Application of Sinc-collocation method for solving an inverse problem

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ABSTRACT

In this article, the identification of an unknown time-dependent source term in an inverse problem of parabolic type with nonlocal boundary conditions is considered. The main approach is to change the inverse problem to a system of Volterra integral equations. The resulting integral equations are convolution-type, which by using Sinc-collocation method, are replaced by a system of linear algebraic equations. The convergence analysis is included, and it is shown that the error in the approximate solution is bounded in the infinity norm by the norm of the inverse of the coefficient matrix multiplied by a factor that decays exponentially with the size of the system. To show the efficiency of the present method, an example is presented. The method is easy to implement and yields very accurate results.

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1. Introduction

In this paper, we consider the inverse problem of determination a pair of functions \( \{u, p\}\) in the following parabolic equation

\[
-u_t = u_{xx} + p(t)u + q(x, t), \quad 0 < x < 1, \quad 0 < t \leq T, \tag{1}
\]

with the initial condition

\[
u(x, 0) = f(x), \quad 0 < x < 1, \tag{2}
\]

boundary conditions

\[
\alpha_1(t)u_t(0, t) + \beta(t)u(0, t) = g_1(t), \quad 0 < t \leq T, \tag{3}
\]

\[
\alpha_2(t)u_t(1, t) + \gamma(t)u(1, t) = g_2(t), \quad 0 < t \leq T, \tag{4}
\]

and the energy overspecified condition

\[
\int_0^1 u(x, t)dx = E(t), \quad 0 \leq t \leq T, \tag{5}
\]

where \(q, f, E, \alpha_i, \beta, \gamma, g_i, i = 1, 2\) are known functions.

The existence and uniqueness, and continuous dependence of the solution upon the data for this problem are demonstrated in [1] under the following assumptions:

\[
\begin{align*}
f & \in C^1[0, 1], \\
q & \in C^{a, a/2}([0, 1] \times [0, T]), \quad \text{where} \ 0 < a \leq 1, \quad q(x, t) \geq 0, \\
E & \in C^1[0, 1], \quad E(t) > 0, \quad E(0) = \int_0^1 f(x)dx > 0.
\end{align*}
\]

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In the following statement, without loss of generality, we assume that \( \alpha_1, \alpha_2 \neq 0 \),
\[
\alpha_1(0)f_1(0) + \beta(0)f(0) = g_1(0),
\]
\[
\alpha_2(0)f_2(1) + \gamma(0)f(1) = g_2(0),
\]
\[
f_{\min} > \frac{4c_0\sqrt{T}}{1 - 4c_0\sqrt{T}(\|\beta\| + \|\gamma\|)} \left[ \|r\|\|g\| + 2(\|\beta\| + \|\gamma\|)(\|f\| + c_0\sqrt{T}\|q\|) \right],
\]
where \( c_0 \) and \( \varepsilon \) are positive constants, \( \varepsilon \) is sufficiently small and
\[
f_{\min} = \min\{f(x); 0 \leq x \leq 1\}, \quad r(t) = \exp\left\{ -\int_0^t p(s)ds \right\}.
\]
These kinds of problems have many important applications in heat transfer, thermoelasticity, control theory, population dynamics, nuclear reactor dynamics, medical sciences, biochemistry etc. [2–13]. For example, in a heat transfer process, if we let \( u \) represent the temperature distribution, then (1)–(5) can be regarded as a control with a source control. A source control parameter \( p(t) \) needs to be determined so that a desired thermal energy can be obtained for a portion of the spatial domain.

In an environment where heat transfer takes place between liquids and solids, the heat flux is often taken to be proportional to the difference to the boundary temperature of the solid and the temperature of the liquid, and here, \( \alpha_i, \beta, \gamma; i = 1, 2 \) represent those proportionality factors [1]. More examples in this regard are presented in [2–5]. Recently, many authors have investigated similar problems to (1)–(5) [10–13].

