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# COMMUTATIVE BASES OF DERIVATIONS IN POLYNOMIAL AND POWER SERIES RINGS

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In this note we give at first a description of commutative basis of modules $\mathrm{Der}_k(k[x_1,\dots,x_n])$ and $\mathrm{Der}_k(k[[x_1,\dots,x_n]])$ in characteristic zero (Theorem 2), and next we prove, using a theorem of Nousiainen and Sweedler [3, Theorem 3.3], an equivalent version of the Jacobian Conjecture (Theorem 5).

Let $k$ be a commutative ring containing the field $\mathbb{Q}$ of rational numbers and let $R$ denote either the ring $k[x_1,\dots,x_n]$ of polynomials over $k$ or the ring $k[[x_1,\dots,x_n]]$ of formal power series over $k$. We denote by $\Delta_1,\dots,\Delta_n$ the partial derivatives $\partial/\partial x_1,\dots,\partial/\partial x_n$, and by $\mathrm{Der}_k(R)$ the $R$-module of all $k$-derivations from $R$ to $R$.

It is well-known [2] that $\mathrm{Der}_k(R)$ is a free $R$-module on the basis $\Delta_1,\dots,\Delta_n$.

**Proposition 1.** *Let $d_1,\dots,d_n$ be k-derivations of R. The set $\{d_1,\dots,d_n\}$ is a basis of* $\mathrm{Der}_k(R)$ *if and only if the matrix $[d_i(x_j)]$ is invertible.*

**Proof.** If $d_1,\dots,d_n$ form a basis of $\mathrm{Der}_k(R)$, then there exist elements $b_{ij}\in R$ ($i,j=1,\dots,n$) such that $\sum_{p=1}^n b_{ip}d_p=\Delta_i$ for any $i=1,\dots,n$. Hence, for each $i,j=1,\dots,n$, we have

$$\delta_{ij}=\Delta_i(x_j)=\sum_{p=1}^n b_{ip}d_p(x_j),$$

where $\delta_{ij}$ is the Kronecker delta. Hence the matrix $[d_i(x_j)]$ is invertible.

Conversely, if the matrix $[d_i(x_j)]$ is invertible, then there exists an invertible matrix $[b_{ij}]$ of elements of $R$ such that $\delta_{ij}=\sum_{p=1}^n b_{ip}d_p(x_j)$, for each $i,j=1,\dots,n$.

Denote by $D_i$ (for $i=1,\dots,n$) the map $\sum_{p=1}^n b_{ip}d_p$. Since $D_i$ is a $k$-derivation of $R$ and $D_i(x_j)=\delta_{ij}$ for $,j=1,\dots,n$, we have $D_i=\Delta_i$. Hence $\sum_{p=1}^n b_{ip}d_p=\Delta_i$, for any $i=1,\dots,n$, and so the derivations $d_1,\dots,d_n$ form a basis of $\mathrm{Der}_k(R)$. □

We say that a basis $\{d_1,\dots,d_n\}$ of $\mathrm{Der}_k(R)$ is *commutative* if $d_id_j=d_jd_i$ for any $i,j=1,\dots,n$. We have the following characterization of commutative basis of $\mathrm{Der}_k(R)$.

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**Theorem 2.** *Let $d_1, \dots, d_n$ be k-derivations of R. Then the following conditions are equivalent*

(i) *The set $\{d_1, \dots, d_n\}$ is a commutative basis of* $\mathrm{Der}_k(R)$.

(ii) *There exist elements $F_1, \dots, F_n \in R$ such that $d_i(F_j) = \delta_{ij}$, for any $i, j = 1, \dots, n$, where $\delta_{ij}$ is the Kronecker delta.*

For the proof of this theorem we need two lemmas.

**Lemma 3.** *Let $f_1, \dots, f_n$ be elements of R. Then the following conditions are equivalent:*

(a) *There exists $F \in R$ such that $\Delta_i(F) = f_i$ for $i = 1, \dots, n$.*

(b) *$\Delta_i(f_j) = \Delta_j(f_i)$ for any $i, j = 1, \dots, n$.*

**Proof.** (a)$\Rightarrow$(b) follows from the equality $\Delta_i \Delta_j = \Delta_j \Delta_i$, for any $i, j = 1, \dots, n$.

