mathbf{J} = 0$, leads to $J_z =$ constant, where $\mathbf{J} = (J_x, J_y, J_z)$. As in the case of vertical velocity, we immediately see that $J_z = 0$. Thus, equation (6) yields

$$J_x = \sigma B_0^2 v, \qquad J_y = -\sigma B_0^2 u. \tag{13}$$

In view of the above consideration, equation (2) can be rewritten in the component form

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x} + \left(\nu + \beta_1\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial z^2} + 2\Omega v - \frac{\sigma B_0^2 u}{\rho}, \tag{14}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p^*}{\partial y} + \left(\nu + \beta_1\frac{\partial}{\partial t}\right)\frac{\partial^2 v}{\partial z^2} - 2\Omega u - \frac{\sigma B_0^2 v}{\rho}, \tag{15}$$

where ν is the kinematic viscosity, $\beta_1 = \alpha_1/\rho$ and the modified pressure

$$p^* = p - \frac{\rho}{2}r^2\Omega^2 - (2\alpha_1 + \alpha_2)\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right]. \tag{16}$$

The boundary conditions for the present problem are

$$\begin{aligned} u = v = 0, &\qquad \text{at } z = 0, \\ u = U(t) = U_0(1 + \epsilon\cos\omega t), \quad v = 0, &\qquad \text{at } z = d, \end{aligned} \tag{17}$$

where ω is the frequency of oscillations and ϵ is a constant.

Now eliminating the modified pressure gradient p^*, under the boundary layer approximation, the resulting equation of (14) and (15) can be combined as

$$\frac{\partial q}{\partial t} = \left(\nu + \beta_1 \frac{\partial}{\partial t}\right) \frac{\partial^2 q}{\partial z^2} + \frac{\partial U}{\partial t} - 2i\Omega\,(q - U) - \frac{\sigma B_0^2}{\rho}\,(q - U), \tag{18}$$

and the corresponding boundary conditions (17) are

$$\begin{aligned} q &= 0, && \text{at } z = 0, \\ q &= U(t), && \text{at } z = d, \end{aligned} \tag{19}$$

where

$$q = u + iv \tag{20}$$

is the fluid velocity in the complex form. It should be noted that equation (18) includes the Navier-Stokes fluid as a special case for $\beta_1 = 0$. If $\Omega = 0$, the equation reduces to that of second grade fluid in an inertial frame. Moreover, if $B_0 = 0$, the equation governing the flow of a nonconducting second grade fluid is obtained.

3. SOLUTION OF THE PROBLEM

In order to solve equation (18) subject to the boundary conditions (19), we look for the solution of the form [17]

$$q\,(\eta, t) = U_0 \left[q_0\,(\eta) + \frac{\epsilon}{2}\left\{q_1\,(\eta)\,e^{i\omega t} + q_2\,(\eta)\,e^{-i\omega t}\right\}\right], \tag{21}$$

where

$$\eta = \frac{z}{d}, \qquad q_0\,(\eta) = u_0(\eta) + iv_0(\eta) \quad \text{and} \quad q_1\,(\eta)\,e^{i\omega t} + q_2\,(\eta)\,e^{-i\omega t} = u_1\,(\eta, t) + iv_1\,(\eta, t)\,. \tag{22}$$

Using equation (21) into equation (18) and boundary condition (19) and then collecting harmonic and nonharmonic terms, we obtain

$$\frac{d^2 q_0}{d\eta^2} - \left(2iK + M^2\right) q_0 = -\left(2iK + M^2\right), \tag{23}$$

$$\frac{d^2 q_1}{d\eta^2} - \left(1 + \frac{i\omega\beta_1}{\upsilon}\right)^{-1} \left[2iK + M^2 + i\lambda\right] q_1 = -\left(1 + \frac{i\omega\beta_1}{\upsilon}\right)^{-1} \left[2iK + M^2 + i\lambda\right], \tag{24}$$

$$\frac{d^2 q_2}{d\eta^2} - \left(1 + \frac{i\omega\beta_1}{\upsilon}\right)^{-1} \left[2iK + M^2 - i\lambda\right] q_2 = -\left(1 + \frac{i\omega\beta_1}{\upsilon}\right)^{-1} \left[2iK + M^2 - i\lambda\right], \tag{25}$$

$$\begin{aligned} q_0 &= q_1 = q_2 = 0, && \text{at } \eta = 0, \\ q_0 &= q_1 = q_2 = 1, && \text{at } \eta = 1. \end{aligned} \tag{26}$$

In the above equations, $K = \Omega d^2/\nu$ is the rotation parameter, $\lambda = \omega d^2/\nu$ is an oscillatory Reynolds number and $M = B_0 d(\sigma/\mu)^{1/2}$ is the Hartmann number. Solving equations (23)–(25) under boundary conditions (26), we get

$$q_0\,(\eta) = 1 - \frac{\sinh l\,(1 - \eta)}{\sinh l}, \tag{27}$$

$$q_1\,(\eta) = 1 - \frac{\sinh m\,(1 - \eta)}{\sinh m}, \tag{28}$$

$$q_2\,(\eta) = 1 - \frac{\sinh n\,(1 - \eta)}{\sinh n}, \tag{29}$$

where

$$l = \left(2iK + M^2\right)^{1/2}, \qquad m = \left[\frac{2iK + M^2 + i\lambda}{(1 + i\omega\beta_1/\upsilon)}\right]^{1/2}, \qquad n = \left[\frac{2iK + M^2 - i\lambda}{(1 + i\omega\beta_1/\upsilon)}\right]^{1/2}.$$

We note from equations (27)–(29) that if $\beta_1 = 0$, then results of Singh [10] are recovered.

