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MORSE THEORY OF HARMONIC FORMS

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We consider the problem of whether it is possible to improve the Novikov inequalities for closed 1-forms, or any other inequalities of a similar nature, if we assume, additionally, that the given 1-form is harmonic with respect to some Riemannian metric. We show that, under suitable assumptions, it is impossible. We use a theorem of Calabi [1], characterizing 1-forms which are harmonic with respect to some metric, in an essential way. We also study some interesting examples illustrating our results.

1. INTRODUCTION

The Morse theory of closed 1-forms was begun by Novikov [8, 9]; a survey can be found in [10]. The standard Morse inequalities for functions were generalized, by Novikov, to forms using a twisted cohomology defined by the 1-form in place of the usual cohomology. In some cases it is known that these inequalities are sharp [3, 10].

In this paper we will address the problem of whether it is possible to improve these Novikov inequalities (or any other inequalities of a similar nature) *assuming additionally that the given closed 1-form is harmonic with respect to some Riemannian metric*. This problem was first considered in the paper of Calabi [1] even before Novikov's theory. There is one obvious restriction—namely, any non-constant harmonic form has no local minima and maxima—and Calabi asked in [1] if there are any other restrictions on the critical point structure of Morse harmonic 1-forms.

Motivated by this, Calabi solved instead in [1] another fundamental problem: he gave a complete topological criterion for a closed Morse 1-form to be harmonic with respect to some Riemannian metric. We will call such forms *intrinsically harmonic* or *Calabi forms*. We will describe the theorem of Calabi briefly in Section 3.

The thrust of the main results in this paper is that, at least under certain quite general conditions, *there exist no restrictions on the Morse numbers of harmonic forms except for the trivial restrictions mentioned above*.

The conditions we need to impose are closely tied to the topological nature of the (singular) foliation which is associated to a Morse 1-form (see below for the definition). In particular, compactness of the leaves is a crucial item. Our results will show that if the foliation defined by a form ω is made up entirely of *non-compact* leaves or, at the other extreme, belongs to a cohomology class which is a scalar multiple of an integral class (in which case all the leaves of the foliation are compact), then there is a Calabi form $\tilde{\omega}$ which is *contiguous* to ω, i.e. it is in the same cohomology class and has the same number of critical points of each index.[†]

We will also present some results concerning two issues related to the existence of compact leaves in the foliation of a closed Morse 1-form. In Section 8.1 we show that such

[†] Recently Ko Monda has shown this to be true for any form ω.

a form defines a natural decomposition of the manifold into pieces, each of whose interiors consists of all compact or all non-compact leaves. The combinatorics of this decomposition plays a crucial role in determining whether the form is contiguous to a Calabi form.

In Sections 8.2 and 8.3 we present two results which give restrictions on the cohomology class of a form when either all or none of the leaves of its foliation are compact. In Section 9 we describe several examples for surfaces which will illuminate and illustrate our results.

It is a theorem of Lalonde–Polterovich [6] that, given any non-trivial class in $H^1(M, \mathbb{R})$, then, for a generic Riemannian metric, the harmonic form representing that class has only Morse singularities. In other words, generically harmonic forms have only Morse type singularities. This allows us to avoid considering more complicated singularities and leaves us with the question addressed here of how many critical points these Morse harmonic forms can have.

2. THE MAIN RESULTS

Here we state in a precise form our principal results.

THEOREM 1. *Suppose that ω is a closed Morse 1-form on a closed manifold M which represents a scalar multiple of an integral cohomology class and such that ω has no critical points of index 0 or $n = \dim M$. Then there exists an intrinsically harmonic Morse 1-form $\tilde{\omega}$ which has the same numbers of critical points of all indices as ω and which is in the same cohomology class as ω.*

This theorem implies that, if there exist Morse type inequalities for harmonic forms estimating the numbers of critical points by information depending only on the cohomology class of the form, they have to be also true for any closed 1-form representing a scalar multiple of an *integral* cohomology class. We refer to such cohomology classes as being of *rank one*. Following Novikov [8], we say that a cohomology class $\xi \in H^1(M, \mathbb{R})$ *is of rank k* if the image of the homomorphism $\pi_1(M) \to \mathbb{R}$ determined by ξ is a free abelian group of rank k.

We recall from [8] that, for a closed 1-form ω of rank one on a closed manifold M, the Novikov inequalities take the form:

$$m_p(\omega) \geqslant b_p([\omega]) + q_p([\omega]) + q_{p-1}([\omega])$$

where $m_p(\omega)$ is the number of critical points of index p and $b_p([\omega])$, $q_p([\omega])$ are the rank and torsion Novikov numbers of the cohomology of M with local coefficients defined by $[\omega]$ (see [3, 8] for precise definitions). In [3] it is shown that these inequalities are exact. Combining this with Theorem 1 gives:

COROLLARY. *If $\pi_1(M) = \mathbb{Z}$ and $n = \dim M \geqslant 6$, then for any nonzero class $\xi \in H^1(M, \mathbb{R})$, there exists an intrinsically harmonic More form ω representing ξ such that $m_p(\omega) = b_p(\xi) + q_p(\xi) + q_{p-1}(\xi)$.*

This follows by combining the main theorem of [3] and Theorem 1 above and observing that, since $b_0(\xi) = q_0(\xi) = b_n(\xi) = q_{n-1}(\xi) = 0$, the 1-form given by [3] realizing the given non-zero cohomology class ξ has no critical points of indices 0 and n.

It is an easy fact that rank one More forms must have all compact leaves. In the next theorem we consider the opposite situation: we assume that all leaves are non-compact. In addition we need to assume that ω is *generic*, which we define to mean that any singular leaf contains only one critical point. It is easy to see that any Morse form can be perturbed slightly so that it becomes generic without changing the numbers of critical points or the cohomology class.

