Some properties of semiconcave functions with general modulus

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Received 13 June 2001

Submitted by H. Frankowska

Abstract

The class of semiconcave functions represents a useful generalization of the one of concave functions. Such an extension can be achieved requiring that a function satisfies a suitable one-sided estimate. In this paper, the structure of the set of points at which a semiconcave function fails to be differentiable—the singular set—is studied. First, we prove some results on the existence of arcs contained on the singular set. Then, we show how these abstract results apply to semiconcave solutions of Hamilton–Jacobi equations.

Keywords: Semiconcave functions; Singularities; Viscosity solutions; Hamilton–Jacobi equations

0. Introduction

Semiconcave functions naturally arise in optimal control theory and in the theory of Hamilton–Jacobi equations. It is well known that, in general, the value function associated to an optimal control problem fails to be differentiable. Moreover, for several optimal control problems, such a function can be characterized as the unique weak solution (for instance, in the viscosity sense) of a suitable Hamilton–Jacobi–Bellman equation (see, e.g., [8,14,16]).
Similarly, even for smooth data, Hamilton–Jacobi equations have, in general, no global smooth solutions.

On the other hand, if the data of the optimal control problem (or the data of a general Hamilton–Jacobi equation) are smooth one expects the value function (respectively, a weak solution) to be semiconcave (see, e.g., [9,12–15]). In some sense, semiconcavity represents an useful intermediate regularity between Lipschitz continuity and differentiability.

Loosely speaking, the points of nondifferentiability of the value function of an optimal control problem are related with the points that are starting points of multiple optimal trajectories. This fact can be easily understood thinking to the value function for the simplest minimum time optimal control problem: the Euclidean distance function $d_S$ from a closed nonempty set $S$. One can prove that $d_S(x)$ is semiconcave for $x \notin S$. Moreover, it is well known that a point $x \notin S$ is a point of nondifferentiability for the distance function if and only if $x$ possesses a not unique projection onto $S$. In this case an optimal trajectory starting from a given point, say $x$, is a line segment joining $x$ with a point $y$ which is the projection of $x$ onto $S$.

The main object of the present paper is the set, $\Sigma(u)$, of the points of non-differentiability (or the singular points) of a semiconcave function $u$ defined on an open set of $\mathbb{R}^n$.

Some information on the structure of $\Sigma(u)$ is available. In fact, one can show that $\Sigma(u)$ can be covered by countably many Lipschitz hypersurfaces of dimension $n - 1$ (see, e.g., [2,6,17,18]). Such a property can be considered as upper bounds for the dimension of the singular set. To provide a more complete description of $\Sigma(u)$ one also need to find lower bounds; i.e., an analysis of the local structure of the singular set is required. For this purpose, we introduce a very weak concept of propagation of singularities as follows. We will say that a singularity for $u$ at $x_0$ propagates if there exists a nonconstant map $x : [0, \sigma) \to \Sigma(u)$ continuous at 0, with $x(0) = x_0$. Propagation of singularities was first studied in [11] for semiconcave solutions to Hamilton–Jacobi–Bellman equations with semiconcavity modulus $\omega(r) = Cr^\alpha$, for some $\alpha \in ]0, 1]$, and in [7] for semiconcave functions with general modulus. Then, in [2] and [3] some stronger results on the singular set have been obtained in the more restrictive class of semiconcave functions with modulus $\omega(r) = Cr$. Finally, [1,4,5] are devoted to propagation of singularities for solutions to Hamilton–Jacobi equations (also in this case with a semiconcavity modulus of the form $\omega(r) = Cr$).

More precisely, in [11], singularities were shown to propagate along a sequence of points. In [7], conditions were given to derive estimates for the Hausdorff dimension of the singular set in a neighborhood of a point $x_0 \in \Sigma(u)$.

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1 See Definition 2.1 in Section 2.
Such conditions were expressed in terms of the superdifferential of $u$ at $x_0$, $D^+ u(x_0)$. We point out that in [7] the assumption $\dim D^+ u(x_0) < n$ is required.

In [2] and [3], under a suitable topological condition on $D^+ u(x_0)$ that implies no restriction on the dimension of this set, it is proved that there exists a positive number $\nu$ (related to the geometry of $D^+ u(x_0)$) such that $\Sigma(u)$ contains the Lipschitz image of a $\nu$-dimensional convex set, and that such an image has positive $\nu$-dimensional Hausdorff density at $x_0$. In particular, this result gives propagation of singularities along Lipschitz arcs. Finally, in [1,3–5] the above (abstract) results have been applied to the analysis of propagation of singularities to several Hamilton–Jacobi equations such as the Eikonal equation (in [3,5]), the Hamilton–Jacobi equation associated to a problem in calculus of variations (in [4, 5]) and to a Mayer optimal control problem (in [1,2]).