Sinc methods are highly efficient numerical methods developed by Frank Stenger, the pioneer of this field, people in his school and others [14]. The books [14, 15] provide excellent overviews of methods based on Sinc functions for solving ordinary and partial differential equations and integral equations. Sinc methods have increasingly been recognized as powerful tools for attacking problems in applied physics and engineering [14, 16]. They have also been employed as forward solvers in the solution of inverse problems [17, 18]. In this work, we use the Sinc-collocation method for solving problems (1)–(5) and provide an accurate estimate for the solution \( u(x, t) \) and \( p(t) \).

This paper is organized as follows. In Section 2 we transform the inverse problem (1)–(5) into a system of Volterra integral equations. In Section 3 we will give some properties of Sinc function and Sinc method to approximate the convolution integrals. Section 4 contains the construction of the new Sinc-collocation method to replace the system of integral equations obtained in Section 2 by an explicit system of linear algebraic equations. In Section 5, convergence and error estimates are proved. Finally, some numerical results are presented in Section 6.

2. Reformulation of the problem

Employing a pair of transformations
\[
r(t) = \exp\left( -\int_0^t p(s)ds \right),
\]
\[
v(x, t) = r(t)u(x, t),
\]
the problem (1)–(5) will become
\[
v_t = v_{xx} + r(t)q(x, t), \quad 0 < x < 1, \quad 0 < t \leq T,
\]
\[
v(x, 0) = f(x), \quad 0 < x < 1,
\]
\[
\alpha_1(t)v_0(0, t) + \beta(t)v(0, t) = r(t)g_1(t), \quad 0 < t \leq T,
\]
\[
\alpha_2(t)v_0(1, t) + \gamma(t)v(1, t) = r(t)g_2(t), \quad 0 < t \leq T,
\]
\[
\int_0^1 v(x, t)dx = r(t)E(t), \quad 0 \leq t \leq T.
\]

Obviously, if we obtain \( \{v, r\} \) from (6)–(10) then \( \{u, p\} \) can be found as
\[
u(x, t) = \frac{v(x, t)}{r(t)}, \quad 0 \leq x \leq 1, \quad 0 < t \leq T,
\]
\[
p(t) = -\frac{r'(t)}{r(t)}, \quad 0 < t \leq T.
\]
In the following statement, without loss of generality, we assume that \( \alpha_1(t) = \alpha_2(t) = 1 \).

If we assume that the function \( r(t) \) is known, then, the direct problem (6)–(9) has the following solution [19]
\[
v(x, t) = w(x, t) - 2 \int_0^t \theta(x, t - \tau)[r(\tau)g_1(\tau) - \beta(\tau)\varphi(\tau)]d\tau + 2 \int_0^t \theta(x - 1, t - \tau)[r(\tau)g_2(\tau) - \gamma(\tau)\psi(\tau)]d\tau
\]
\[
+ \int_0^t \int_0^1 G(x, t, \xi, \tau)r(\tau)q(\xi, \tau)d\xi d\tau,
\]
where \( \varphi \) and \( \psi \) are piecewise-continuous solutions of the following system of integral equations

\[
\begin{align*}
\varphi(t) &= w(0, t) - 2 \int_0^t \theta(0, t - \tau)\{r(\tau)g_1(\tau) - \beta(\tau)\varphi(\tau)\}d\tau + 2 \int_0^t \theta(-1, t - \tau)\{r(\tau)g_2(\tau) - \gamma(\tau)\psi(\tau)\}d\tau \\
&\quad + \int_0^t \int_0^1 G(0, t, \zeta, \tau)r(\tau)q(\zeta, \tau)d\zeta d\tau, \\
\psi(t) &= w(1, t) - 2 \int_0^t \theta(1, t - \tau)\{r(\tau)g_1(\tau) - \beta(\tau)\varphi(\tau)\}d\tau + 2 \int_0^t \theta(0, t - \tau)\{r(\tau)g_2(\tau) - \gamma(\tau)\psi(\tau)\}d\tau \\
&\quad + \int_0^t \int_0^1 G(1, t, \zeta, \tau)r(\tau)q(\zeta, \tau)d\zeta d\tau,
\end{align*}
\]

where

\[
\begin{align*}
w(x, t) &= \int_0^1 G(x, t, \zeta, 0)f(\zeta)d\zeta, \\
G(x, t, \zeta, \tau) &= \theta(x - \zeta, t - \tau) + \theta(x + \zeta, t - \tau), \\
\theta(x, t) &= \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \exp \left\{ -\frac{(x + 2n)^2}{4t} \right\}.
\end{align*}
\]