THEOREM 2. *If ω is a generic closed Morse 1-form all of whose leaves are non-compact then ω is intrinsically harmonic.*

Theorem 2 is false if ω is not generic. See Example 4 in Section 9.
Theorems 1 and 2 will be proved in Sections 6 and 7.

3. CALABI'S THEOREM

In this section we will recall Calabi's theorem [1].
Let ω be a fixed closed 1-form on a closed manifold M.

A smooth path $\gamma:[0,1]\to M$ will be called *ω-positive* if $\omega(\dot\gamma(t))>0$ for any $t\in[0,1]$. Calabi considered the following condition on ω which he calls *transitivity*:

(a) For any ordered pair of points x and y in M, which are not singular points of ω, there exists an ω-positive path from x to y.

Evidently, (a) implies:

(b) For any non-singular point $x\in M$ there exists a *closed ω-positive* path γ through x.

We shall see that, in fact, (a) is equivalent to (b).

The main result of [1] can be reformulated as follows.

THEOREM (Calabi [1]). *A closed Morse 1-form ω on M is harmonic with respect to some Riemannian metric if and only if condition* (b) *holds.*

Remark. It is pointed out in [1] that (a) precludes the existence of critical points of index 0 or $n=\dim M$ and that non-singular forms satisfy (a).

4. SINGULAR FOLIATION OF A MORSE FORM

Let again ω denote a closed 1-form with only Morse-type singularities.

For any simply connected open set $U\subset M$ we have $\omega|_U = \mathrm{d}f_U$ for some Morse function $f_U: U\to\mathbb{R}$ determined up to a constant. We will consider the foliation in U determined by the level sets of f_U. Choosing a covering $\{U\}$ of M we see that all these foliations match together to form a foliation $\mathscr{F}$ of M. We will call it *the foliation determined by the form ω.*

Note this foliation has finitely many singular points which are the critical points of ω. Locally, the structure of the singular foliation $\mathscr{F}$ around a critical point of index d has the form

$$-x_1^2-\cdots-x_d^2+x_{d+1}^2\cdots+x_n^2=c.$$

A *leaf of $\mathscr{F}$* is, by definition, any maximal subset $\mathscr{L}$ of M such that for any two points x, y in $\mathscr{L}$ there exists a smooth path $\gamma:[0,1]\to M$ which connects x with y and such that $\omega(\dot\gamma(t))=0$ for all t. There are finitely many leaves containing the singular points; we will call those leaves *singular*.

Note that the *non-singular* leaves, i.e. those containing none of the critical points of ω, are smooth while the *singular* leaves are smooth except at each of the critical points which has a neighborhood homeomorphic to a cone over $S^{d-1}\times S^{n-d-1}$, where d is the index of the critical point. Each nonsingular leaf $\mathscr{L}$ has a natural topology (which will be called *the leaf topology*) and smooth manifold structure such that the inclusion $\mathscr{L} \to M$ is an immersion; similarly for the singular leaves. We will be particularly interested in those leaves containing critical points of index 1 or $n - 1$. In these cases removing a critical point locally disconnects the leaf and we will find it useful to define a *singular leaf component* to be the closure (in the leaf topology) of a connected component of $\mathscr{L} - \Sigma(\mathscr{L})$, where $\Sigma(\mathscr{L})$ denotes the set of all singular points of the leaf $\mathscr{L}$.

5. CALABI GRAPHS

5.1. We now make some preliminary constructions to set up the proof of Theorem 1. In section 5.2 we will see that the Calabi properties of closed 1-forms can be interpreted by analogous properties of oriented graphs.

Let Γ be an *oriented* connected finite graph. Consider the following two properties.

(a′) If $x, y \in \Gamma$ (it suffices to consider only vertices) then there is a path from x to y that traverses edges of Γ only in the positive direction.

(b′) For any point $x \in \Gamma$, there exists a closed path through x that traverses edges of Γ only in the positive direction.

Clearly (a′) implies (b′).

LEMMA 1. *If* Γ *satisfies* (b′), *then it satisfies* (a′).

Proof. Condition (b′) tells us that Γ is the union of closed edge-paths which can be traversed in a positive direction will respect to the orientation. Since Γ is connected, then, for any two points $x, y \in \Gamma$, there is a sequence of points $x = x_1, x_2, \dots, x_k = y$ such that x_i and x_{i+1} are on a common closed edge path C_i. Clearly x_i and x_{i+1} can be joined by a positive path on C_i. Putting these together gives the desired path from x to y. □

Definition. We shall say that Γ *is a Calabi graph if it satisfies* (a′) *and* (b′).

In Fig. 1 we show an example of a Calabi graph and a non-Calabi graph.

5.2. With any closed Morse 1-form ω on a closed manifold M, generating a singular foliation with *all compact leaves*, we shall associate an oriented connected graph Γ_ω. With this goal in mind, we introduce an equivalence relation in M. We declare two points in M to be equivalent if they lie on the same leaf of the foliation $\mathscr{F}_\omega$, generated by ω. When all the

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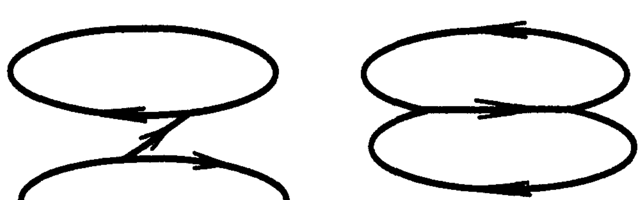

NON-CALABI GRAPH

CALABI GRAPH

Fig. 1.