In this paper, we show that a suitable adaptation of the methods introduced in [3] to treat semiconcave functions with linear modulus can be used to derive some new results on the singular set for semiconcave functions with a general modulus. More precisely, the same geometrical assumption introduced in [3] ensures that the singularity (of a semiconcave function with general modulus) at $x_0$ propagates in the sense described above. Basically, this result improves the one given in [7] since also the case $\dim D^+ u(x_0) = n$ is admitted.

In the particular case of semiconcave functions with semiconcavity modulus $\omega(r) = Cr^\alpha$, for some $\alpha \in [0, 1]$, one can give a result on propagation of singularities along sets described by maps differentiable at the “starting” point $x_0$.

The above abstract results immediately apply to the singular set for a semiconcave solutions to first-order nonlinear partial differential equations. In particular, one recover the result given in [11] plus an additional information on the right derivative of the singular arc. An outline of the paper follows. Section 1 contains the main notation used in the sequel. In Section 2, we recall the basic properties of semiconcave functions. The results on the propagation of singularities are given in Sections 3 and 4. Finally, applications to Hamilton–Jacobi equations are discussed in Section 5.

1. Notation

The symbol $\mathbb{R}^+$ stands for the set $\{r \in \mathbb{R}: r \geq 0\}$. Let $n$ be a positive integer. We denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the Euclidean scalar product and norm in $\mathbb{R}^n$. For any $R > 0$ and $x_0 \in \mathbb{R}^n$ we set

$$B_R(x_0) = \{x \in \mathbb{R}^n: |x - x_0| < R\}$$

and we abbreviate $B_R = B_R(0)$. We denote by $\overline{B_R(x_0)}$ the closure of $B_R(x_0)$.

Let $A$ be a subset of $\mathbb{R}^n$. We use the notation $\text{diam}(A)$ for the diameter of $A$. We write

$$A \ni x \to x_0$$
to mean that \( x \in A \) and \( x \to x_0 \). Moreover,
\[
T(A, x) := \left\{ r \theta : r \geq 0, \, \theta = \lim_{k \to \infty} \frac{x_k - x}{|x_k - x|}, \, A \setminus \{x\} \ni x_k \to x \right\}
\]
is the so-called contingent cone to \( A \) at \( x \).

If \( A \) is convex, we denote by \( N_A(x) \) the normal cone to \( A \) at \( x \); that is,
\[
N_A(x) = \left\{ q \in \mathbb{R}^n : \langle q, y - x \rangle \leq 0, \, y \in A \right\}, \quad \forall x \in A.
\]

For any real number \( \nu \in [0, n] \), the \( \nu \)-dimensional Hausdorff measure of \( A \) is defined as
\[
H_\nu(A) := \frac{\alpha_\nu}{2^\nu} \sup_{\delta > 0} \inf \left\{ \sum_{j=0}^{\infty} (\text{diam}(A_j))^\nu : A \subset \bigcup_{j=0}^{\infty} A_j, \, \text{diam}(A_j) < \delta \right\},
\]
where \( \alpha_\nu = \sqrt{\pi^\nu / \Gamma(\nu/2 + 1)} \) and \( \Gamma(t) = \int_0^{+\infty} e^{-st} s^{t-1} \, ds \) is the Euler’s function. Moreover, the (Hausdorff) dimension of \( A \) is defined as
\[
\text{dim} A = \inf \{ \nu > 0 : H_\nu(A) = 0 \}.
\]
If \( A \) is convex, then \( \text{dim} A \) coincides with the dimension of the smaller affine hyperplane containing \( A \).

Finally, the symbol \( Du \) stands for the gradient of \( u \) while \( D_x u \) denotes the gradient of \( u \) w.r.t. the variables \( x \).

### 2. Preliminaries and definitions

Let \( n \geq 1 \) and let \( A \) be a subset of \( \mathbb{R}^n \).

**Definition 2.1.** A function \( u : A \to \mathbb{R} \) is said to be semiconcave if there exists a nondecreasing upper semicontinuous function \( \omega : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( \lim_{r \to 0} \omega(r) = 0 \) and

\[
\lambda u(x) + (1 - \lambda) u(y) - u(\lambda x + (1 - \lambda)y) \leq \lambda(1 - \lambda)|x - y| \omega(|x - y|)
\]
for any \( x, y \in A \) such that \([x, y] \subset A\) and \( \forall \lambda \in [0, 1] \).

In the sequel, a function \( \omega \) satisfying the above properties will be called a semiconcavity modulus for \( u \) in \( A \).

