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Connecting the quenched and unquenched worlds via the large N_c world

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Abstract

In the large N_c (number of colors) limit, quenched QCD and QCD are identical. This implies that, in the effective field theory framework, some of the low energy constants in ($N_c = 3$) quenched QCD and QCD are the same up to higher-order corrections in the $1/N_c$ expansion. Thus the calculation of the non-leptonic kaon decays relevant for the $\Delta I = 1/2$ rule in the quenched approximation is expected to differ from the unquenched one by an $\mathcal{O}(1/N_c)$ correction. However, the calculation relevant to the CP-violation parameter ϵ'/ϵ would have a relatively big higher-order correction due to the large cancellation in the leading order. Some important weak matrix elements are poorly known that even constraints with 100% errors are interesting. In those cases, quenched calculations will be very useful.

Quantum chromodynamics (QCD) is the underlying theory of strong interaction. At high energies ($\gg 1$ GeV), the consequences of QCD can be studied in systematic, perturbative expansions. Good agreement with experiments is found in the electron–hadron deep inelastic scattering and hadron–hadron inelastic scattering (Drell–Yan process) [1]. At low energies ($\lesssim 1$ GeV), however, QCD becomes non-perturbative. Quantitative results not based on symmetry properties along can only be calculated directly from QCD via computer simulations on a Euclidean lattice.

In lattice QCD simulations, it is common to perform the quenched approximation which is to drop the internal quark loop contributions by setting the fermion determinant arising from the path integral to be one. This approximation cuts down the computing time tremendously and by construction, the results obtained are close to the unquenched ones when the quark masses are heavy ($\gg \Lambda_{\mathrm{QCD}} \sim 200$ MeV) so that the internal quark loops are suppressed. Thus the quenched approximation is an efficient and reliable tool to use as long as all the quark masses are heavy. However, in the real world, the light quark (u, d and s) masses are less than Λ_{QCD} so the justification for quenching does not apply. In this range of quark mass, quenching effects can be systematically studied by quenched chiral perturbation theory (QχPT) [2], an effective field theory of quenched QCD (QQCD). In QχPT, quenching could induce a double pole to the flavor singlet meson propagator and could make quenched quantities more singular than

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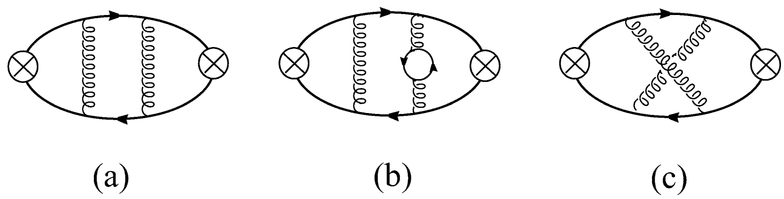

Fig. 1. Feynman diagrams contributing to the pion two point function. The solid lines and curly lines denote the quark and gluon propagators, respectively.

the unquenched ones [2] in the infrared region. In the ultraviolet region, QCD and QQCD are different as well because QQCD can be considered as having infinite quark masses for the internal loops so its ultraviolet behavior is different from that of QCD. Those uncontrolled effects make it hard to see what we can learn about QCD from QQCD.

However, there is a known connection between QCD and QQCD in the limit that the number of colors (N_c) becomes infinite; the large N_c limit of QQCD is indistinguishable to the large N_c limit of QCD. In this paper, we study the finite N_c quenching errors systematically using effective field theories with $1/N_c$ expansions. Generally, in the real world where $N_c = 3$, some of the contributions which are formally higher order in the $1/N_c$ expansion could become numerically important for certain observables near the chiral limit (zero quark mass limit). Fortunately, these big contributions are typically non-analytic in quark masses and can be computed in effective field theories of QCD and QQCD. After these big non-analytic contributions in QQCD are replaced by those in QCD, it is plausible that the errors which are formally $O(1/N_c)$ in the quenched approximation could imply numerically $\sim 30\%$ errors. Sometimes even smaller errors can be achieved by forming ratios of observables, such that the $O(1/N_c)$ errors from the leading terms in the chiral expansion cancel. The ratio of B meson decay constants f_{B_s}/f_B [3,4] is an explicit example. We then study the quenched calculations of the non-leptonic kaon decays $K \to \pi\pi$ [5,6]. We find that the quenching error in the part relevant to the $\Delta I = 1/2$ rule is consistent with $O(1/N_c) \sim 30\%$. However, the CP violation parameter ϵ'/ϵ calculation could have a big quenching error due to the large cancellation in the leading terms in the $1/N_c$ expansion. Finally, for illustration, we identify some cases that quenched calculations can be very useful. Those are calculations of important but poorly known weak matrix elements for which even 100% error constraints are interesting.

