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The area quantum and Snyder space

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Abstract

We show that in the Snyder space the area of the disc and of the sphere can be quantized. It is also shown that the area spectrum of the sphere can be related to the Bekenstein conjecture for the area spectrum of a black hole horizon.

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1. Introduction

One of the fundamental problems in physics is the quantization of gravity. A satisfactory solution to it has not been given yet. Nevertheless, there are signs that the theory must imply discretization of some geometric quantities. For instance, it can be shown from heuristic arguments that the area of the event horizon of a black hole must have a discrete spectrum. The proof follows from Ehrenfest principle [1], which states that to every classical adiabatic-invariant there corresponds a discrete spectrum in its quantum version. Thus, as this area is proportional to the entropy of the black hole, which is a classical adiabatic-invariant, Bekenstein proposed that it must have a discrete spectrum in the quantum level of the form [2,3]

$$A_n = 4\pi r^2 \approx \gamma l_p^2 n, \quad n = 1, 2, \ldots \tag{1}$$

with γ being a proportionality constant and $l_p = \sqrt{\hbar G/c^3}$ the Planck length. It can be seen that (1) is essentially the area of a sphere of radius r.

It is expected that the correct version of quantum gravity be consistent with proposal (1). That implies a space–time consistent with discretization of some geometric quantities in the level of quantum gravity. In this sense, G. ’t Hooft showed that in quantum gravity in $(2+1)$ dimensions a discrete space–time is obtained in an effective fashion [4]. This result suggests that in other dimensions quantum gravity implies discrete spaces. It is worth pointing out that the discrete space obtained by G. ’t Hooft is the so-called Snyder space, which is interesting because it is discrete, noncommutative and compatible with the Lorentz symmetry [5]. In fact, based on it other Lorentz-invariant discrete spaces can be constructed [6–8]. A proposal to quantize gravity based on discrete spaces can be seen in [9].

In this Letter we show that, starting from the Snyder space, the area of the disc and of the sphere can be quantized. We also show that the area spectrum of the sphere can be related to the Bekenstein conjecture (1).

The manuscript is organized as follows: in Section 2 we study some properties of the Snyder space. We then use these properties to calculate the area quantum: of the disc in Section 3 and of the sphere in Section 4. Finally, in Section 5 we summarize our results.

2. The Snyder space

We shall express the Snyder space in terms of the $\zeta^A = (\zeta^0, \zeta^1, \ldots, \zeta^D, \zeta^{D+1})$ variables which are considered in a flat metric with signature $\mathrm{sig}(\eta_{AB}) = (1, -1, \ldots, -1, -1)$. In this fashion, the actual coordinates of the Snyder space [5] and their

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commutation rules are given by

$$\hat{x}^{\mu} = -ia\left(\zeta^{D+1}\frac{\partial}{\partial\zeta_{\mu}} - \zeta^{\mu}\frac{\partial}{\partial\zeta_{D+1}}\right), \tag{2}$$

$$\left[\hat{x}^{\mu}, \hat{x}^{\nu}\right] = ia\hat{l}^{\mu\nu}, \tag{3}$$

$$\hat{l}^{\mu\nu} = -ia\left(\zeta^{\nu}\frac{\partial}{\partial\zeta_{\mu}} - \zeta^{\mu}\frac{\partial}{\partial\zeta_{\nu}}\right), \tag{4}$$

$$\mu, \nu = 0, 1, \ldots, D. \tag{5}$$

where a is a constant with units of length and ζ^{D+1} is a parameter invariant under the $SO(D,1)$ group [5]. In particular, for the $D=3$ case, the commutation rules of the spatial variables can be represented by the matrix

$$C_{ij} = \begin{pmatrix} -\hat{y} & \hat{x} & \hat{l}^{12} \\ -\hat{z} & \hat{y} & \hat{l}^{23} \\ -\hat{x} & \hat{z} & \hat{l}^{31} \end{pmatrix}, \quad \hat{x} = \hat{x}^1,\ \hat{y} = \hat{x}^2,\ \hat{z} = \hat{x}^3. \tag{6}$$

Notice that the rows of this matrix satisfy the commutation rules

$$[C_{li}, C_{lj}] = ia\epsilon_{ijk}C_{lk}. \tag{7}$$

That is, for every pair of spatial variables and their commutator the $SO(3)$ algebra holds, i.e. a noncommutative space of the type of the fuzzy sphere [10].

It can be shown that $\hat{l}^{\mu\nu}$ is the generator of Lorentz transformations in the Snyder space [8]. In fact, for the spatial part of the commutation rules, (3) can be written as

$$[\hat{x}_i, \hat{x}_j] = ia\epsilon_{ijk}\hat{l}_k, \qquad \hat{l}_i = -ia\epsilon_{ijk}\zeta^j\frac{\partial}{\partial\zeta^k}, \tag{8}$$

where $\hat{l}_i$ is the generator of rotations which satisfies

$$[\hat{l}_i, \hat{l}_j] = ia\epsilon_{ijk}\hat{l}_k, \tag{9}$$

$$[\hat{x}_i, \hat{l}_j] = ia\epsilon_{ijk}\hat{x}_k. \tag{10}$$

By using $\hat{l}_i$ and $\hat{x}_j$, one may define the operators

$$\hat{M}_i = \frac{1}{2}(\hat{l}_i + \hat{x}_j), \tag{11}$$

$$\hat{N}_i = \frac{1}{2}(\hat{l}_i - \hat{x}_j), \tag{12}$$

which satisfy

$$[\hat{M}_i, \hat{M}_j] = ia\epsilon_{ijk}\hat{M}_k, \tag{13}$$

$$[\hat{N}_i, \hat{N}_j] = ia\epsilon_{ijk}\hat{N}_k, \tag{14}$$

$$[\hat{N}_i, \hat{M}_j] = 0. \tag{15}$$

That is, the algebra of $SO(3) \times SO(3)$. In this sense the spatial sector of the Snyder space can be regarded as a combination of two fuzzy spheres.