**Remark 2.1.** An interesting class of semiconcave functions is the one with linear semiconcavity modulus in \( A \), i.e., \( \omega(r) = Cr \), for \( r \geq 0 \) and a suitable constant \( C > 0 \). One can prove the following structure result:

\[
u \text{ is semiconcave with linear modulus } \iff \quad u(\cdot) - C|\cdot|^2 \text{ satisfies (1) with } \omega \equiv 0
\]
for some $C \geq 0$. In other words, such a function $u$ can be decomposed as the sum of a “concave” function plus a smooth one. This property could suggest the following conjecture: a function $u : A \to \mathbb{R}$, with semiconcavity modulus $\omega(r) = Cr^\alpha$ ($\alpha \in ]0, 1[$), can be decomposed as the sum of a concave function plus a function of class $C^1$.

Unfortunately, the above conjecture is false (see, e.g., [10]). In particular, there is no hope to reduce the analysis of the structure of the set of the points of nondifferentiability for general semiconcave functions to the case of the concave ones.

For an open domain $\Omega \subset \mathbb{R}^n$, one can extend the above class of functions as follows.

**Definition 2.2.** We say that $u : \Omega \to \mathbb{R}$ is **locally semiconcave** in $\Omega$ if $u$ is semiconcave in every compact set $A \subset \subset \Omega$. We denote by $SC(\Omega)$ the class of all locally semiconcave functions defined in $\Omega$. Moreover, we will write $u \in SC^\alpha(\Omega)$ if, for some $\alpha \in ]0, 1[$, $u : \Omega \to \mathbb{R}$ satisfies (1) in every compact set $A \subset \subset \Omega$ with $\omega(r) = CAr^\alpha$, for some $CA > 0$.

The next result provides a kind of regularization of a semiconcavity modulus.

**Proposition 2.1.** Let $u \in SC(\Omega)$ and let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be a semiconcavity modulus for $u$ in $A \subset \subset \Omega$. Then, there exists a nondecreasing function $\omega_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

(i) $\omega(r) \leq \omega_1(r)$, for any $r \geq 0$;
(ii) $\lim_{r \to 0} \omega_1(r) = 0$;
(iii) $g(r) := r\omega_1(r) \in C^1([0, +\infty[)$ and satisfies $g'(0) = 0$.

**Proof.** It is easy to verify that the function

$$\omega_1(r) := \begin{cases} \frac{1}{r} \int_{2r}^{2^2 r} \frac{1}{s} \int_{s}^{2s} \max_{\sigma \in [0, r]} \omega(\sigma) \, dt \, ds & \text{if } r > 0, \\ 0 & \text{if } r = 0, \end{cases}$$

possesses all the required properties. $\square$

Locally semiconcave functions share many properties with concave functions. For instance, it is easy to show that any locally semiconcave function in $\Omega$ is locally Lipschitz continuous. This fact can be proved arguing as in [6]. Consequently, $u$ is differentiable a.e. in $\Omega$ by Rademacher’s theorem, and the gradient of $u$, $Du$, is locally bounded. Now, set

$$D^*u(x) := \left\{ \lim_{i \to \infty} Du(x_i) : \Omega \ni x_i \to x \right\} \quad (x \in \Omega).$$

In view of the above remarks we have that $D^*u(x) \neq \emptyset$ for any $x \in \Omega$. 
Just like the concave case, locally semiconcave functions possess a natural notion of generalized gradient, given by the superdifferential

\[ D^+ u(x) = \left\{ p \in \mathbb{R}^n : \limsup_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\}, \quad \forall x \in \Omega. \]

Actually, a similar generalization of the gradient is the subdifferential of \( u \) defined as

\[ D^- u(x) = \left\{ p \in \mathbb{R}^n : \liminf_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \geq 0 \right\}. \]

However, for a locally semiconcave function, the superdifferential is much more interesting than the subdifferential. Indeed, in view of the proposition below, \( D^+ u \) is nonempty at every point. Therefore, either \( D^- u \) is empty, or \( u \) turns out to be differentiable.

**Proposition 2.2.** Let \( u \in SC(\Omega) \) and let \( x \in \Omega \). Then \( D^+ u(x) \) is nonempty since

\[ D^+ u(x) = \text{co} \ D^* u(x), \quad (2) \]

where \( \text{co} \) denotes the convex hull. Moreover, \( p \in D^+ u(x) \) if and only if, for any \( y \) such that \([x, y] \subset \Omega\),

\[ u(y) - u(x) - \langle p, y - x \rangle \leq |y - x| \omega(|y - x|), \quad (3) \]

where \( \omega(\cdot) \) is a semiconcavity modulus for \( u \) in \( \Omega \). Furthermore,

\[ \langle p - q, y - x \rangle \leq 2|y - x| \omega(|y - x|) \]

for all \( x, y \in \Omega \) with \([x, y] \subset \Omega\) and all vectors \( p \in D^+ u(y) \) and \( q \in D^+ u(x) \).