leaves of $\mathscr{F}_\omega$ are compact, then the singular leaves are isolated (i.e. each singular leaf has a neighborhood, free of other singular leaves) and the quotient space $\Gamma_\omega := M/\sim$ is a graph. Indeed, any non-singular leaf $\mathscr{L}$ has a collar, consisting of non-singular leaves. Hence, its neighborhood in the quotient space Γ_ω is an open interval. Suppose that $\mathscr{L}_0$ is a singular leaf corresponding to a point $v \in \Gamma_\omega$. Then form ω is exact in some neighborhood of $\mathscr{L}_0$. Suppose $\omega = df$ and $f^{-1}(a) = \mathscr{L}_0$. Then, for some $\varepsilon, f|_{f^{-1}(a, a+\varepsilon)}$ and $f|_{f^{-1}(a-\varepsilon, a)}$ are fibrations and the fibers are non-singular leaves. Thus the complement of $\mathscr{L}_0$ in this neighborhood is a product of an open interval with a finite number of compact manifolds which define a finite number of edges of Γ_ω with v viewed as a vertex.

Suppose $x_i \in \mathscr{L}_0$ is a critical point of index s_i. If $1 < s_i < n - 1$, then it follows from the local structure of a non-degenerate critical point that the intersection of each leaf with a small neighborhood of x_i is connected. We conclude that if all the indices s_i for a given singular leaf $\mathscr{L}_0$ satisfy $1 < s_i < n - 1$, then the leaves on both sides of $\mathscr{L}_0$ are connected and so v is just an interior point of an edge of Γ_ω. If $s_i = 1$ or $n - 1$ and $n > 2$, then locally we have two non-singular leaves coalescing into the singular leaf from one side and one emerging on the other side (see Fig. 2). (Of course it is possible that the what looks like two leaves locally may actually be part of the same leaf globally.)

Thus when one or more of the indices s_i equal 1 or $n - 1$, then v may be a *true* vertex of Γ_ω. If $s_i = 0$ or n, then we have non-singular fibers on only one side and v will again be a *true* vertex.

In Fig. 3 we show two examples of Morse 1-forms with all compact leaves whose associated graphs are those in Fig. 1. The forms are the pullbacks of the canonical 1-form $d\theta$ on the circle S via the obvious projection maps onto S.

The following lemma is obvious from the preceding discussion.

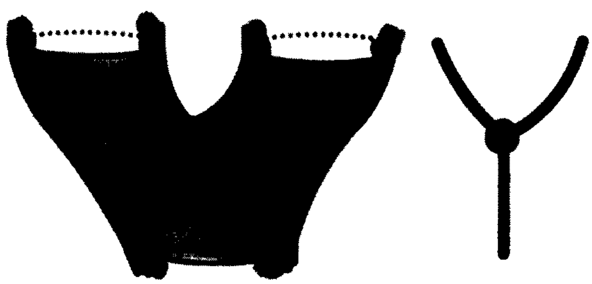

Fig. 2.

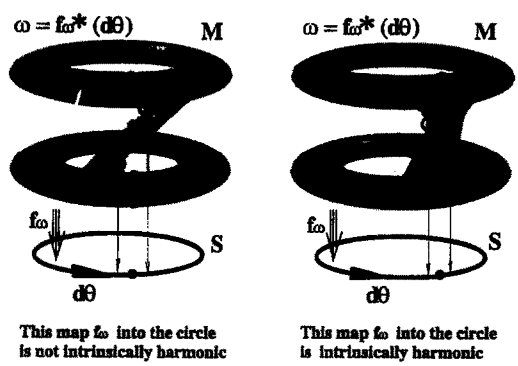

Fig. 3.

LEMMA 2. (M, ω) *(with all leaves compact) satisfies properties* (a) *and* (b) *of Section* 3 *iff the associated oriented graph* Γ_ω *satisfies properties* (a′) *and* (b′) *of Section* 5.1, *respectively.*

6. PROOF OF THEOREM 1

First, we remark that we may assume that $n = \dim M > 2$ without loss of generality. In fact, if $n = 2$, then from the Euler–Poincare theorem it follows that any 1-form which has no maxima and minima, has precisely $-\chi(M)$ critical points of index 1. Thus we will assume in the sequel that $n > 2$. In fact, with some extra care, all our arguments will work in dimension 2. Our figures are all in dimension 2 so the reader should interpret them accordingly.

Multiplying the form ω by a scalar, we may assume that it is the differential of a (multivalued) Morse function $f: M \to S$, where S is an oriented circle. More precisely, this means that $\omega = f^*(d\phi)$, where $d\phi$ is the standard angular form on the circle.

We may also assume, after a perturbation, that all the critical values of f are distinct. Then, since there are no critical points of index 0 or n, every vertex of the graph Γ_ω (cf. Section 5.2) is trivalent.

Consider the map $\psi_f: \Gamma_\omega \to S$ induced by f. Define the *complexity* of ω to be the smallest cardinality of any $\psi_f^{-1}(a)$, where a ranges over the regular values of f. For example in Fig. 3 the complexities of the two graphs are 2 and 1, respectively.

We will show that either:

(1) Γ_ω has no vertices, or
(2) there exists another closed 1-form ω' with strictly smaller complexity such that ω and ω' are *contiguous.* Recall, that this means, that ω and ω' have the same numbers of critical points of all indices and belong to the same cohomology class.

Thus, using induction, we will eventually reach case (1) i.e. we will find a closed 1-form $\tilde{\omega}$ contiguous to ω and such that $\Gamma_{\tilde{\omega}}$ is a circle and ψ_f is a covering map. In this case it is clear that $\Gamma_{\tilde{\omega}}$ is a Calabi graph and we are done.