Integrating (6) on \([0, 1]\), yields

\[
\int_0^1 v_1(x, t)dx = \int_0^1 v_{\text{ex}}(x, t)dx + r(t) \int_0^1 q(x, t)dx.
\]

Using energy overspecification (10), and by boundary conditions (8) and (9) we obtain

\[
E(t)\dot{r}(t) + [g_1(t) - g_2(t) - Q(t) + E(t)]r(t) = -\gamma(t)\psi(t) + \beta(t)\varphi(t),
\]

where \( Q(t) = \int_0^1 q(x, t)dx \). Note that we have \( r(0) = 1 \). Therefore, we get

\[
r(t) = e^{-h(t)} \left\{ 1 + \int_0^t \frac{e^{h(\tau)}}{E(\tau)}\{\beta(\tau)\varphi(\tau) - \gamma(\tau)\psi(\tau)\}d\tau \right\}, \tag{14}
\]

where \( h(t) \) is a known function as

\[
h(t) = \int_0^t \frac{g_1(\tau) - g_2(\tau) - Q(\tau) + E(\tau)}{E(\tau)} d\tau.
\]

Thus, by solving the system of integral equations (12)–(14), we obtain \( \{\varphi, \psi, r\} \), then, if we replace them in (11), we will have the solution \( v(x, t) \). Therefore, in the remaining part of this paper we try to solve the system of Eqs. (12)–(14), by using the Sinc-collocation method.

### 3. Sinc function

Sinc function properties have been discussed thoroughly in [14]. In this section, an overview of the basic formulation of the Sinc function is presented, and we describe the Sinc-collocation procedure for convolution integrals needed for our subsequent development.

#### 3.1. Sinc interpolation

The Sinc function is defined on the whole real line by

\[
\text{Sinc}(x) = \begin{cases} 
\frac{\sin(\pi x)}{\pi x}, & x \neq 0 \\
1, & x = 0.
\end{cases}
\]

For positive \( h \), the translated Sinc functions with evenly spaced nodes are given by

\[
S(k, h)(x) = \text{Sinc} \left( \frac{x - kh}{h} \right),
\]

where \( k \) is an integer and are called the \( k \)th Sinc functions.

The Sinc functions form an interpolatory set of functions, i.e.,

\[
S(j, h)(kh) = \delta_{jk} = \begin{cases} 
1, & k = j \\
0, & k \neq j.
\end{cases}
\]
Let

\[ \delta_{jk}^{(-1)} = \frac{1}{Z} + \int_{0}^{j-k} \sin(\pi x) \frac{1}{\pi x} \, dx, \]  

(15)

then we define a matrix whose \((j, k)\)th entry is given by \(\delta_{jk}^{(-1)}\) as \(I^{(-1)} = [\delta_{jk}^{(-1)}]\).

For further explanation of the procedure, we consider the following definitions and theorems in [14].

**Definition 1.** Let \(D\) be a simply connected domain in the complex plane \((z = x + iy)\) having boundary \(\partial D\). Let \(a\) and \(b\) denote two distinct points of \(\partial D\) and \(\phi\) denote a conformal map of \(D\) onto \(D_d\), where \(D_d = \{\omega \in \mathbb{C} : |\text{Im}(\omega)| < d\}\), such that \(\phi(a) = -\infty\) and \(\phi(b) = \infty\). Let \(\mu = \phi^{-1}\) denote the inverse map. Let \(\Gamma\) be defined by \(\Gamma = \{z \in \mathbb{C} : 0 = H_d, x \in \mathbb{R}\}\) when either of \(\Gamma\) has singularities at one or both of the endpoints of its interval of definition, or in the case that \(\Gamma\) has singularities at one or both endpoints of \((a, b)\).