For the proof of the above proposition the reader is referred to [10]. It is shown in [11] that \( D^* u(x) \) is contained in the (topological) boundary of \( D^+ u(x) \), i.e.,

\[ D^* u(x) \subset \partial D^+ u(x), \quad \forall x \in \Omega. \quad (5) \]

From (2) it follows that \( D^+ u(x) \) is a nonempty compact convex set and that

\[ \max\{|p| : p \in D^+ u(x)\} \leq L, \quad (6) \]

where \( L \) is any Lipschitz constant for \( u \) in a neighborhood of \( x \). Inequality (3) is also useful to check the validity of many calculus rules for the superdifferential, such as Fermat’s rule, that is \( 0 \in D^+ u(x) \) at any local maximum or minimum point \( x \) for \( u \), and the sum rule

\[ D^+ u(x) + D^+ v(x) \subset D^+ (u + v)(x), \quad \forall x \in \Omega. \]

Notice that the above inclusion reduces to an equality if at least one of the functions \( u, v \) is continuously differentiable at \( x \). Another easy consequence of (3) is the upper semicontinuity of \( D^+ u \) as a set-valued map, that is

\[ \Omega \ni x_i \to x \quad D^+ u(x_i) \ni p_i \to p \quad \Rightarrow \quad p \in D^+ u(x). \quad (7) \]
We complete this preliminary section with the following

**Definition 2.3.** Let \( u \in SC(\Omega) \). We denote by \( \Sigma(u) \) the set of all points \( x \in \Omega \) at which \( u \) fails to be differentiable. In other words, \( x \in \Sigma(u) \) if and only if \( D^+u(x) \) is not a singleton or, equivalently,

\[
\Sigma(u) := \{ x \in \Omega : \dim D^+u(x) \geq 1 \}.
\]

The points of \( \Sigma(u) \) are the *singular points*, or singularities, of \( u \).

3. Propagation of singularities in \( SC(\Omega) \)

Let \( u \) be a locally semiconcave function defined in an open domain \( \Omega \subset \mathbb{R}^n \) and let \( x_0 \) be a singular point for \( u \). As recalled in the previous section, we have that

\[
D^*u(x_0) \subset \partial D^+u(x_0).
\]

We want to show that if \( D^*u(x_0) \) is strictly contained in \( \partial D^+u(x_0) \) then there exists a nonconstant map, \( x : [0, \sigma] \to \Omega \), with \( x(0) = x_0 \), continuous at 0 and taking values in \( \Sigma(u) \). We start with an easy preliminary result.

**Lemma 3.1.** Let \( u \in SC(\Omega) \). Then,

\[
\partial D^+u(x_0) \setminus D^*u(x_0) \neq 0
\]

if and only if there exist two vectors, \( p_0 \in \mathbb{R}^n \) and \( \theta \in \mathbb{R}^n \setminus \{0\} \), such that

\[
p_0 \in D^+u(x_0) \setminus D^*u(x_0),
\]

\[
\langle \theta, p - p_0 \rangle \geq 0, \quad \forall p \in D^+u(x_0).
\]

**Proof.** It is clear that (9) and (10) imply that \( p_0 \in \partial D^+u(x_0) \setminus D^*u(x_0) \). Conversely, if (8) holds true, then taking \( p_0 \in \partial D^+u(x_0) \setminus D^*u(x_0) \) (9) is trivially satisfied and (10) follows choosing \(-\theta\) in the normal cone to the convex set \( D^+u(x_0) \) at \( p_0 \). This completes the proof. \( \square \)

**Theorem 3.1.** Let \( x_0 \in \Omega \) be a singular point of a function \( u \in SC(\Omega) \). Suppose that

\[
\partial D^+u(x_0) \setminus D^*u(x_0) \neq \emptyset
\]

and let \( p_0 \) and \( \theta \) as in Lemma 3.1. Then there exists a map \( x : [0, \sigma] \to \mathbb{R}^n \), with \( x(0) = x_0 \), such that

\[
\lim_{s \to 0^+} x(s) = x_0 \quad \text{and} \quad x(s) \neq x_0, \quad \forall s \in [0, \sigma],
\]

\[
x(s) \in \Sigma(u), \quad \forall s \in [0, \sigma],
\]

\[
\lim_{s \to 0^+} \frac{x(s) - x_0}{|x(s) - x_0|} = \frac{\theta}{|\theta|} \in -N_{D^+u(x_0)}(p_0).
\]
Proof. Let $R > 0$ be fixed and let $\rho \in [0, R]$ be such that

$$\omega(s) \leq 1, \quad \forall s \in [0, \rho].$$

(15)

Without loss of generality, in view of Proposition 2.1, we can suppose that there exists a semiconcavity modulus for $u$ in $\overline{B}_R(x_0)$, $\omega : \mathbb{R}^+ \to \mathbb{R}^+$, such that $r \mapsto g(r) := r\omega(r)$ belongs to $C^1([0, \infty[)$ and $g'(0) = 0$.