Suppose that Γ_ω has at least one vertex. Choose a regular value a such that the cardinality of $\psi^{-1}(a)$ equals the complexity of ω. Cut open M along $M_a = f^{-1}(a)$ to obtain a manifold $\bar{M}$. The map f induces a Morse function $\bar{f}: \bar{M} \to [0, 2\pi]$ such that the two identical boundary components M_a^+ and M_a^- map to 0 and 2π respectively. The leaves of the foliation defined by $\bar{f}$ are the same as those of ω and the graph $\Gamma_{\bar{f}}$ it defines (cf. Section 5.2) can be obtained from Γ_ω by cutting some of the edges. We now change the function $\bar{f}$, using the Morse–Smale reindexing technique (see [7]), to obtain a self-indexing Morse function $\bar{g}: \bar{M} \to [0, 2\pi]$ with the same number of critical points of each index as $\bar{f}$ and such that $\bar{g}$ agrees with $\bar{f}$ in a neighborhood of $\partial\bar{M}$. The self-indexing property means that for any two critical points x, y, $\text{index}(x) > \text{index}(y)$ implies $\bar{g}(x) > \bar{g}(y)$. We may also assume that all the critical values of $\bar{g}$ are distinct. If we reglue M_a^+ and M_a^-, the function $\bar{g}$ induces a new Morse 1-form ω' on M which is contiguous to ω. See Fig. 4.

Now consider the associated oriented trivalent graphs $\Gamma_{\omega'}$ and $\Gamma_{\bar{g}}$. The vertices of $\Gamma_{\bar{g}}$ are of two types: those with two ingoing and one outgoing edge (they necessarily belong to singular leaves containing critical points of index 1) and those with one ingoing and two outgoing edges (they necessarily belong to singular leaves containing critical points of index $n-1$). The self-indexing property implies (since we assume that $n > 2$ and so $1 < n - 1$) that the induced map $\psi_{\bar{g}}: \Gamma_{\bar{g}} \to [0, 2\pi]$ has the property that there is some $b \in (0, 2\pi)$ such that

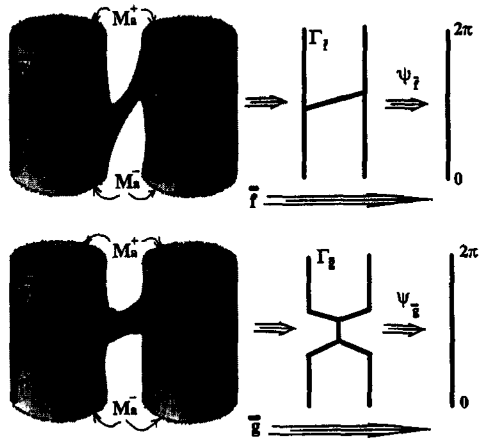

Fig. 4.

$\Psi_{\bar{g}}$ maps each vertex of the first type into $(0, b)$ and each vertex of the second type into $(b, 2\pi)$. As a consequence, if there are any vertices at all, then the cardinality of $\psi_{\bar{g}}^{-1}(b)$ is strictly smaller than that of $\psi^{-1}(a)$. But this means that the complexity of ω' is smaller than the complexity of ω.

For example, applying this modification to the non-Calabi example in Fig. 3 and its graph in Fig. 1 will produce the form given by the other example in Fig. 3.

This completes the proof. □

7. PROOF OF THEOREM 2

Suppose the Morse 1-form ω has all non-compact leaves. We will show that it satisfies the Calabi condition (b) of Section 3. Let $x \in M$. If x belongs to a non-singular leaf, we will denote it by $\mathscr{L}_x$; if x belongs to a singular leaf, then we will denote by $\mathscr{L}_x$ the *singular leaf component* (cf. Section 4) containing x.

By statement (2) of Proposition 1 (cf. below in Section 8.1), since ω is assumed to be generic, we know that $\mathscr{L}_x$ is non-compact.

First of all we point out that if any non-singular point x' of $\mathscr{L}_x$ satisfies Calabi condition (b), i.e. there is a ω-positive closed path γ passing through x', then any other non-singular point x'' in $\mathscr{L}_x$ satisfies Calabi condition (b) as well. To show this we connect x'' to x' by a path in $\mathscr{L}_x$. Now, γ crosses $\mathscr{L}_x$ transversely at x' and so we can pull γ, as it passes through x', along γ, so that it now passes through x''. It is clear that we may keep γ positive during this deformation. Thus, it suffices to find an ω-positive closed path intersecting $\mathscr{L}_x$ anywhere.

Since $\mathscr{L}_x$ is not compact there exists a limit point y of $\mathscr{L}_x$ not in $\mathscr{L}_x$. Since we have explicit models for the foliation near y, for the singular and non-singular case, we can see that all points on $\mathscr{L}_y$ close to y are also limit points of $\mathscr{L}_x$. So we may assume that y is not a critical point. If U is a sufficiently small coordinate neighborhood of y, then the foliation on U consists of parallel hyperplanes and so $\mathscr{L}_x \cap U$ must contain a sequence of these hyperplanes converging to the one containing y. Choose any two points x', x'' of $\mathscr{L}_x \cap U$ in different hyperplanes. We can connect these two points by a path $\rho: [-1, 1] \to \mathscr{L}_x$ with $\rho(-1) = x''$ and $\rho(1) = x'$. We can also obviously connect them by a positive path ξ in U (say, running from x' to x''). We now push ρ slightly off $\mathscr{L}_x$ to make it into a positive path.

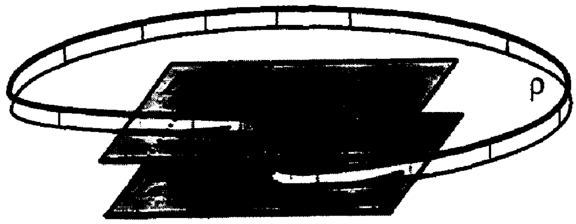

Fig. 5.