It is known that the quenched world has some similarity to the large N_c world. Take a pion two-point function $\langle 0|\pi(x)\pi(0)|0\rangle$, for example: there are contributions from diagrams shown in Fig. 1. For convenience, the quarks connected to the external sources are called "valence quarks"; the quarks not connected to the external sources are called "sea quarks". In the quenched approximation, the diagrams with sea quark loops such as the one in Fig. 1(b) are zero. (Then the gluon–quark coupling in surviving diagrams such as the ones in Fig. 1(a), (c) are usually rescaled to mimic the situation that the sea quark masses are set to infinity. We will not do this rescaling at this moment but leave it for later discussion.) In contrast, in the large N_c limit proposed by 't Hooft [7],

$$N_c \to \infty, \qquad g \to 0, \qquad N_c g^2 \to \text{constant}, \tag{1}$$

where g is the quark gluon coupling, both the Fig. 1(b) diagram with one sea quark loop and the "non-planar" diagram of Fig. 1(c) with one crossing in gluon lines are suppressed by one power of $1/N_c$ compared to that of Fig. 1(a). The diagrams with more sea quark loops or more gluon crossings are further suppressed by even higher powers of $1/N_c$. So, in the large N_c limit of QCD, only the planar diagrams such as Fig. 1(a) survive, while in QQCD both Fig. 1(a) and (c) survive. Now if we further take the large N_c limit of QQCD, the non-planar diagrams vanish as well. From this we conclude that QQCD and QCD have the same large N_c limit—at least in the mesonic two-point function case. It is easy to generalize the above observation to an arbitrary n-point function even involving baryons. To generalize the baryons to the large N_c case, we follow the prescription suggested by Dashen and Manohar [8] such that there is no valence strange quark in a proton.

The identity between the large N_c QCD and large N_c QQCD imposes some constraints on the low energy effective field theories of QCD and QQCD, which are the chiral perturbation theory (χPT) [9–12] and the quenched chiral perturbation theory (QχPT) [2], respectively. χPT and QχPT have the same symmetries and symmetry breaking patterns as QCD and QQCD. In both cases, low energy observables are expanded in a power series of the small expansion parameters $(p, M_{\mathrm{GB}})/\Lambda_\chi$, where p is the characteristic momentum transfer in the problem, M_{GB} is the Goldstone boson mass and Λ_χ is an induced chiral perturbation scale. In QCD, $\Lambda_\chi \sim 1$ GeV and the Goldstone boson masses are $M_\pi \sim 140$ MeV, $M_K \sim 500$ MeV and $M_\eta \sim 550$ MeV. In the QCD case ($N_c = 3$), χPT gives [10]

$$M_\pi^2 = 2m_u B\left\{1 + \frac{1}{(4\pi f_\pi)^2}\left[M_\pi^2 \log\left(\frac{M_\pi^2}{\mu^2}\right) - \frac{1}{3}M_\eta^2 \log\left(\frac{M_\eta^2}{\mu^2}\right)\right] + 2m_u K_3 + \frac{2m_u + m_s}{3}K_4\right\} + O\big(m_q^3\big), \tag{2}$$

where m_q is the mass of the flavor q $(= u, d, s)$ quark and we have used $m_d = m_u$. The pion decay constant $f_\pi = 93$ MeV and the renormalization scale μ dependence is absorbed by the low energy constants (or counterterms) K_3 and K_4 defined in [10]. When N_c becomes large, $f_\pi = \mathcal{O}(\sqrt{N_c})$, $K_4 = \mathcal{O}(1/N_c)$, while B, quark masses and K_3 are $\mathcal{O}(1)$. Thus as $N_c \to \infty$, only the B and K_3 terms contribute.

$$M_\pi^2 \to 2m_u B(1 + 2m_u K_3) + O\big(m_q^3\big). \tag{3}$$

Note that the μ dependence of K_3 is $\mathcal{O}(1/N_c)$, so K_3 becomes μ independent as $N_c \to \infty$. In the real world, the chiral logarithms could be numerically enhanced even though they are formally higher order in $1/N_c$.