Fix $p_0 \in D^+ u(x_0)$ and $-\theta \in N_{D^+ u(x_0)}(p_0)$, with $|\theta| = 1$, as in Lemma 3.1. Let us define, for any $s \in [0, \rho)$,

$$\phi_s(x) = u(x) - u(x_0) - \langle p_0 - \theta, x - x_0 \rangle - \frac{2}{\omega(s)}|x - x_0|\omega(|x - x_0|),$$

$x \in \overline{B}_R(x_0)$.

Let us denote by $x_s$ a maximum point of $\phi_s$ in $\overline{B}_R(x_0)$ and set

$$x(s) = \begin{cases} x_0 & \text{if } s = 0, \\
 x_s & \text{if } s \in (0, \rho]. \end{cases}$$

Now, we proceed to show that $x$ possesses all the required properties. First, we claim that the arc $x$ possesses properties (12). By the characterization of $D^+ u$ given in (3), we have that

$$\phi_s(x) \leq \langle \theta, x - x_0 \rangle + \left(1 - \frac{2}{\omega(s)}\right)|x - x_0|\omega(|x - x_0|)$$

(16)

for any $x \in \overline{B}_R(x_0)$. Moreover, $p_0 - \theta \notin D^+ u(x_0)$ in view of condition (10). Since this fact implies that there are points in $\overline{B}_R(x_0)$ at which $\phi_s$ is positive, we conclude that $\phi_s(x(s)) > 0$. The last estimate implies that for $s \in [0, \rho]$,

$$0 < \langle \theta, x(s) - x_0 \rangle + \left(1 - \frac{2}{\omega(s)}\right)|x(s) - x_0|\omega(|x(s) - x_0|);$$

(17)

so, $x(s) \neq x_0$, for $s \in [0, \rho]$, and

$$\omega(|x(s) - x_0|) \leq \omega(s), \quad \forall s \in [0, \rho],$$

(18)

by (17) and (15). Now, (18) and the fact that $\omega$ is nondecreasing yield that

$$|x(s) - x_0| \leq s, \quad \forall s \in [0, \rho].$$

(19)

Hence, (12) follows.

Now, we show that, possibly taking a subinterval $[0, \sigma] \subset [0, \rho]$, property (13) holds. First, we observe that estimate (19) implies that $x(s) \in B_R(x_0)$, for any $s \in [0, \rho]$. Moreover, $x(s)$ is also a local maximum point of $\phi_s$. So, by the calculus rules for $D^+ u$ recalled in Section 2, for any $s \in [0, \rho]$, we have that

$$0 \in D^+ \phi_s(x(s)) = D^+ u(x(s)) - p_0 + \theta - 2\frac{x(s) - x_0}{|x(s) - x_0|}\omega(s)g'(|x(s) - x_0|).$$
The above computation shows that, for any \( s \in [0, \rho] \),
\[
D^+ u(x(s)) \ni p(s) := p_0 - \theta + 2 \frac{x(s) - x_0}{|x(s) - x_0| \omega(s)} g'(|x(s) - x_0|).
\]
We claim that (13) follows from
\[
\lim_{s \to 0} p(s) = p_0. \tag{20}
\]
Indeed, let us suppose that there exists a sequence \( s_k \downarrow 0 \) such that \( u \) is differentiable at \( x(s_k) \). Then, \( p(s_k) = Du(x(s_k)) \) and (20) should imply the contradiction \( p_0 \in D^* u(x_0) \).

Now, we show that (20) holds. Let \( \bar{p} = \lim_{k \to \infty} p(s_k) \) for some sequence \( s_k \downarrow 0 \). Then, taking the scalar product of both sides of the identity
\[
p(s_k) - p_0 + \theta = 2 \frac{x(s_k) - x_0}{|x(s_k) - x_0| \omega(s_k)} g'(|x(s_k) - x_0|),
\]
with \( p(s_k) - p_0 \) and recalling property (4), we obtain that
\[
|p(s_k) - p_0|^2 + \langle \theta, p(s_k) - p_0 \rangle
= 2 \frac{g'(|x(s_k) - x_0|)}{|x(s_k) - x_0| \omega(s_k)} \langle p(s_k) - p_0, x(s_k) - x_0 \rangle
\leq 4 \frac{g'(|x(s_k) - x_0| \omega(|x(s_k) - x_0|)}{\omega(s_k)} \leq 4 g'(|x(s_k) - x_0|). \tag{21}
\]
Observe that in the last estimate we used (18). Now, the last term in (21) tends to 0 as \( k \to \infty \), in view of (18) and by the definition of \( g \). Moreover, \( \bar{p} \in D^+ u(x_0) \) since \( D^+ u \) is upper semicontinuous; so, \( \langle \theta, \bar{p} - p_0 \rangle \geq 0 \) by assumption (10).
Therefore, (21) yields \( |\bar{p} - p_0|^2 \leq 0 \) in the limit as \( k \to \infty \). This proves that \( \bar{p} = p_0 \) as required.