(b)$\Rightarrow$(a). Let

$$f_i = \sum_{k_1, \dots, k_n} [k_1, \dots, k_n]_i x_1^{k_1} \cdots x_n^{k_n},$$

for $i = 1, \dots, n$, where the coefficients of the form $[k_1, \dots, k_n]_i$ are elements of $k$. Since $\Delta_i f_j = \Delta_j f_i$, for any $k_i$, $k_j \geq 1$ we have

$$\begin{aligned}(1/k_i)&[k_1, \dots, k_{i-1}, k_i - 1, k_{i+1}, \dots, k_n]_i \\ &= (1/k_j)[k_1, \dots, k_{j-1}, k_j - 1, k_{j+1}, \dots, k_n]_j.\end{aligned}$$

Put

$$F = \sum_{k_1, \dots, k_n} a_{k_1 \cdots k_n} x_1^{k_1} \cdots x_n^{k_n},$$

where $a_{0 \cdots 0} = 0$ and, if $k_i \geq 1$ for some $i$, then

$$a_{k_1 \cdots k_n} = (1/k_i)[k_1, \dots, k_{i-1}, k_i - 1, k_{i+1}, \dots, k_n]_i.$$

It is easy to check, that $\Delta_i(F) = f_i$ for $i = 1, \dots, n$. □

**Lemma 4.** *Let $\{d_1, \dots, d_n\}$ be a commutative basis of* $\mathrm{Der}_k(R)$. *Assume that $\Delta_i = \sum_{j=1}^n b_{ij} d_j$ for all $i = 1, \dots, n$, where $b_{ij}$ $(i, j = 1, \dots, n)$ are elements of R. Then $\Delta_p(b_{qj}) = \Delta_q(b_{pj})$ for any $p, q, j = 1, \dots, n$.*

**Proof.** It suffices to prove that

$$\sum_{j=1}^{n} (\Delta_p(b_{qj}) - \Delta_q(b_{pj})) d_j = 0.$$

From the commutativity of the basis $\Delta_1, \dots, \Delta_n$ we have

$$\begin{aligned}0 &= \Delta_p \Delta_q - \Delta_q \Delta_p \\ &= \Delta_p \left( \sum_{j=1}^{n} b_{qj} d_j \right) - \Delta_q \left( \sum_{j=1}^{n} b_{pj} d_j \right)\end{aligned}$$

$$= \sum_{j=1}^{n} (\Delta_p(b_{qj})d_j + b_{qj}\Delta_p d_j - \Delta_q(b_{pj})d_j - b_{pj}\Delta_q d_j).$$

Therefore

$$\sum_{j=1}^{n} (\Delta_p(b_{qj}) - \Delta_q(b_{pj}))d_j = \sum_{j=1}^{n} (b_{pj}\Delta_q d_j - b_{qj}\Delta_p d_j)$$

$$= \sum_{j=1}^{n} b_{pj}\left(\sum_{i=1}^{n} b_{qi}d_i d_j\right) - \sum_{j=1}^{n} b_{qj}\left(\sum_{i=1}^{n} b_{pi}d_i d_j\right)$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} b_{pj}b_{qj}(d_i d_j - d_j d_i) = 0.$$

This completes the proof. □

**Proof of Theorem 2.** (i)⇒(ii). Assume that the set $\{d_1, \ldots, d_n\}$ is a commutative basis of $\mathrm{Der}_k(R)$. Then, by Proposition 1, the matrix $A = [d_i(x_j)]$ is invertible. Put $A^{-1} = [b_{ij}]$, where $b_{ij} \in R$ for $i, j = 1, \ldots, n$. Then $\Delta_i = \sum_{j=1}^{n} b_{ij}d_j$, for any $i = 1, \ldots, n$ (see proof of Proposition 1).

Now let us fix $j \in \{1, \ldots, n\}$ and consider the elements $b_{1j}, \ldots, b_{nj}$. Since $\Delta_q(b_{pj}) = \Delta_p(b_{qj})$ for any $p, q = 1, \ldots, n$ (Lemma 4), there exists an element $F_j \in R$ such that $\Delta_p(F_j) = b_{pj}$ for any $p = 1, \ldots, n$ (by Lemma 3).

Moreover, we have

$$d_i(F_j) = \sum_{p=1}^{n} \Delta_p(F_j)d_i(x_p)$$

$$= \sum_{p=1}^{n} b_{pj}d_i(x_p) = (A \cdot A^{-1})_{ij} = \delta_{ij}.$$

(ii)⇒(i). Since $d_i(F_j) = \delta_{ij}$, we have $\sum_{k=1}^{n} \Delta_k(F_j)d_i(x_k) = \delta_{ij}$. Thus the matrix $[d_i(x_j)]$ is invertible, and hence, by Proposition 1, the derivations $d_1, \ldots, d_n$ form a basis of $\mathrm{Der}_k(R)$.