3. Area quantum of the disc

To obtain this quantum we use the spatial part of the commutation rules (3), which are equivalent to (6). In particular, for the x–y plane we define the operators

$$\hat{L}^2 = \hat{x}^2 + \hat{y}^2 + \left(\hat{l}^{12}\right)^2, \tag{16}$$

$$\hat{\mathcal{A}} = \pi\left(\hat{L}^2 - \left(\hat{l}^{12}\right)^2\right) = \pi\left(\hat{x}^2 + \hat{y}^2\right), \tag{17}$$

where $\hat{\mathcal{A}}$ can be identified with the area operator of a disc. By using the fact that the rows of matrix (6) satisfy the $SO(3)$ algebra, Eq. (7), one finds the spectra

$$\hat{L}^2|nm\rangle = L^2_{nm}|nm\rangle = a^2 n(n+1)|nm\rangle, \tag{18}$$

$$\hat{\mathcal{A}}|nm\rangle = \mathcal{A}_{nm}|nm\rangle = \pi a^2\left[n(n+1) - m^2\right]|nm\rangle, \tag{19}$$

$$\text{for } n = 0, 1, 2, \ldots \quad \text{and} \quad -n \leqslant m \leqslant n, \tag{20}$$

so that the area of the disc, A_{nm}, is discrete.

It can be shown that the uncertainty principle

$$\Delta x \Delta y \geqslant \frac{1}{2}\left|[\hat{x}, \hat{y}]\right|, \tag{21}$$

for this case becomes

$$\frac{a^2}{2}\left[n(n+1) - m^2\right] \geqslant \frac{a^2}{2}|m|. \tag{22}$$

Notice that the equality holds when $m = \pm n$, which is the case of the coherent states. Thus, for the coherent states the area of the disc has spectrum

$$\mathcal{A}_{nn} = \pi a^2 n. \tag{23}$$

We remark that, for a given n, the uniformly spaced spectrum holds when $|m|$ reaches its largest value, which occurs when the area is a minimum.

All the previous results hold equally for the x–z and y–z planes. Another noncommutative space that allows to quantize the area of the disc can be seen in [11].

4. Area quantum of the sphere

To get this quantum let us recall that in Minkowski space

$$ds^2 = c^2(dt)^2 - (dr)^2 - r^2\, d^2\Omega, \tag{24}$$

whereas in Schwarzschild space

$$ds^2 = \left(1 - \frac{2MG}{c^2 r}\right)c^2(dt)^2 - \left(1 - \frac{2MG}{c^2 r}\right)^{-1}(dr)^2 - r^2\, d^2\Omega, \tag{25}$$

so that the area of the sphere in each case is

$$A = 4\pi r^2. \tag{26}$$

Therefore, in principle, the spectrum of A must be the same in either the Minkowski or Schwarzschild space. Then, as quantizing A is equivalent to quantizing $r^2 = x^2 + y^2 + z^2$, we use Eqs. (11)–(12) to find

$$\hat{r}^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2 = (\hat{M} - \hat{N})^2 = \hat{M}^2 + \hat{N}^2 - 2\hat{N}\cdot\hat{M}, \tag{27}$$

$$\hat{l}^2 = (\hat{M} + \hat{N})^2 = \hat{M}^2 + \hat{N}^2 + 2\hat{N}\cdot\hat{M}, \tag{28}$$

which can be combined to get

$$\hat{A} = 4\pi\hat{r}^2 = 4\pi\left(2\left(\hat{M}^2 + \hat{N}^2\right) - \hat{l}^2\right). \tag{29}$$

By using the addition rule of two angular momenta one gets to the spectrum [12]

$$A_{l_1 l_2} = 4\pi a^2\left(2\left(l_1(l_1+1) + l_2(l_2+1)\right) - l(l+1)\right), \tag{30}$$

where $l_1, l_2 = 0, 1, 2, 3, \ldots$ and $|l_1 - l_2| \leqslant l \leqslant l_1 + l_2$. Notice that in the case $l_1 = l_2 = n$ and $l = 2l_1 = 2n$, one finds the spectrum

$$A_n = 8\pi a^2 n, \tag{31}$$

which coincides with Bekenstein conjecture (1). Comparison between spectra (31) and (1) suggests that the constant a must be of the order of the Planck length.

Analogously to the disc, for $l_1 = l_2$ the uniformly spaced spectrum is obtained when l is maximum; that is when the area is a minimum.

It should be pointed out that, starting from the Snyder space, Yang constructed a discrete space–time which, apart from being compatible with the Lorentz symmetry, is also compatible with the Poincaré symmetry [6]. The commutation rules in the spatial coordinates of the Yang space coincide with those of the Snyder space and so the results here presented are also valid for the Yang space.

The fact that the Snyder space is compatible with Bekenstein conjecture (1) reinforces the idea that in some limit quantum gravity in $(3+1)$ dimensions implies the Snyder space. Other works relating quantum gravity and noncommutative spaces can be seen in [13].

5. Summary

It was shown that the spatial sector of the Snyder space can be regarded as an array of fuzzy spheres. In addition, the area quantum of the disc and of the sphere were obtained. It was also shown that a uniformly spaced spectrum may be obtained for the sphere which coincides with the Bekenstein conjecture for the area of a black hole horizon. This last result reinforces the idea that in some limit quantum gravity in $(3+1)$ dimensions may imply the Snyder space.

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