To complete the proof we only need to check condition (14). Notice that Eq. (20) can be rewritten as
\[
\lim_{s \to 0} 2 \frac{x(s) - x_0}{|x(s) - x_0| \omega(s)} g'(|x(s) - x_0|) = \theta;
\]
so,
\[
\lim_{s \to 0} 2 \frac{|g'(|x(s) - x_0|)|}{\omega(s)} = |\theta|,
\]
and recalling that \( g' \geq 0 \) the conclusion follows. \( \square \)

It is immediate to see that the above theorem can be (partially) recast in a more geometrical way.

**Corollary 3.1.** Let \( u \in SC(\Omega) \) and let \( \partial D^+ u(x) \setminus D^* u(x) \neq \emptyset \), for some \( x \in \Omega \). Then, for every \( p \in \partial D^+ u(x) \setminus D^* u(x) \),
\[
-N_{D^+ u(x)}(p) \subset T(\Sigma(u), x).
\]
4. Propagation of singularities in $SC^\alpha(\Omega)$

In this section we restrict our attention to locally semiconcave functions of class $SC^\alpha(\Omega)$, for some $\alpha \in ]0, 1]$. We show that if $u \in SC^\alpha(\Omega)$, $x_0 \in \Sigma(u)$ and $\partial D^+u(x_0) \setminus D^*u(x_0) \neq \emptyset$ then we can find a subset of $\Sigma(u)$ (containing $x_0$) described by a map differentiable at $x_0$. More precisely, the following result holds.

**Theorem 4.1.** Let $x_0 \in \Omega$ be a singular point of a function $u \in SC^\alpha(\Omega)$, for some $\alpha \in ]0, 1]$. Suppose that

$\partial D^+u(x_0) \setminus D^*u(x_0) \neq \emptyset$

and, having fixed a point $p_0 \in \partial D^+u(x_0) \setminus D^*u(x_0)$, define

$\nu := \dim N_{D^+u(x_0)}(p_0)$.

Then a number $\sigma > 0$ and map $f : N_{D^+u(x_0)}(p_0) \cap B_\sigma \to \Sigma(u)$ exists such that

$f(\theta) = x_0 - \theta + |\theta| h(\theta)$ with

$h(\theta) \to 0$ as $N_{D^+u(x_0)} \cap B_\sigma \ni \theta \to 0$.

(22)

**Remark 4.1.** We observe that if $\alpha = 1$, a finer result can be proved. In fact, in this case, one can also show that $f$ is Lipschitz continuous and that

$$
\liminf_{r \to 0^+} r^{-\nu} \mu^\nu \left( f \left( N_{D^+u(x_0)}(p_0) \cap B_\sigma \right) \cap B_r(x_0) \right) > 0
$$

(see [3]).

**Proof of Theorem 4.1.** For the sake of brevity, let us set

$N := N_{D^+u(x_0)}(p_0)$.

Let $R > 0$ be fixed and let $\omega(r) = Cr^{-\alpha}$ a semiconcavity modulus for $u$ in $\overline{B}_R(x_0)$.

For any $\theta \in N \setminus \{0\}$, let us define

$$\phi_\theta(x) = u(x) - u(x_0) - \left( p_0 + \frac{\theta}{\theta}, x - x_0 \right) - \frac{1}{(1 + \alpha)|\theta|\alpha} |x - x_0|^{1+\alpha},$$

$x \in \overline{B}_R(x_0)$.

Let $x_\theta$ be such that

$$\phi_\theta(x_\theta) = \max_{x \in \overline{B}_R(x_0)} \phi_\theta(x),$$

and define

$$f(\theta) := \begin{cases} x_0 \text{ if } \theta = 0, \\ x_\theta \text{ if } \theta \in N \cap B_\rho \setminus \{0\}, \end{cases}$$
where
\[ \rho := \min \left\{ \frac{1}{(4C)^{1/\alpha}}, \frac{R}{41^{1/\alpha}} \right\}. \]

Arguing as in the proof of Theorem 3.1 we obtain that, for any \( \theta \in N \cap B_{\rho} \setminus \{0\} \),
\[ |f(\theta) - x_0| \leq 4|\theta| \quad (23) \]
and
\[ p(\theta) := p_0 + \frac{\theta}{|\theta|} + |f(\theta) - x_0|^{\alpha-1} \frac{f(\theta) - x_0}{|\theta|^\alpha} \in D^+ u(f(\theta)). \quad (24) \]

Now, we must show that
\[ p(\theta) \to p_0 \quad \text{as} \quad N \cap B_{\rho} \ni \theta \to 0. \quad (25) \]