In the QQCD ($N_c = 3$) case, QχPT gives a similar result for the pion mass [2]

$$M_\pi^{Q^2} = 2m_u B^Q\left\{1 - \frac{2}{3(4\pi f_\pi^Q)^2}M_0^2 \log\left(\frac{M_\pi^{Q^2}}{\mu^2}\right) + 2m_u K_3^Q\right\} + \cdots, \tag{4}$$

where the low energy constants B^Q, f_π^Q and K_3^Q have different values than those in QCD and where M_0^2 is the strength of the $\eta'\eta'$ coupling. In QCD, M_0^2 can be iterated to all orders to develop a shifted η' pole such that η' becomes non-degenerate with the Goldstone bosons in the chiral limit. However, in QQCD, all the sea quark loops are dropped such that the iteration stops at one insertion of M_0^2. This gives the η' propagator a double pole dependence and makes physical quantities in QQCD more singular than those in QCD in the infrared region [2]. The quenched pion mass in Eq. (4) is an explicit example. The quenched chiral logarithm (the M_0^2 term) is more singular than the QCD chiral logarithm in Eq. (2). As N_c becomes large, M_0^2 scales as $1/N_c$ while the other quenched low energy constants scale the same way as the corresponding unquenched ones, $f_\pi^Q = \mathcal{O}(\sqrt{N_c})$, B^Q and $K_3^Q = \mathcal{O}(1)$. Thus as $N_c \to \infty$,

$$M_\pi^{Q^2} \to 2m_u B^Q\big(1 + 2m_u K_3^Q\big) + O\big(m_q^3\big). \tag{5}$$

Since QCD and QQCD have the same large N_c limits, Eqs. (3) and (5) imply

$$B = B^Q, \qquad K_3 = K_3^Q \quad \text{as} \quad N_c \to \infty. \tag{6}$$

Thus some low energy constants (to be more precise, those that are leading in the $1/N_c$ expansion) of χPT and QχPT are identical in the large N_c limit. This implies their differences are higher order in $1/N_c$:

$$B^Q = B\left(1 + \mathcal{O}\left(\frac{1}{N_c}\right)\right),$$
$$K_3^Q = K_3\left(1 + \mathcal{O}\left(\frac{1}{N_c}\right)\right). \tag{7}$$

Now we can have a clear idea about how large the quenching error is. The error could come from the quenched logarithms which might be formally higher order in the $1/N_c$ expansion but numerically significant. Fortunately, those chiral logarithms can be calculated using effective field theories so that one can just replace the quenched logarithms by the real ones. Another source of error comes from the difference between the low energy constants which is an $\mathcal{O}(1/N_c) \sim 30\%$ effect. This is consistent with the $\sim 25\%$ quenching error seen in simulations (see [4], for example).

Recently kaon weak matrix elements relevant to the $K \to \pi\pi$ process have been simulated using the quenched approximation by two groups [5,6]. In both calculations, QχPT low energy constants are extracted from $K \to \pi$ and $K \to$ vacuum amplitudes in order to recover the desired $K \to \pi\pi$ amplitude. Based on the above discussion that leads to Eq. (7), the errors of these analyses are $\mathcal{O}(1/N_c)$ and naively are $\sim 30\%$. Indeed this is the case for the $\Delta I = 1/2$ rule. The simulations give 25.3 ± 1.8 [5] and 10 ± 4 [6] compared to the experimental value 22.2. However, for the case of CP violation ϵ'/ϵ parameter, Ref. [5] obtained $(-4.0 \pm 2.3) \times 10^{-4}$, consistent with Ref. [6], compared to the current experimental average of $(17.2 \pm 1.8) \times 10^{-4}$. In the ϵ'/ϵ case, a large accidental cancellation happens between the $I = 0$ and $I = 2$ contributions such that the estimated $\sim 30\%$ errors from the two pieces add up to $\sim 8 \times 10^{-4}$. Besides, in the ϵ'/ϵ quenched calculation, there is an additional quenched artifact that can be fixed by dropping some "eye diagrams" [13]. Doing this will increase the ϵ'/ϵ by $\sim 4 \times 10^{-4}$ [14]. Thus in this case the quenching error is significant.

The above examples suggest that in principle, the quenched approximation is typically expected to give a $\sim 30\%$ error result if there are no big cancellations between different contributions and if the quenched chiral logarithms (or non-analytic contributions, in general) are replaced by the unquenched ones. Now we will try to identify some matrix elements that can be reliably computed in the quenched approximation with 100% errors and are still worth computing even with that size of uncertainties.

