Acute triangulations of flat tori

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\textbf{ABSTRACT}

In this paper, we investigate the acute triangulations of the family of flat tori. We prove that every flat torus can be triangulated into at most 16 acute triangles.

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1. Introduction

A triangulation of a two-dimensional space means a collection of (full) triangles covering the space, such that the intersection of any two triangles is either empty or consists of a vertex or of an edge. A triangle is called geodesic if all its edges are segments, i.e., shortest paths between the corresponding vertices. We are interested only in geodesic triangulations, all the members of which are, by definition, geodesic triangles. An acute triangulation is a triangulation whose triangles have all their angles less than $\frac{\pi}{2}$. The number of triangles in a triangulation is called size.

The discussion of acute triangulations has one of its origins in a problem of Stover reported in 1960 by Gardner in his Mathematical Games section of the \textit{Scientific American} (see [3–5]). There the question was raised of whether a triangle with one obtuse angle can be cut into smaller triangles, all of them acute. In the same year, independently, Burago and Zalgaller [1] investigated in considerable depth acute triangulations of polygonal complexes, being led to them by the problem of their isometric embedding into $\mathbb{R}^3$. In 1980, Cassidy and Lord [2] considered acute triangulations of the square. Recently, Maehara investigated acute triangulations of quadrilaterals [9] and other polygons [10], and a result for the latter was improved by Yuan [12].

On the other hand, compact convex surfaces have also been triangulated. Acute triangulations of all Platonic surfaces, which are surfaces of the five well-known Platonic solids, were investigated in [6–8].

The following problem first raised in [6] is natural, and not easy.

\textbf{Problem 1.} Does there exist a number $N$ such that every compact convex surface in $\mathbb{R}^3$ admits an acute triangulation with at most $N$ triangles?
As remarked in [8], Problem 1 can be transferred to other families of Alexandrov surfaces, with or without boundary.

Besides the platonic surfaces, other surfaces homeomorphic to the sphere have also been acutely triangulated. Zamfirescu considered acute triangulations of the doubly covered triangles [15]. Subsequently, acute triangulations of the doubly covered quadrilaterals [14] and pentagons [11] have also been investigated. We remark that Problem 1 can be quite difficult even for small classes of surfaces like, for example, the family of all tetrahedral surfaces, for which it is still open.

In this paper, we consider the acute triangulations of flat tori. Among flat surfaces, besides planar polygons, only the case of flat Möbius strips [13] has been solved completely. It was proved that a flat Möbius strip can always be triangulated into at most nine acute triangles, and sometimes (but not always) that many triangles are really needed. It is well known that any triangulation of the torus contains at least 14 triangles. As regards acute triangulations, we prove that every flat torus admits an acute triangulation with size at most 16. Furthermore, we found out that often 14 triangles do indeed suffice! In Section 3, we present an example of a flat torus admitting an acute triangulation of size 14.

2. Main result

**Theorem.** Every flat torus can be triangulated into 16 acute triangles.

**Proof.** We refer to the standard planar representation of the flat torus. There are two cases to consider:

Case 1. There is a rectangle $A_1A_2A_3A_4$, where $A_1, A_2, A_3$ and $A_4$ are identical on $T$. We may assume that $|A_1A_2| \leq |A_2A_3|$.

Let $B_1, B_2, D_1, D_2$ be the mid-points of $A_1A_2, A_2A_4, A_4A_1, A_1A_3$ respectively. Keep in mind that $B_i$ (resp. $D_i$) $(i = 1, 2)$ are identical on $T$. Let $A_1B_2 \cap OD_1 = F$, as shown in Fig. 1. Let $G \in B_1O$ satisfy $GF \perp A_1B_2$. Let $B_2FGE$ be a rectangle and let $C_1, C_2, E_1, E_2$ be the orthogonal projections of $H$ on $A_1A_2, A_3A_4, A_1A_4, A_2A_3$ respectively. Keep in mind again that $C_i$ (resp. $E_i$) $(i = 1, 2)$ are identical on $T$. Noticing that $\angle D_2C_1A_2 > \angle GC_1B_1 > \frac{\pi}{4} \Rightarrow \angle GC_1D_2 < \frac{\pi}{2}$, $\angle GD_2O < \angle GFO = \angle FA_1D_1 < \angle FE_1D_1 = \angle HD_2E_2 \Rightarrow \angle GD_2H < \frac{\pi}{2}$ and $\angle FB_2E_1 < \frac{\pi}{2}$, we see that $\triangle C_1D_2G \cong \triangle HGD_2 \cong \triangle FB_2E_1$ are acute. It is easy to check that all the edges are segments, i.e., shortest paths between the corresponding vertices. Thus $\mathcal{T}$ admits a non-obtuse triangulation with size 16 as shown in Fig. 1.

Now, in six steps, we successively move vertices in order to transform right angles into acute ones. Once an angle is acute, the next steps will be performed so gently that the angle remains acute.

Step 1: Slide $D_2$ in direction $D_2O$ such that $FD_1A_1, E_1D_1F, C_1A_2D_2$ and $D_2E_2H$ become acute.