For this purpose, let \( \{\theta_i\} \) be an arbitrary sequence in \( N \cap B_{\rho} \setminus \{0\} \) such that \( \theta_i \to 0 \).
Since \( p(\theta) \) is bounded, we can extract a subsequence (still termed \( \{\theta_i\} \)) such that
\[ \lim_{i \to \infty} p(\theta_i) \text{ exists and } \lim_{i \to \infty} \theta_i \frac{\theta_i}{|\theta_i|} = \bar{\theta}, \]
for some \( \bar{\theta} \in N \) satisfying \( |\bar{\theta}| = 1 \). We claim that \( \lim_{i \to \infty} p(\theta_i) = p_0 \), which in turn implies (25). Indeed, let us set
\[ \bar{p} := \lim_{i \to \infty} p(\theta_i) \]
and observe that \( \bar{p} \in D^+ u(x_0) \) as \( D^+ u \) is upper semicontinuous and \( f \) is continuous at 0. Taking the scalar product of both sides of the identity
\[ p(\theta_i) - p_0 - \frac{\theta_i}{|\theta_i|} = |f(\theta_i) - x_0|^{\alpha-1} \frac{f(\theta_i) - x_0}{|\theta_i|^\alpha} \]
with \( p(\theta_i) - p_0 \) and applying inequality (4), we deduce that
\[ |p(\theta_i) - p_0|^2 - \left( \frac{\theta_i}{|\theta_i|}, p(\theta_i) - p_0 \right) \]
\[ = \frac{|f(\theta_i) - x_0|^{\alpha-1}}{|\theta_i|^\alpha} \left( f(\theta_i) - x_0, p(\theta_i) - p_0 \right) \]
\[ \leq 2C \frac{|f(\theta_i) - x_0|^{2\alpha}}{|\theta_i|^\alpha} \leq 2C |\theta_i|^{2\alpha}, \]
where the last estimate follows from (23). In the limit as \( i \to \infty \), the above inequality yields
\[ |\bar{p} - p_0|^2 - \langle \bar{\theta}, \bar{p} - p_0 \rangle \leq 0. \]

Hence, recalling that \( \bar{\theta} \in N \), we conclude that \( \bar{p} = p_0 \). Our claim is thus proved.
Formula (22) is a direct consequence of the above computations. Indeed,

\[ p(\theta) - p_0 = \frac{\theta}{|\theta|} + |f(\theta) - x_0|^{\alpha-1} \frac{f(\theta) - x_0}{|\theta|^\alpha} \to 0, \]

as \( N \cap B_\rho \ni \theta \to 0 \), implies that

\[ \frac{\theta}{|\theta|} + \frac{1}{(1 + o(1))} \frac{f(\theta) - x_0}{|\theta|} \to 0, \]

for a suitable function \( o(1) \) converging to 0 as \( N \cap B_\rho \ni \theta \to 0 \). So, (22) follows with

\[ h(\theta) = (p(\theta) - p_0)(1 + o(1)) - \frac{\theta}{|\theta|} o(1). \]

Finally, arguing by contradiction—as in the proof of Theorem 3.1—it is easy to see that a suitable restriction of \( f \) to \( N \cap B_\sigma, 0 < \sigma \leq \rho \), satisfies

\[ f(\theta) \in \Sigma(u), \quad \theta \in N \cap B_\sigma. \]

This completes the proof. \( \square \)

In particular, the following result holds.

**Theorem 4.2.** Let \( x_0 \in \Omega \) be a singular point of a function \( u \in SC^\alpha(\Omega) \), for some \( \alpha \in ]0, 1]. \) Suppose that

\[ \partial D^+ u(x_0) \setminus D^* u(x_0) \neq \emptyset. \quad (26) \]

Let \( p_0 \in \partial D^+ u(x_0) \setminus D^* u(x_0) \) and fix

\[ \theta \in N_{D^+ u(x_0)}(p_0). \]

Then there exists a map \( x : [0, \sigma] \to \mathbb{R}^n \), with \( x(0) = x_0 \), such that

\[ \lim_{s \downarrow 0} x(s) = x_0 \quad \text{and} \quad x(s) \neq x_0, \quad \forall s \in ]0, \sigma[, \]

\[ x(s) \in \Sigma(u), \quad \forall s \in ]0, \sigma[, \]

\[ x(s) = x_0 - \theta s + o(s) \quad \text{with} \quad o(s)/s \to 0 \quad \text{as} \quad s \to 0. \]

5. Applications to Hamilton–Jacobi equations

In this section we show how the abstract results on the propagation of singularities for locally semiconcave functions apply to solutions of first-order nonlinear partial differential equations. For this purpose, let \( u : \Omega \to \mathbb{R} \) be a semiconcave solution of the equation

\[ F(x, u(x), Du(x)) = 0 \quad \text{a.e. in} \quad \Omega. \quad (27) \]
We need the following regularity assumptions on $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$:

$$\begin{align*}
F &= F(x, u, p) \text{ is a continuous function;} \\
F &= F(x, u, p) \text{ is convex and differentiable w.r.t. the variables } p; \\
\forall (x, u) \in \Omega \times \mathbb{R} \text{ the set } \{ p \in \mathbb{R}^n : F(x, u, p) = 0 \} \text{ contains no lines segment.}
\end{align*}$$

(28)

**Remark 5.1.** It is well known that under assumption (28) a semiconvex function $u$ satisfying (27) a.e. in $\Omega$ is also a viscosity solution of the same equation and vice versa.