We assume that \(g \in H^1(D_f)\), (see [14]) and that \(f\) is analytic in a domain \(D_f\), with \(\phi_f: (0, c) \to \mathbb{R}\), \(c\) being an arbitrary number on the interval \([2(b - a), 0]\). Corresponding to a positive integer \(N\) we set \(m = 2N + 1\), and we determine \(h\) via the formula \(h = (\frac{\pi d}{an})^{\frac{1}{2}}\). Define square matrix \(A_m\) by

\[ A_m = \begin{bmatrix} \frac{1}{\phi'(z_{-N})}, & \ldots, & \frac{1}{\phi'(z_N)} \end{bmatrix}. \]

Throughout this paper, the Laplace transformation means the function \(F\) defined by

\[ F(s) = \int_{0}^{c} f(t) e^{-st} \, dt, \]

where \(c\) is as defined above, and we shall assume that the Laplace transformation exists for some \(c \in [2(b - a), \infty]\), for all \(s\) on the right half of the complex plane i.e., \(\Omega^+ = \{z \in \mathbb{C} : \text{Re}(z) > 0\}\).

Now, by above assumptions, we describe the approximation procedure for \(p\) in (16). If the nonsingular matrix \(X_m\), and complex numbers \(s_j\) are determined such that

\[ A_m = X_m \text{diag}[s_{-N}, \ldots, s_N] X_m^{-1}, \]

then, square matrix \(F(A_m)\) may be defined via the equation

\[ F(A_m) = X_m \text{diag}[F(s_{-N}), \ldots, F(s_N)] X_m^{-1}. \]
Now, define column vectors $G_m$ and $P_m$ by

\[ G_m = [g(z_{-N}), \ldots, g(z_N)]^T, \]
\[ P_m = [p_{-N}, \ldots, p_N]^T = F(A_m)G_m. \]

Then, the component $p_j$ of vector $P_m$ approximates the value $p(x)$ at the Sinc point $x = z_j$. Thus, the approximation of $p$ on $(a, b)$ takes the form

\[ p(x) \approx \sum_{j=-N}^{N} p_j \omega_j(x) = \{F(A_m)G_m\}^TW(x), \quad x \in (a, b), \quad (17) \]

where $W(x) = [\omega_{-N}(x), \ldots, \omega_N(x)]^T$, and $\{\omega_j\}$ is a Sinc basis as follows

\[ \lambda_j(x) = \frac{1}{1 + \rho(x)} - \sum_{j=-N+1}^{N} \frac{\lambda_j(x)}{1 + e^{\text{th}}}, \]
\[ \omega_{-N}(x) = \{1 + e^{-N\eta}\} \left\{ \frac{1}{1 + \rho(x)} - \sum_{j=-N+1}^{N} \frac{\lambda_j(x)}{1 + e^{\text{th}}} \right\}, \]
\[ \omega_N(x) = \{1 + e^{-N\eta}\} \left\{ \frac{\rho(x)}{1 + \rho(x)} - \sum_{j=-N}^{N-1} \frac{e^{\text{th}} \lambda_j(x)}{1 + e^{\text{th}}} \right\}. \]

Note that the functions $\omega_j$ defined above satisfy the relation $\omega_j(z_k) = \delta_{jk}$.

**Theorem 3.** Let $p$ be defined as (16) with the above assumptions, let $F(s) = O(s)$ as $s \rightarrow \infty$ in $\Omega^+$. Set

\[ P(r, \tau) = \int_{\tau}^{\pi} f(r + \tau - \eta)g(\eta)d\eta \]

and assume that $P(r, \cdot) \in M_a(D')$ [14], uniformly, for $r \in [0, b - a]$, and also that

\[ |P_r(r, \tau)| \leq c_4 \frac{[\rho(r)]^D \rho'(r)}{[1 + \rho(r)]^2\tau}, \]

for all $r \in [0, b - a]$, and for all $\tau \in D$, with $c_4$ a constant independent of $r$ and $\tau$. Let $h = (\frac{a}{\Omega})^{\frac{1}{2}}$, then there exists a constant $c_5$ which is independent of $N$ such that

\[ \|p - \{F(A_m)G_m\}^TW\| \leq c_5N^{\frac{1}{2}}e^{-(\pi daN)^{\frac{1}{2}}}. \]