To do so choose a vector field X on $\mathscr{L}_x$ so that $\omega(X)$ is a positive constant. We can assume X is tangent to ξ at x' and x''. Define $\eta(t) = \gamma(t) + \varepsilon t X(\gamma(t))$ for ε small enough, so that $\eta(-1)$ lies ahead of $\eta(1)$ on ξ. Now we can put η together with the portion of ξ that runs from $\eta(1)$ to $\eta(-1)$ to create a positive closed path which intersects $\mathscr{L}_x$ at $\eta(0)$. Of course, we need to round corners. See Fig. 5. □

8. COMPACT AND NON-COMPACT LEAVES

8.1. It is clear from Theorems 1 and 2 that the general problem of deciding whether a given closed Morse 1-form is contiguous to a Calabi form may depend on the structure of the foliation defined by ω. We will now explain how the manifold M breaks up, in a nice way, into pieces made up (roughly) of all compact leaves or all non-compact leaves.

It is a theorem of Novikov, mentioned in [11], that when ω is non-singular (and M is connected) then either all the leaves are compact, or all the leaves are non-compact. This is certainly not true for general foliations (e.g. the Reeb foliation) but, in Proposition 1, we will generalize Novikov's theorem to foliations coming from a closed Morse 1-form, showing that the compact and non-compact leaves give a nice decomposition of M. The terms used in the statement were defined above in sections 4 and 2.

PROPOSITION 1. *If ω is a closed Morse 1-form on M, then M is the union of two compact n-dimensional submanifolds M_c and M_∞, with a common (but generally singular) boundary, satisfying:*

(1) Int M_c *is a union of all of the compact leaves plus some compact singular leaf components of non-compact leaves.*
(2) Int M_∞ *is a union of all non-compact non-singular leaves, all non-compact singular leaf components and some compact singular leaf components of non-compact leaves.*
(3) $\partial M_c = \partial M_\infty = M_c \cap M_\infty$ *is a subvariety which is smooth except at a finite number of points, which are critical points of ω, and is a union of some compact singular leaf components of non-compact singular leaves.*
(4) *If ω is generic, then $M_c \subset M_\infty$ contains all compact singular leaf components of non-compact leaves.*

The theorem of Novikov follows from Proposition 1 since if there are no critical points, then by statement (3)

$$\partial M_c = \partial M_\infty = \emptyset$$

and so $M = M_c$ or $M = M_\infty$.

Proof. (1) Let U be the union of all the compact leaves of the foliation determined by the form ω. We show that U is open. If $\mathscr{L}$ is a compact leaf, then it has a neighborhood W with compact closure and containing no critical points other than those on $\mathscr{L}$, such that $\omega | \bar{V} = df$ for some smooth function on $\bar{V}$ with $f^{-1}(0) = C$. Then there exists $\varepsilon > 0$ such that $f^{-1}(-\varepsilon, \varepsilon) \subseteq V$ and so $f^{-1}(a)$ is a compact leaf for $|a| < \varepsilon$.

(2) Let X be the union of all compact leaves and compact leaf components of non-compact leaves. Then X is closed. Indeed if we consider the non-singular foliation $\mathscr{F}_0$ defined by ω on $M_0 = M -$ critical points of ω, we can apply a theorem of Haefliger [4, p. 386] to conclude that the union X_0 of all the closed leaves of $\mathscr{F}_0$ is a closed subset of M_0. But then X is just the closure of X_0 in M.

Let $M_c = \bar{U}$ and $V = M - \bar{U}$. From (2) we conclude that $Y = M_c - U$ is a union of some compact leaf components of non-compact leaves. Y is a codimension one submanifold of M except for singularities at the critical points of ω in Y. If x is such a critical point of index q then, for some neighborhood W of x, $(W, Y \cap W) \cong$ cone over $(S^{n-1}, S^{q-1} \times S^{n-q-1})$ if $1 < q < n-1$, while if $q = 1$ or $n-1$ then $(W, Y \cap W) \cong$ cone over (S^{n-1}, S^{n-2}) or $(W, Y \cap W) \cong$ cone over $(S^{n-1}, S^0 \times S^{n-2})$. In this last case we call x *special.* It is clear that M_c is a manifold with boundary Y which is smooth except at the critical points and at those points we have a very explicit model of the singularity. At every point of Y, except the special points, Y locally separates M into two components exactly one of which must lie in U and the other lies in V. At the special points Y locally separates M into three components if $n > 2$ or four components if $n = 2$. If $n > 2$ then either one or two of the components lie in U. See Fig. 6.

It remains to discuss the generic case. Suppose C is a compact leaf component of a non-compact leaf $\mathscr{L}$. Then C contains exactly one critical point x which must be of index 1 or $n-1$. As above we have a neighborhood W of x so that $(W, \mathscr{L} \cap W) \cong$ cone over $(S^{n-1}, S^0 \times S^{n-2})$ where $C \cap W$ is one of the two cones over S^{n-2}. Now we can choose a regular neighborhood R of C so that $R = W$ near x but $(R - W, C - W) \cong ((C - W) \times [-1, 1], (C - W) \times 0)$ since there are no other critical points on C. See Fig. 7. It is clear that the leaves of the foliation defined by ω on one side of C must lie entirely in R and are homeomorphic to C. Since C is compact this shows that $C \subseteq \bar{U}$. □

8.2. The next two propositions describe some relations between the cohomology class represented by a Morse 1-form and the presence of compact leaves in its foliation.