Since $d_p d_q - d_q d_p$ is an element of $\mathrm{Der}_k(R)$, there exist elements $a_1, \ldots, a_n$ of $R$ such that

$$d_p d_q - d_q d_p = a_1 d_1 + \cdots + a_n d_n.$$

But we have

$$a_i = (a_1 d_1 + \cdots + a_n d_n)(F_i) = (d_p d_q - d_q d_p)(F_i) = 0$$

for any $i = 1, \ldots, n$. Therefore $d_p d_q = d_q d_p$. This completes the proof of Theorem 2. □

Throughout the rest of the note we assume tha $R$ is the ring $k[x_1, \ldots, x_n]$ of polynomials over $k$.

We recall from [3] that a $k$-derivation $d$ of $R$ is called *locally nilpotent* if for each

$r \in R$ there exists a natural number $n$ such that $d^n(r)=0$, and is called *locally finite* if for any $r \in R$ there exists a finite generated $k$-module $M$ such that $r \in M$ and $d(M) \subseteq M$.

We say that a basis $\{d_1, \dots, d_n\}$ of $\mathrm{Der}_k(R)$ is *locally nilpotent* (resp. *locally finite*) if every derivation $d_i$ $(i=1, \dots, n)$ is locally nilpotent (resp. locally finite).

We recall that the *Jacobian Conjecture* states that if $F=(F_1, \dots, F_n)$ is a polynomial map $(F_1, \dots, F_n \in R)$ such that the Jacobian matrix $\mathrm{Jac}(F)=\mathrm{Jac}(F_1, \dots, F_n)=[\Delta_i(F_j)]$ is invertible, then $F$ has a polynomial inverse (see [1]).

The following theorem is a modification of a result of Nousiainen and Sweedler [3].

**Theorem 5.** *Let $k$ be a commutative ring containing the field $\mathbb{Q}$ of rational numbers and let $R=k[x_1, \dots, x_n]$ be the polynomial ring in $n$ variables over $k$. The following conditions are equivalent.*

(1) *The Jacobian Conjecture is true in the n-variable case.*

(2) *Every commutative basis of the R-module* $\mathrm{Der}_k(R)$ *is locally nilpotent.*

(3) *Every commutative basis of the R-module* $\mathrm{Der}_k(R)$ *is locally finite.*

**Proof.** (1)$\Rightarrow$(2) and (3). Let $\{d_1, \dots, d_n\}$ be a commutative basis of $\mathrm{Der}_k(R)$. Then, by Theorem 2, there exist polynomials $F_1, \dots, F_n \in R$ such that $d_i(F_j)=\delta_{ij}$, that is $\sum_{p=1}^{n} \Delta_p(F_j) d_i(x_p)=\delta_{ij}$ for $i, j=1, \dots, n$. Therefore the Jacobian matrix $\mathrm{Jac}(F_1, \dots, F_n)$ is invertible and it is easy to see that the following matrix equality holds

$$(*) \qquad \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = \mathrm{Jac}(F_1, \dots, F_n)^{-1} \begin{bmatrix} \Delta_1 \\ \vdots \\ \Delta_n \end{bmatrix}.$$

Hence, by [3, Theorem 3.3], the basis $\{d_1, \dots, d_n\}$ is locally nilpotent and locally finite.

(2) or (3)$\Rightarrow$(1). Let $F_1, \dots, F_n \in R$ be polynomials such that the Jacobian matrix $\mathrm{Jac}(F_1, \dots, F_n)$ is invertible. We define derivations $d_1, \dots, d_n$ of $R$ by the equality $(*)$. Then we can see, by [3, Proposition 2.4], that $\{d_1, \dots, d_n\}$ is a commutative basis of $\mathrm{Der}_k(R)$. Therefore, by [3, Theorem 3.3], the polynomial map $(F_1, \dots, F_n)$ has a polynomial inverse. This completes the proof. $\square$

**Remark.** There exist bases (non-commutative) of the module $\mathrm{Der}_k(k[x_1, \dots, x_n])$ which are not locally finite (and hence are not locally nilpotent). The following is an example of those:

$$\{d_1=\Delta_1+x_2^2\Delta_2, d_2=\Delta_2, \dots, d_n=\Delta_n\}.$$

## References

[1] H. Bass, E.H. Connell and D. Wright, The Jacobian Conjecture: Reduction of degree and formal expansion of the inverse, Bull. Amer. Math. Soc. 7 (1982) 287–330.
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[3] P. Nousiainen and M.E. Sweedler, Automorphisms of polynomial and power series rings, J. Pure Appl. Algebra 29 (1983) 93–97.