**Proof ([14]).**

**4. The Sinc-collocation method**

Let $\Gamma = (0, T)$, and $\phi$ be a conformal map onto $D$. By exploiting Definition 1, we have

\[ D = \left\{ z \in \mathbb{C} : \arg \left( \frac{z}{r - z} \right) < d \leq \frac{\pi}{2} \right\}, \]
\[ \phi(z) = \ln \left( \frac{z}{r - z} \right), \]

and the Sinc grid points are

\[ x_k = \frac{Te^{kh}}{1 + e^{kh}}, \quad k = -N, \ldots, N. \]

Now we approximate the functions $\varphi$ and $\psi$ in Eqs. (12) and (13) by a linear combination of $\omega_j(t)$. We take

\[ \varphi(t) \approx \sum_{j=-N}^{N} \varphi_j \omega_j(t) = \Phi^TW(t), \quad t \in (0, T), \quad (18) \]
\[ \psi(t) \approx \sum_{j=-N}^{N} \psi_j \omega_j(t) = \Psi^TW(t), \quad t \in (0, T). \quad (19) \]
where

\[ \Phi = [\varphi_{-N}, \ldots, \varphi_N]^T, \quad \Psi = [\psi_{-N}, \ldots, \psi_N]^T, \]

such that the unknown coefficients \( \{\varphi_j\} \) and \( \{\psi_j\} \) must be determined. The Laplace transformation defined in 3.2 for \( \theta(0, t) \), with \( c = \infty \), can be determined as

\[ F_0(s) = \int_0^\infty \theta(0, t) e^{-st} dt = \sqrt{s} \left( \frac{1}{2} + \sum_{i=1}^\infty e^{-\frac{s^2}{2}} \right). \]

By using the numerical procedure in the previous section for convolution integrals we may write

\[ \int_0^t \theta(0, t - \tau) [r(\tau) g_1(\tau) - \beta(\tau) \varphi(\tau)] d\tau \approx \{F_0(A_m)(\bar{g}_1R - \bar{\beta} \Phi)\}^T W(t), \tag{20} \]

\[ \int_0^t \theta(0, t - \tau) [r(\tau) g_2(\tau) - \gamma(\tau) \psi(\tau)] d\tau \approx \{F_0(A_m)(\bar{g}_2R - \bar{\gamma} \Psi)\}^T W(t), \tag{21} \]

where \( \bar{u} \) is the diagonal matrix defined by \( \bar{u} = \text{diag}[u(x_{-N}), \ldots, u(x_N)] \). Similarly the Laplace transformations for \( \theta(-1, t) \) and \( \theta(1, t) \) are

\[ F_1(s) = \int_0^\infty \theta(1, t) e^{-st} dt = \int_0^\infty \theta(-1, t) e^{-st} dt, \]

and we find

\[ \int_0^t \theta(-1, t - \tau) [r(\tau) g_1(\tau) - \beta(\tau) \varphi(\tau)] d\tau \approx \{F_1(A_m)(\bar{g}_1R - \bar{\beta} \Phi)\}^T W(t), \tag{22} \]

\[ \int_0^t \theta(1, t - \tau) [r(\tau) g_2(\tau) - \gamma(\tau) \psi(\tau)] d\tau \approx \{F_1(A_m)(\bar{g}_2R - \bar{\gamma} \Psi)\}^T W(t). \tag{23} \]

Depending on \( q \), the Laplace transformations \( F_2 \) and \( F_3 \) can be found, then we have

\[ \int_0^t \int_0^1 G(0, t, \zeta, \tau) r(\tau) q(\zeta, \tau) d\zeta d\tau \approx \{F_2(A_m)\bar{q}R\}^T W(t), \tag{24} \]

\[ \int_0^t \int_0^1 G(1, t, \zeta, \tau) r(\tau) q(\zeta, \tau) d\zeta d\tau \approx \{F_3(A_m)\bar{q}R\}^T W(t). \tag{25} \]

Substituting relations (18), (20), (22) and (24) in Eq. (12) we obtain

\[ \{\Phi - 2F_0(A_m)(\bar{\beta} \Phi - \bar{g}_1R) - 2F_1(A_m)(\bar{g}_2R - \bar{\gamma} \Psi) - F_2(A_m)\bar{q}R\}^T W(t) = w(0, t). \tag{26} \]

Also, by substituting relations (19), (21), (23) and (25) in Eq. (13), we get

\[ \{\Psi - 2F_1(A_m)(\bar{\beta} \Phi - \bar{g}_1R) - 2F_0(A_m)(\bar{g}_2R - \bar{\gamma} \Psi) - F_3(A_m)\bar{q}R\}^T W(t) = w(1, t). \tag{27} \]

Eqs. (26) and (27) are collocated at \( 2N + 1 \) points. For suitable collocation points we use the Sinc grid points \( x_k, k = -N, \ldots, N \).