PROPOSITION 2. *A class $\xi \in H^1(M, \mathbb{R})$ has a representative Morse 1-form with all compact leaves if and only if the homomorphism $\pi_1(M) \to \mathbb{R}$ determined by the cohomology class ξ can be factorized $\pi_1(M) \to F \to \mathbb{R}$ through a free group F.*

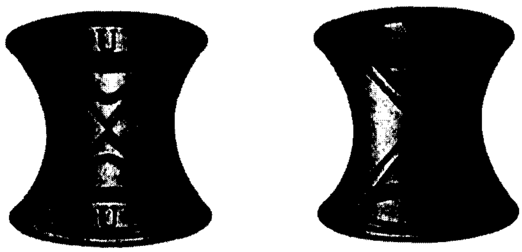

Fig. 6.

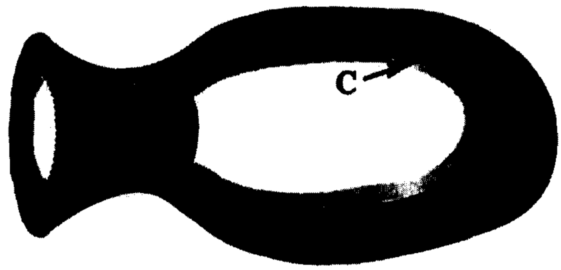

Fig. 7.

It will be convenient for us to refer to the cohomology classes satisfying the condition of Proposition 2 as *split* cohomology classes. A more geometric description of this condition is the following. The manifold M can be cut into several pieces by disjoint codimension one submanifolds so that the restriction of the class ξ to each piece is of rank one.

Remarks

(i) As pointed out in Section 2, a rank one class has the property that *every* representative Morse 1-form has all compact leaves. In Section 9 we will give examples that suggest that only rank one classes have this stronger property.

(ii) It seems to be difficult to give a purely cohomological criterion for ξ to be split, but a necessary condition is that there exist a finite subset $\{\xi_i\} \subseteq H^1(M, \mathbb{Z})$ such that ξ is a linear combination of the ξ_i and all Massey products (see [2]) of the $\{\xi_i\}$ vanish. However *this condition is not sufficient.* At example can be obtained as follows. Choose a slice link L of two components which is not a homology boundary link. According to [2, p. 72] the link in Fig. 8 is such an example. Define M to be the result of 0-surgery on S^3 along L. Then $H^1(M, \mathbb{Z}) \cong \mathbb{Z}^2$ and all Massey products vanish. But there is no map $\pi_1(M) \to F$, F a free group inducing an injection on H^1 and so, any class ξ of rank two is not split.

Proof of Proposition 2. Let ω be a Morse 1-form representing ξ with all compact leaves and Γ_ω the associated oriented graph as defined in Section 5.2. Let $\phi: M \to \Gamma_\omega$ be the identification map. It is clear that $\xi \in \phi^* H^1(\Gamma_\omega; \mathbb{R})$ and half of Proposition 2 follows since any connected graph has a free fundamental group.

Now suppose we are given ξ and a map $f: M \to W$, where W is a wedge of circles, such that $\xi = f^*(\xi')$ for some $\xi' \in H^1(W, \mathbb{R})$. Choose a point a_i in each circle S_i of W; we may assume that a_i is a regular value of f. Then $M_i = f^{-1}(a_i)$ is a smooth compact submanifold of M. Let $\bar{M}$ be the result of cutting M open along the M_i. Then $\partial \bar{M}$ consists of two copies M_i^+ and M_i^- of each M_i. Obviously $\xi|_{\bar{M}} = 0$. Now assume each S_i is oriented and set $p_i = \langle \xi', [S_i] \rangle$, the period of the class ξ' on the circle S_i. We may assume that the orientations are such that a short path in M which crosses M_i, hitting M_i^+ before M_i^-, maps into S_i with positive orientation. Now choose a Morse function $g: \bar{M} \to \mathbb{R}$ with no critical points on $\partial \bar{M}$, which is constant on each boundary component so that $g(M_i^+) - g(M_i^-) = p_i$. We may also assume that $g|_{\partial \bar{M}}$ fits together smoothly to form a smooth 1-form ω on M. Clearly ω is Morse and closed, has zero periods in $\bar{M}$ and has period p_i around any closed curve which crosses M_i "positively" but no other M_j. Thus ω represents ξ. Furthermore, the leaves of the foliation defined by ω are components of $f^{-1}(a)$ for some $a \in \mathbb{R}$ and so they are compact. □

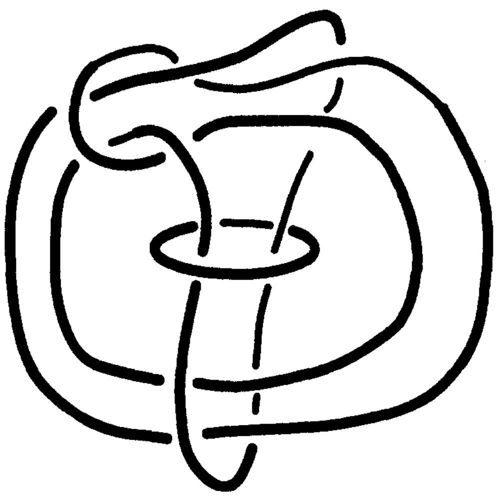

Fig. 8.

8.3.

PROPOSITION 3. *If ω is a closed intrinsically harmonic Morse 1-form whose associated foliation has a compact leaf, then there exists some non-zero $\theta \in H^1(M, \mathbb{Z})$ such that $\theta \cup [\omega] = 0 \in H^2(M, \mathbb{R})$.*

We will show in Section 9 that Proposition 3 is false if ω is not assumed to be intrinsically harmonic.

Proof. If ω has a compact leaf, then, by statement (1) of Proposition 1, there exist *non-singular* compact leaves. Suppose $\mathscr{L}$ is a non-singular compact leaf of ω. Then $\mathscr{L}$ is a closed submanifold of M and the normal bundle to $\mathscr{L}$ is trivialized by the form ω. Also, $\mathscr{L}$ cannot bound in M since this would contradict the Calabi condition.