4. RESULTS AND DISCUSSION

We note that steady solution (27) is independent on β_1. It means that primary and secondary velocity components u_0 and v_0, respectively, for present steady flow do not depend upon the nature of the fluid. The amplitudes and phase differences in terms of u_0 and v_0 are given by

$$R_0 = \sqrt{u_0^2 + v_0^2}, \qquad \theta_0 = \tan^{-1} \frac{v_0}{u_0}.$$

From equation (27), we have for large K

$$\begin{aligned} u_0 &\approx 1 - e^{-l_r \eta} \cos l_i \eta, \\ v_0 &\approx e^{-l_r \eta} \sin l_i \eta, \end{aligned} \tag{30}$$

where

$$l_r = \frac{1}{\sqrt{2}} \left[M^2 + \sqrt{M^4 + 4K^2} \right]^{1/2}, \qquad l_i = \frac{1}{\sqrt{2}} \left[-M^2 + \sqrt{M^4 + 4K^2} \right]^{1/2}. \tag{31}$$

Equations (30) represent spiral distribution. The thickness of boundary layer is of $O(l_r^{-1})$ in the neighborhood of plates. We conclude with the help of equations (31) that the thickness is reduced as M or K is increased.

Solutions (28) and (29) together give the unsteady part of the flow. These solutions depend on β_1. For large K, the primary and secondary velocity components u_1 and v_1, respectively, for the fluctuating flow are given by

$$u_1(\eta, t) \approx 2 \cos \omega t - e^{-m_r \eta} \cos (m_i \eta - \omega t) - e^{-n_r \eta} \cos (n_i \eta + \omega t), \tag{32}$$

$$v_1(\eta, t) \approx e^{-m_r \eta} \sin (m_i \eta - \omega t) + e^{-n_r \eta} \sin (n_i \eta + \omega t), \tag{33}$$

where

$$m_r = (C_1)^{-1} \left[\sqrt{\sqrt{A_1^2 + B_1^2} + A_1} \right],$$

$$m_i = (C_1)^{-1} \left[\sqrt{\sqrt{A_1^2 + B_1^2} - A_1} \right],$$

$$n_r = (C_1)^{-1} \left[\sqrt{\sqrt{A_2^2 + B_2^2} + A_2} \right],$$

$$n_i = (C_1)^{-1} \left[\sqrt{\sqrt{A_2^2 + B_2^2} - A_2} \right],$$

$$A_1 = M^2 + \frac{(2K + \lambda)\, \omega \beta_1}{\nu},$$

$$B_1 = (2K + \lambda) - \frac{M^2 \omega \beta_1}{\nu},$$

$$A_2 = M^2 + \frac{(2K - \lambda)\, \omega \beta_1}{\nu},$$

$$B_2 = (2K - \lambda) - \frac{M^2 \omega \beta_1}{\nu},$$

$$C_1 = \sqrt{2 \left[1 + \frac{\omega^2 \beta_1^2}{\nu^2} \right]}.$$

Expressions (32) and (33) for u_1 and v_1 show the emergence of a boundary layer of thickness of order $O(m_r^{-1})$ superimposed with a boundary layer of thickness of order $O(n_r^{-1})$. These boundary layers which are a direct consequence of the cyclonic and anticyclonic components of the impressed harmonic oscillations decrease with increase in M, β_1, and K. It may be noted that in second grade fluid the boundary layer thickness decrease more rapidly than for viscous fluid. Also, the present analysis exhibits a striking difference between the structure of hydrodynamic and the hydromagnetic boundary layers.

In case of resonance ($2\Omega - \omega = 0$ or $2K - \lambda = 0$), the solution of equation (25) is

$$q_2(\eta) = 1 - \frac{\sinh \tilde{M}\,(1-\eta)}{\sinh \tilde{M}}, \tag{34}$$

where

$$\tilde{M}^2 = M^2 \left(1 + \frac{i\omega\beta_1}{v}\right)^{-1}. \tag{35}$$

We note that when $M = 0$, then $q_2(\eta)$ for viscous and second grade fluids is the same and is given by

$$q_2(\eta) = \eta.$$

This above solution is the same as obtained by Singh [10]. Even in the case of resonance, differential equation (5) in [9] yields $q_2(\eta) = \eta \neq 0$. This is a contradiction to the claim of Ganapathy that $q_2(\eta) = 0$ and the solution

$$q(\eta, t) = U_0 \left[q_0(\eta) + \frac{\epsilon}{2} q_1(\eta)\, e^{i\omega t}\right],$$

of Mazumder is valid for the special case.

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