Now, consider Eq. (14), in order to discretize it by using Sinc collocation, first of all, we approximate \( r(t) \) as

\[ r(t) \approx \sum_{j=-N}^N r_j \omega_j(t), \quad t \in (0, T). \tag{28} \]

Substituting (28) in Eq. (14) and evaluating the results at \( t = x_k \), and by using Theorems 2 and 3 we obtain

\[ r_k = e^{-h(x_k)} \left\{ 1 + h \sum_{j=-N}^N \frac{\delta_{kj}(-1)}{\phi(x_j)E(x_j)} \frac{e^{h(x_k)}}{\beta(x_j)\varphi_j - \gamma(x_j)\psi_j} \right\}. \tag{29} \]

Thus, by collocating Eqs. (26) and (27) at \( 2N + 1 \) points and by using (29), we have \( 3(2N + 1) \) linear algebraic equations which can be solved for the unknown coefficients \( \varphi_j, \psi_j \) and \( r_j, j = -N, \ldots, N \). This system in the matrix form is given by

\[ HX = Y, \tag{30} \]
where

\[
H = \begin{bmatrix}
H_{11} & H_{12} & H_{13} \\
H_{21} & H_{22} & H_{23} \\
H_{31} & H_{32} & H_{33}
\end{bmatrix},
\]

\[
H_{11} = I - 2F_0(A_m)\tilde{\beta}, \quad H_{12} = 2F_1(A_m)\tilde{\gamma}, \quad H_{13} = 2F_0(A_m)\tilde{g}_1 - 2F_1(A_m)\tilde{g}_2 - F_2(A_m)\tilde{q},
\]

\[
H_{21} = -2F_1(A_m)\tilde{\beta}, \quad H_{22} = I + 2F_0(A_m)\tilde{\gamma}, \quad H_{23} = 2F_1(A_m)\tilde{g}_1 - 2F_0(A_m)\tilde{g}_2 - F_3(A_m)\tilde{q},
\]

\[
H_{33} = -I
\]

and the square matrix \(H_{31}\) as \(H_{31} = [a_{ij}]\) and \(H_{32} = [b_{ij}]\) such that

\[
a_{ij} = e^{-h(x_i)}h_{ij}^{(-1)} \frac{e^{h(x_j)}}{\varphi_j(x_j)E_j(x_j)} \beta(x_i),
\]

\[
b_{ij} = -e^{-h(x_i)}h_{ij}^{(-1)} \frac{e^{h(x_j)}}{\varphi_j(x_j)E_j(x_j)} \gamma(x_i),
\]

and

\[
X = [\varphi_{-N}, \ldots, \varphi_N, \psi_{-N}, \ldots, \psi_N, r_{-N}, \ldots, r_N]^T,
\]

\[
Y = [w(0, x_{-N}), \ldots, w(0, x_N), w(1, x_{-N}), \ldots, w(1, x_N), -e^{-h(x_{-N})}, \ldots, -e^{-h(x_N)}]^T.
\]

By solving the linear system (30), we obtain approximate solutions \(\psi_j, \psi_j\) and \(r_j, j = -N, \ldots, N\). Then we employ a method similar to Nyström’s idea and write

\[
\psi_m(t) = w(0, t) + [2F_0(A_m)(\tilde{\beta}\Phi - \tilde{g}_1R) + 2F_1(A_m)(\tilde{g}_2R - \tilde{\gamma}\Psi) + F_2(A_m)\tilde{q}R]^TW(t),
\]

\[
\psi_m(t) = w(1, t) + [2F_0(A_m)(\tilde{\beta}\Phi - \tilde{g}_1R) + 2F_0(A_m)(\tilde{g}_2R - \tilde{\gamma}\Psi) + F_3(A_m)\tilde{q}R]^TW(t),
\]

\[
r_m(t) = e^{-h(t)}\left\{ 1 + \int_0^t e^{h(s)} \beta(s)\varphi_m(s) - \gamma(s)\psi_m(s) ds \right\},
\]

where \(m = 2N + 1\).