The first quantity we look at is the proton strange magnetic moment μ_s. It is measured in the electron–proton and electron–deuteron parity violation experiments [15]. The measured value has a large uncertainty

$$\mu_s^{\exp} = \big[0.01 \pm 0.29(\text{stat}) \pm 0.31(\text{sys}) \pm 0.07(\text{theor.})\big] \text{ n.m.} \tag{8}$$

This implies that the strange quark only contributes around $-0.1 \pm 5.1\%$ of the proton's magnetic moment. In χPT, the first two orders in chiral expansion for μ_s is [16]

$$\mu_s = \mu_s^0 + \frac{M_N M_K}{24\pi f_\pi^2}\big(5D^2 - 6DF + 9F^2\big), \tag{9}$$

where D and F are Goldstone boson–nucleon couplings. Using the standard values for the parameters, the second term which is the kaon cloud contribution is ~ 2.0 n.m. This combined with the experimental data implies the leading order low energy constant μ_s^0 is ~ -2.0 n.m. Thus a quenched calculation would have a quenching error of ~ 0.8 n.m. Again, the large cancellation makes the quenching error relatively big and, hence, the result is not reliable within a 100% error.

The second quantity that is also interesting but even less well measured is the longest range parity violating (PV) isovector pion nucleon coupling, h_π. For many years, serious attempts have been made to measure this quantity in many-body systems where the PV effects are enhanced. However, currently even its order of magnitude is still not well constrained (see [17] for reviews) largely due to the theoretical uncertainties. Few-nucleon experiments, such as the new $\vec{n}p \to d\gamma$ experiment at LANSCE [18], are expected to have small theoretical uncertainties and will set a tight constraint on h_π. Also, recent studies in single nucleon systems in Compton scattering [19] and pion productions [20] show that the near threshold $\vec{\gamma}p \to \pi^+ n$ process is both a theoretically clean and experimentally feasible way to extract h_π [20]. Given the intensive interests and the level of difficulty involved in the experiments, a lattice QCD determination of h_π with a 100% error will be very valuable. In fact, issues such as chiral extrapolation has been addressed in [21] in the context of partially quenched approximations [22] in which sea and valence quark masses are different. Since one does not expect large cancellations between the first few orders—as what happened in the μ_s case—we suggest that a quenched simulation of h_π is enough to give a 100% error result.

Another interesting PV quantity is the longest range PV photon–nucleon–delta ($\gamma N \Delta$) coupling. Zhu et al. suggested a model, which was inspired by the hyperon decays, to argue that there could be enhanced PV mixing in the initial and final state wave functions such that the PV $\gamma N \Delta$ coupling is 25–100 times larger than its size from the naive dimensional analysis [23]. If correct, this enhancement will introduce an easily measurable PV signal in $\vec{\gamma}p \to \pi^+ n$ at the delta peak. To check if the large PV enhancement due to the wave function mixing really takes place, one can compute the PV mixing between $\Delta(\frac{3}{2}^+)$ and $N(1520; \frac{3}{2}^-)$ and between the nucleon $(\frac{1}{2}^+)$ and $N(1535; \frac{1}{2}^-)$ in QQCD. One expects the mixing between these states to be 5–20 times larger than their sizes from the naive dimensional analysis according to the model of Zhu et al. This can be checked easily by a quenched calculation with a 100% error.

Recently, Young et al. [24] studied the lattice results of nucleon and delta masses. They suggested the difference between the quenched and unquenched cases is largely due to the one-loop self-energy contributions calculated in a certain cut-off scheme, while the difference due to the low energy constants seems to be smaller than a generic estimation of $\sim 30\%$. This might be because in the real simulations, only the mass ratios are computed. The absolute mass scale is set by assuming zero quenching error for a certain quantity, then the coupling constant and, equivalently, the lattice spacing, is determined accordingly. It is conceivable that the ratio of certain quantities can be computed more reliably because the leading $\mathcal{O}(1/N_c)$ errors cancel. An explicit example is the ratio of B meson decay constants f_{B_s}/f_B, where $B_s(B)$ denotes a low-lying pseudoscalar meson with the same quantum number as a $\bar{b}s(\bar{b}d)$ state [3]. However, to argue that all the mass ratios have smaller than $\mathcal{O}(1/N_c)$ quenching errors is beyond the scape of effective field theory. To see if certain error suppression really takes place, it would be very interesting to study the N_c dependence for quenched approximations by direct simulations.

There is another class of ratios in large N_c QCD that the finite N_c correction is $\mathcal{O}(1/N_c^2)$ to the leading order. It would be interesting to see whether the correction is still $\mathcal{O}(1/N_c^2)$ in QQCD.

In summary, quenched QCD and QQCD have the same large N_c limit, such that some of the low energy constants in their corresponding effective field theories are identical in the large N_c limit. This implies that some physical quantities can be determined in the quenched approximation with $\mathcal{O}(1/N_c) \sim 30\%$ accuracy. A few interesting applications are explored.

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