We begin with a result connecting the geometry of the superdifferential of a semiconcave solution to (27) with the function $F$.

**Lemma 5.1.** Under assumption (28) let $u \in SC(\Omega)$ be a solution of Eq. (27) and let $x_0 \in \Sigma(u)$. Let us suppose that

$$D_p F(x_0, u(x_0), p_0) \neq 0$$

(29)

for some $p_0 \in D^+ u(x_0)$ such that

$$\min_{p \in D^+ u(x_0)} F(x_0, u(x_0), p) = F(x_0, u(x_0), p_0).$$

Then,

$$-D_p F(x_0, u(x_0), p_0) \in N_{D^+ u(x_0)}(p_0).$$

(30)

**Proof.** The definition of $p_0$ and the convexity of $D^+ u(x_0)$ imply that

$$\langle D_p F(x_0, u(x_0), p_0), p - p_0 \rangle \geq 0, \quad \forall p \in D^+ u(x_0).$$

So, the conclusion holds. $\square$

The following result is a straightforward consequence of Theorem 3.1 and Lemma 5.1.

**Theorem 5.1.** Under assumption (28) let $u \in SC(\Omega)$ be a solution of Eq. (27) and let $x_0 \in \Sigma(u)$. Let us suppose that

$$D_p F(x_0, u(x_0), p_0) \neq 0$$

(31)

for some $p_0 \in D^+ u(x_0)$ such that

$$\min_{p \in D^+ u(x_0)} F(x_0, u(x_0), p) = F(x_0, u(x_0), p_0).$$

Then, the singularity at $x_0$ propagates.
Remark 5.2. An interesting class of problem satisfying assumption (31) are the evolutional Hamilton–Jacobi of the form

\[ u_t(t, x) + H(t, x, u(t, x), D_xu(t, x)) = 0. \]

For such an equation, it is easy to see that part of the assumption (28) concerning the 0-level set of \( p \mapsto F(t, x, u, p) \) is trivially fulfilled if \( H \) above is strictly convex.

Proof of Theorem 5.1. Equation (27) and the continuity of \( F \) imply that

\[ F(x, u(x), p) = 0, \quad \forall x \in \Omega, \forall p \in D^*u(x). \quad (32) \]

On the other hand, recalling the convexity of \( F \) and property (2) we deduce that

\[ F(x, u(x), p) \leq 0, \quad \forall x \in \Omega, \forall p \in D^+u(x). \quad (33) \]

Now, since \( x_0 \in \Sigma(u) \) then \( D^+u(x_0) \) is a convex set of dimension greater or equal to 1. Hence, using the fact that the 0-level set of \( F \) contains no line segment it follows that there exists \( q \in D^+u(x_0) \) such that

\[ F(x_0, u(x_0), q) < 0. \]

In other words, we have that

\[ p_0 \in D^+u(x_0) \setminus D^*u(x_0). \quad (34) \]

Now, using Lemma 3.1 after Lemma 5.1, we obtain that

\[ \partial D^+u(x_0) \setminus D^*u(x_0) \neq \emptyset. \]

Hence, the result follows from Theorem 3.1. \( \square \)

If \( u \) belongs to the class \( SC^\alpha(\Omega) \), using Theorem 4.2 and Lemma 5.1, a stronger result can be proved.

Theorem 5.2. Under assumption (28) let \( u \in SC^\alpha(\Omega) \) be a solution of Eq. (27) and let \( x_0 \in \Sigma(u) \). Let us suppose that

\[ D_pF(x_0, u(x_0), p_0) \neq 0 \]

for some \( p_0 \in D^+u(x_0) \) such that

\[ \min_{p \in D^+u(x_0)} F(x_0, u(x_0), p) = F(x_0, u(x_0), p_0). \]

Then, there exists an arc \( x: [0, \sigma] \to \mathbb{R}^n \), with \( x(0) = x_0 \), such that, for any \( s \in [0, \sigma] \),

(i) \( x(s) \in \Sigma(u) \);

(ii) \( \lim_{s \to 0} x(s) = x_0 \) and \( x(s) \neq x_0 \), for \( s \neq 0 \);

(iii) \( x(s) = x_0 + sD_pF(x_0, u(x_0), p_0) + o(s) \), with \( o(s)/s \to 0 \) as \( s \to 0 \).
Acknowledgment

The author is grateful to the referee for carefully reading the manuscript.

References