5. Convergence analysis

In this section, we discuss the convergence of the Sinc-collocation method introduced in the previous section.

**Theorem 4.** Suppose that \(\{\varphi, \psi, r\}\) is the exact solution of the system of integral equations (12)–(14), and let \(\{\psi_m, \psi_m, r_m\}\) be the approximate solution of the system (12)–(14) given by (31)–(33), then there exist some constants \(M_i, i = 1, 2, 3\), independent of \(N\) such that

\[
\sup_{t \in (0, T)} |\varphi(t) - \varphi_m(t)| \leq M_1\|H^{-1}\|N^\frac{1}{2} e^{-(\pi du)N}\frac{1}{2},
\]

\[
\sup_{t \in (0, T)} |\psi(t) - \psi_m(t)| \leq M_2\|H^{-1}\|N^\frac{1}{2} e^{-(\pi du)N}\frac{1}{2},
\]

\[
\sup_{t \in (0, T)} |r(t) - r_m(t)| \leq M_3\|H^{-1}\|N^\frac{1}{2} e^{-(\pi du)N}\frac{1}{2}.
\]

**Proof.** Define

\[
\tilde{X} = [\varphi(x_{-N}), \ldots, \varphi(x_N), \psi(x_{-N}), \ldots, \psi(x_N), r(x_{-N}), \ldots, r(x_N)]^T,
\]

\[
\tilde{\Phi} = [\varphi(x_{-N}), \ldots, \varphi(x_N)]^T,
\]

\[
\tilde{\Psi} = [\psi(x_{-N}), \ldots, \psi(x_N)]^T,
\]

\[
\tilde{R} = [r(x_{-N}), \ldots, r(x_N)]^T,
\]

\[
\tilde{\varphi}(t) = w(0, t) + [2F_0(A_m)(\tilde{\beta}\tilde{\Phi} - \tilde{g}_1\tilde{R}) + 2F_1(A_m)(\tilde{g}_2\tilde{R} - \tilde{\gamma}\tilde{\Psi}) + F_2(A_m)\tilde{q}\tilde{R}]^TW(t),
\]

and

\[
\tilde{\psi}(t) = w(1, t) + [2F_0(A_m)(\tilde{\beta}\tilde{\Phi} - \tilde{g}_1\tilde{R}) + 2F_0(A_m)(\tilde{g}_2\tilde{R} - \tilde{\gamma}\tilde{\Psi}) + F_3(A_m)\tilde{q}\tilde{R}]^TW(t).
\]

Then

\[
|\varphi(t) - \varphi_m(t)| \leq |\varphi(t) - \tilde{\varphi}(t)| + |\tilde{\varphi}(t) - \varphi_m(t)|.
\]
By Theorem 3, there exists a constant $c_0$ such that
\[
\sup_{t \in (0,T)} |\varphi(t) - \tilde{\varphi}_m(t)| = \sup_{t \in (0,T)} \left| -2 \int_0^t \theta(0, t - \tau) [r(\tau) g_1(\tau) - \beta(\tau) \varphi(\tau)] d\tau + 2 \int_0^t \theta(-1, t - \tau) [r(\tau) g_2(\tau) - \gamma(\tau) \psi(\tau)] d\tau + \int_0^t G(0, t, \zeta, \tau) r(\tau) q(\zeta, \tau) d\zeta d\tau - \{2F_0(A_m)(\tilde{\beta} \Phi - \tilde{g}_1 \tilde{R}) + 2F_1(A_m)(\tilde{g}_2 \tilde{R} - \tilde{\psi} \Psi) + \bar{F}_2(A_m) \tilde{\tilde{R}}^T W(t)\} \right| \leq c_0 N^\frac{1}{2} e^{-\frac{1}{2}(\pi d a N)} .
\] (35)