Step 2: Slide $E_2$ in direction $E_2A_3$ such that $C_2HE_2$ becomes acute.

Step 3: Slide $H$ in direction $HG$ such that $HC_2B_2$ and $HB_2F$ become acute.

Step 4: Slide $B_2$ in direction $B_2O$ such that $A_1B_1G, GB_1C_1$ and $B_2A_4E_1$ become acute.
Step 5: Slide $G$ in direction $\overrightarrow{HG}$ (the sliding distance is much less than that of $H$) such that $GFA_1$ and $FGH$ become acute.

Step 6: Slide $A_3$ away from $C_2$ in direction $\overrightarrow{C_2A_3}$ such that $E_2A_3C_2$ becomes acute.

Thus we obtain an acute triangulation of $T$ with size 16.

Case 2. There is a parallelogram $A_1A_2A_3A_4$ with center $O$ and $\angle A_1A_2A_3 \neq \pi/2$, where $A_1$, $A_2$, $A_3$ and $A_4$ are identical on $T$. We may assume that $|A_1A_2| \geq |A_2A_3|$.

Now we construct a hexagon on the basis of $A_1A_2A_3A_4$ as shown in Fig. 2, where $OF_1 \perp B_1C_1$ ($F_1$ is the mid-point of $A_1A_2$), $OA_1 \perp B_1C_1$, $OE_1 \perp C_3B_3$ ($E_1$ is the mid-point of $A_1A_4$), and $B_i$, $C_i$ ($i = 1, 2, 3$), $E_j$, $F_j$ ($j = 1, 2$) are identical on $T$. Clearly the hexagon $B_1C_1B_2C_2B_3C_3$ is an unfolding figure of $T$ on the plane.

If $|B_1C_3| \geq |B_2C_3|$, then we choose a point $M$ on the bisector of angle $A_1OF_2$, which is very close to $O$. Let $N$, $G$ be the orthogonal projections of $M$ on $OA_1$, $OF_2$ respectively. Let $H$ satisfy $NH \parallel MG$ and $HG \parallel NM$. Then $NHGM$ is a rhombus with acute angle $\angle NGM$ and $H$ lies on the bisector of the angle $A_3OF_1$, which means that $HGA_3$ is an acute triangle. Thus we obtain a non-obtuse triangulation of $T$ with size 16, as shown in Fig. 2. Clearly, if we slightly slide $N$ in direction $\overrightarrow{MN}$ and $G$ in direction $\overrightarrow{MG}$, then all triangles become acute.

If $|B_1C_3| < |B_2C_3|$, we can construct an acute triangulation similarly. □

3. Do even 14 triangles suffice?

Since any triangulation of a flat torus contains at least 14 triangles, the following natural problem arises: Does any flat torus admit an acute triangulation of size 14? Unfortunately, we could not settle this problem in all cases. We close our paper by presenting an example of a flat torus admitting an acute triangulation of size 14.

We consider the case when, unfolding the torus $T$ on the plane, we obtain a square $A_1A_2A_3A_4$, where $A_1$, $A_2$, $A_3$ and $A_4$ are identical on $T$. Let $M_1$, $M_2$, $N_1$, $N_2$ be the mid-points of $A_1A_2$, $A_2A_4$, $A_1A_4$, $A_2A_3$ respectively. Keep in mind that $M_i$ (resp. $N_i$) ($i = 1, 2$) are identical on $T$. Let $B_i$ (resp. $E_i$, $C_i$, $D_i$) be the orthogonal projection of $A_i$ (resp. $A_2$, $A_3$, $A_4$) on the line $A_2N_1$ (resp. $A_4M_1$, $A_1N_2$, $A_2M_2$), and $B_2$, $E_2$, $C_1$, $D_1$ be labeled as shown in Fig. 3. Keep in mind again that $B_i$ (resp. $E_i$, $C_i$, $D_i$) ($i = 1, 2$) are identical on $T$. Furthermore, choose $F \in D_1B_2$, $G \in C_1E_2$ such that $|D_1F| = \frac{1}{2}|D_1B_2|$, $|GE_2| = \frac{1}{2}|C_1E_2|$. Then it is not difficult to check that we obtain a non-obtuse geodesic triangulation of $T$ with size 14, as shown in Fig. 3, where only $E_1FA_2$, $C_1FG$, $FGB_2$ and $A_2GD_2$ are acute triangles. Now first we slightly slide $E_2$ in direction $\overrightarrow{E_2A_3}$, such that $B_2GE_2$, $D_2GE_2$, $B_2C_2E_2$, and $D_1E_1F$ become acute triangles. Then we slightly slide $D_1$ in direction $\overrightarrow{D_1A_1}$ such that $D_1FC_1$ and $B_1D_1C_1$ become acute triangles. Thereafter we
slightly slide $B_2$ in direction $B_2C_2$ such that $FA_2B_2$ and $A_1D_1B_1$ become acute triangles. Finally we slightly slide $C_1$ in direction $C_1B_1$ such that $C_1GA_4$ and $E_2C_2A_3$ become acute triangles. Thus we obtain an acute triangulation of the torus $T$, whose size is 14.

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