We may find a cylinder $U = [-1/2, 1/2] \times \mathscr{L}$ in M fibred by the leaves of the foliation determined by ω, such that the initially choosen leaf $\mathscr{L}$ is identified with $0 \times \mathscr{L}$. Now, let us construct a closed 1-form α on M such that:

(1) α vanishes outside the cylinder U;

(2) on the cylinder U the form α is given by $\alpha|_U = d(\phi \circ p)$ where $p: U \to [-1/2, 1/2]$ denotes the projection, and $\phi: [-1/2, 1/2] \to [-1/2, 1/2]$ is a smooth function satisfying:

$$\phi(t) = t \quad \text{for all } |t| < 1/2 - \varepsilon$$

$$\phi(t) = 1/2 \quad \text{for all } t \in [1/2 - \delta, 1/2]$$

$$\phi(t) = -1/2 \quad \text{for all } t \in [-1/2, -1/2 + \delta]$$

where $0 < \delta < \varepsilon$.

Thus we obtain that α represents a nonzero integral cohomology class $\theta \in H^1(M, \mathbb{Z})$ and $\alpha \wedge \omega = 0$. Therefore, on the level of cohomology, we get $\theta \cup [\omega] = 0$ in $H^2(M, \mathbb{R})$. □

Suppose, for example, that the pairing $H^1(M, \mathbb{R}) \times H^1(M, \mathbb{R}) \to H^2(M, \mathbb{R})$ is non-degenerate, i.e. for any $\xi \in H^1(M, \mathbb{R})$ there exists $\eta \in H^1(M, \mathbb{R})$ such that $\xi \cup \eta \neq 0$. Then it follows from Proposition 3 that, if a class $\xi \in H^1(M, \mathbb{R})$ is completely irrational, then any representative harmonic Morse 1-form has all non-compact leaves. (We say that a cohomology class

ξ is *completely irrational* if its rank is equal to the first Betti number of M). This applies to the cases when M is an orientable surface or $M = S^1 \times \cdots \times S^1$.

9. EXAMPLES

We complement and illustrate Theorem 2 and Propositions 2 and 3 with some examples in dimension two. We first summarize the assertions of these results for a class $\alpha \in H^1(M; \mathbb{R})$, where M is a closed orientable surface, and a representative Morse 1-form ω_α.

(1) If rank $\alpha = 1$, then any ω_α has all compact leaves.
(2) α is split if and only if there exists some ω_α with all compact leaves.
(3) If α is completely irrational and ω_α is Calabi, then no leaves are compact.
(4) If ω_α is generic and has no compact leaves, then ω_α is Calabi.

We will construct the following four examples of α and ω_α which will illustrate the sharpness of these results.

Examples. (1) Rank α = one less than maximal and ω_α Calabi with some compact leaves.
(2) Rank $\alpha > 1$, α split and ω_α Calabi with all leaves non-compact.
(3) α completely irrational and ω_α non-Calabi with some compact leaves.
(4) ω_α non-Calabi but all leaves non-compact (and so ω_α is not generic).

We first describe three general connected sum operations which accept as input two closed two-dimensional manifolds M_1, M_2 with closed Morse 1-forms ω_1, ω_2 and constructs a closed Morse 1-form ω on the connected sum $M_1 \# M_2$ such that $\omega | M_i - D_i = \omega_i | M_i - D_i$, where D_i is any disk in M_i containing no critical points of ω_i and within which the connected sum will be constructed.

Choose smooth functions $f_i : D_i \to \mathbb{R}$ such that $\omega_i = df_i$. Suppose that we can identify each D_i with an open rectangle $(a_i, b_i) \times (c_i, d_i)$ in $\mathbb{R}^2$ so that $f_1(x, y) = y$. Note that we can then move the rectangle by any translation in $\mathbb{R}^2$. Now we can perform a connected sum of D_1 and D_2 ambiently by connecting them with a straight tube in $\mathbb{R}^3$ in three different ways.

A. $f_1(D_1) \cap f_2(D_2) = J$ is non-empty, the connecting tube intersects D_i inside $f_i^{-1}(J)$ and is approximately horizontal.
B. $f_1(D_1) \cap f_2(D_2)$ is empty,
C. $f_1(D_1) \cap f_2(D_2)$ is non-empty and the connecting tube has its two critical points at the same level.

See Fig. 9.

We then define $f : D_1 \# D_2 \to \mathbb{R}$ to be the restriction of the height function. Clearly df blends with ω_1 and ω_2 to define a closed 1-form ω on $M_1 \# M_2$. Note that in each of these constructions two new critical points x, y of index 1 are introduced. In construction A, $f(x) < f(y)$; in construction B, $f(x) > f(y)$; in construction C, $f(x) = f(y)$.

If we assume that the ω_i are Calabi then these constructions will have two important properties:

(1) Under construction A, ω is Calabi; under constructions B and C ω is not Calabi.
(2) Under construction A, every leaf of ω intersects at least one leaf of ω_1 and one leaf of ω_2; under construction B there are new compact leaves produced; under construction C there is just one new compact singular leaf component produced.

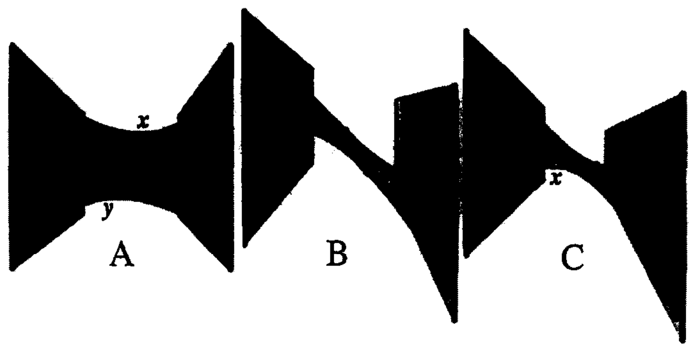

Fig. 9.