Also there exists a constant $c_1$ such that
\[
\sup_{t \in (0,T)} |\tilde{\varphi}_m(t) - \varphi_m(t)| = \sup_{t \in (0,T)} \left| 2F_0(A_m)(\tilde{\beta} \Phi - \tilde{g}_1 \tilde{R}) + 2F_1(A_m)(\tilde{g}_2 \tilde{R} - \tilde{\psi} \Psi) + \bar{F}_2(A_m) \tilde{\tilde{R}}^T W(t) \right| \leq c_1 \| \tilde{X} - X \| .
\] (36)

Since from (30) $X = H^{-1}Y$, we have
\[
\| \tilde{X} - X \| = \| \tilde{X} - H^{-1}Y \| \leq \| H^{-1} \| \| H \tilde{X} - Y \| .
\]

Using Theorem 1
\[
\| H \tilde{X} - Y \| \leq c_0 N^\frac{1}{2} e^{-\frac{1}{2}(\pi d a N)} ,
\]
where $c_0$ is a positive constant. Therefore we have
\[
\sup_{t \in (0,T)} |\varphi_m(t) - \tilde{\varphi}_m(t)| \leq c_0 \| H^{-1} \| N^\frac{1}{2} e^{-\frac{1}{2}(\pi d a N)} .
\] (36)

Finally, substituting (35) and (36) in (34), we obtain
\[
\sup_{t \in (0,T)} |\varphi(t) - \varphi_m(t)| \leq M_1 \| H^{-1} \| N^\frac{1}{2} e^{-\frac{1}{2}(\pi d a N)} ,
\] (37)
where $M_1$ is a constant independent of $N$. We apply a similar approach and obtain
\[
\sup_{t \in (0,T)} |\tilde{\varphi}_m(t) - \psi_m(t)| \leq M_2 \| H^{-1} \| N^\frac{1}{2} e^{-\frac{1}{2}(\pi d a N)} .
\] (38)

Using (37) and (38), the following estimate is valid
\[
\sup_{t \in (0,T)} |r(t) - r_m(t)| \leq M_3 \| H^{-1} \| N^\frac{1}{2} e^{-\frac{1}{2}(\pi d a N)} ,
\]
and the proof is completed. \(\square\)

We observe that $\| H^{-1} \|$ appears in the coefficient of the error terms in Theorem 4. In Fig. 1, we plot norms versus matrix dimension for different coefficient matrices $H$ that come from the test problem in the following section, and dimension $k = 1, \ldots, 200$. (In sinc applications, most examples are satisfactorily solved by matrices in this range.)

6. Numerical results

In this section, we illustrate the use of our algorithm by displaying the results obtained from its application to a test problem. In this example we take $\alpha = 1$ and $d = \frac{T}{2}$, and therefore $h = \frac{T}{\sqrt{2N}}$. 
Consider problem (1)–(5) with
\[
T = 1, \\
F(x) = 1 + \cos x, \\
g_1(t) = t^2 e^{(t^2 - \sin t)}[1 + e^{-t} \sin 1], \\
g_2(t) = e^{(t^2 - \sin t)}[t(1 + e^{-t} \cos 1) - e^{-t} \sin 1], \\
\alpha_i = 1, \quad i = 1, 2, \\
\beta(t) = t^2, \quad \gamma(t) = t, \\
q(x, t) = 0, \\
E(t) = e^{(t^2 - \sin t)}[1 + e^{-t} \sin 1].
\]
This problem has the exact solution \(u(x, t) = e^{(t^2 - \sin t)}(1 + e^{-t} \cos x)\) and \(p(t) = 2t - \cos t\). We report the absolute value of the errors of our method for \(N = 5, 10, 15, 20\) and 25 in Tables 1 and 2.
Table 2
Results for $u(x, t)$ when $t = 0.5$.

<table>
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<tr>
<th>$x$</th>
<th>$u(x, 0.5)$ exact</th>
<th>$N = 5$ error</th>
<th>$N = 10$ error</th>
<th>$N = 15$ error</th>
<th>$N = 20$ error</th>
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References