We now use these general constructions to produce the four examples promised. In these examples we denote the torus by T and θ, ϕ will denote the usual angle coordinates on T.

Example 1. We construct a Calabi form with some compact leaves whose cohomology class has rank one less than the first Betti number. Let $M_1 = T$ and $\omega_1 = d\theta$ (or even $p\,d\theta + q\,d\phi$ for p, q relatively prime integers). Let M_2, ω_2 be arbitrary—for example ω_2 might be a completely irrational Calabi form. Choose $D_1 \subset M_1$ to be small enough so that many of the leaves of ω_1 do not meet D_1. If we use construction A, the result is a Calabi form of non-maximal rank with some compact leaves. See Fig. 10.

Example 2. We construct a Calabi form with all non-compact leaves whose cohomology class is split but has rank > 1. Let $M_1 = M_2 = T$ and $\omega_1 = d\theta$, $\omega_2 = \lambda\,d\theta$, where λ is irrational, and we use construction A to produce a Calabi form. If the disks D_1 and D_2 are small, then every leaf of ω_1 or ω_2 will hit them at most once and the leaves of ω will all be compact. If we choose the disks differently, though, so that they take the form of thin ribbons winding several times around T, then each leaf will hit the disk several times. The increments Δf_1 between different intersections of the same leaf of ω_1 with D_1 will be integral multiples of 2π. The increments Δf_2 between different intersections of the same leaf of ω_2 with D_2 will be integral multiples of $2\pi/\lambda$. Since λ is irrational, the attachments of leaves of ω_1 with leaves of ω_2 will produce non-compact leaves of ω. See Fig. 11.

Example 3. We construct a non-Calabi form with some compact leaves whose cohomology class is completely irrational. Let M_i and ω_i be arbitrary and use construction B. This will produce a non-Calabi Morse 1-form ω with some compact leaves. In this way choosing ω_i's to be completely irrational with corresponding normalization so that ω will be completely irrational as well, we can realize any cohomology class on any surface except for a completely irrational class on a torus.

Example 4. We construct a non-Calabi non-generic form with all non-compact leaves. This is similar to example 3. Let $M_1 = M_2 = T$ and ω_i be any two forms of the form $a\,d\theta + b\,d\phi$, where a/b is irrational. Then use construction C with small disks D_1 and D_2. This will produce a non-Calabi form with one compact singular leaf component and all non-compact leaves. See Fig. 12.

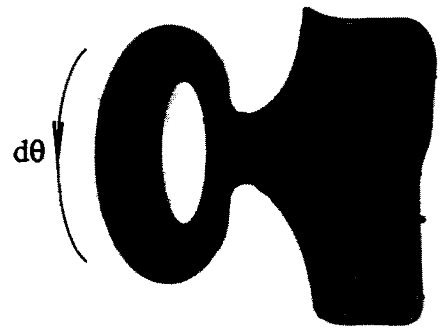

Fig. 10.

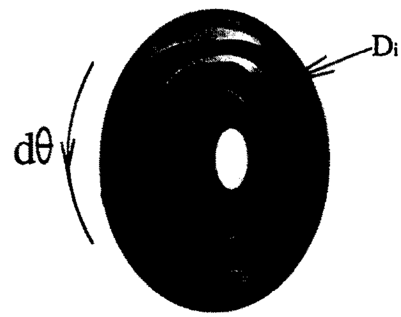

Fig. 11.

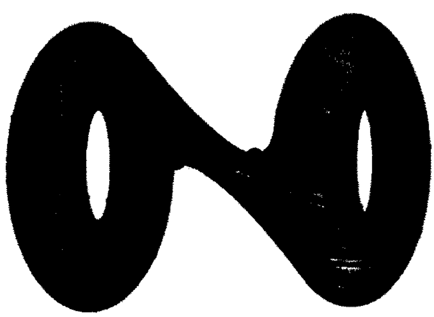

Fig. 12.

10. FINAL REMARKS

Theorems 1 and 2 still leave us with the problem of deciding, in general, whether closed Morse 1-forms are contiguous to harmonic forms.† The next case to consider might be when the form ω has all compact leaves—the issue then reduces to a combinatorial question about the graph Γ_ω. Settling the general compact leaf case and using Theorem 2 might then enable one to use Proposition 1 to deal with the general case.

The results of Sections 8.2 and 8.3 and the examples of Section 9 leave open several interesting questions about the relation between the cohomology class of a closed Morse 1-form and the presence of compact leaves in its foliation.

† Theorem 1 has recently been proved by Ko Monda for any closed Morse 1 form.

(1) What conditions on a cohomology class will assure that it has a representative Morse form with no compact leaves? Example 2 suggests that rank > 1 might be sufficient.
(2) What conditions on a cohomology class are satisfied if *every* representative Morse form has all leaves compact? Example 2 suggests that rank $= 1$ is necessary.
(3) Is there some condition on a cohomology class which implies that *every* representative Morse form has all non-compact leaves? It is not hard to see that, on the torus, rank > 1 will do. Example 1 shows that this is the only case for surfaces. In higher dimensions it is not hard to see that if a cohomology class α is completely irrational and unsplittable, in the sense that it is impossible to separate the manifold M by a codimension one submanifold V so that $\alpha \neq 0$ on each component of $M - V$, and if the pairing $H^1(M) \times H^1(M) \to H^2(M)$ is non-degenerate then α has the desired property.

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