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Another Extremal Property of Some Turán Graphs

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An inequality relating the size and order of a simple graph to the average number of triangles containing a fixed edge is proven. It is shown that the only graphs for which equality holds in this inequality are the Turán graphs with the same number of vertices in each partite set.

The aim of our note is to present an inequality for some parameters of graphs, involving the number of triangles based on a given edge, and to describe the structure of graphs for which equality holds. In particular, our Corollary solves a problem raised by Johnson and Perry [3]. It turns out that the extremal graphs for their question are among the well known Turán graphs that appear so frequently in extremal problems (cf. [1, Ch. 6] and [2]).

Throughout, $G = (V, E)$ is a simple graph with $|V| = n$ vertices and $|E| = m$ edges. For $v \in V$, $N(v)$ denotes the set of vertices adjacent to v, and $\mathrm{d}(v) = |N(v)|$ is the degree of v.

For $e = (uv) \in E$ (where $u, v \in V$), let $t(e)$ denote the number of triangles containing e, i.e., $t(e) = |N(u) \cap N(v)|$. We define

$$t_0 = t_0(G) = \max_{e \in E} t(e), \qquad t_1 = t_1(G) = \frac{1}{m} \sum_{e \in E} t(e), \qquad \text{and}$$

$$\gamma = \gamma(G) = \frac{1}{m} \sum_{v \in V} (d(v))^2 = \frac{1}{m} \sum_{(uv) \in E} (d(u) + d(v)).$$

One can see that the two expressions for γ are identical. Moreover, trivially, $t_0 \geqslant t_1$.

It is shown in [3] that $m \leqslant n(n + t_0)/4$. Here we prove a stronger inequality and also give a structural characterization as follows.

THEOREM. *For every graph G,*

$$t_0(G) \geqslant t_1(G) \overset{(1)}{\geqslant} \gamma(G) - n \overset{(2)}{\geqslant} \frac{4m}{n} - n.$$

Equality holds in (1) if and only if G is a complete r-partite graph, for some r. Equality holds in (2) if and only if G is regular.

PROOF.
$$\begin{aligned} t_1 &= \frac{1}{m} \sum_{e \in E} t(e) = \frac{1}{m} \sum_{(uv) \in E} |N(u) \cap N(v)| \\ &= \frac{1}{m} \sum_{(uv) \in E} (d(u) + d(v) - |N(u) \cup N(v)|) \\ &\geqslant \gamma - n = \frac{1}{m} \sum_{v \in V} (d(v))^2 - n \\ &\geqslant \frac{1}{m} \frac{1}{n} \Big(\sum_{v \in V} d(v) \Big)^2 - n = \frac{4m}{n} - n, \end{aligned}$$

by the inequality between the arithmetic and quadratic mean, and since $\Sigma\, d(v) = 2m$. The condition for equality in (1) follows from the proof above and the next lemma.

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LEMMA. *G is a complete r-partite graph, for some r, if and only if $N(u) \cup N(v) = V$ for all pairs of adjacent vertices $u, v \in V$.*

PROOF. The necessity is obvious.

Suppose $N(u) \cup N(v) = V$ whenever $(uv) \in E$. If $x \notin N(u)$ then, for every v adjacent to u, $x \in N(v)$ must hold. This means that $(ux) \notin E$ implies $N(u) \subseteq N(x)$, and also $N(x) \subseteq N(u)$; so $N(x) = N(x')$ if and only if x and x' are non-adjacent. Consequently, 'non-adjacency' is an equivalence relation on V, therefore the complement of G is a disjoint union of some complete graphs.

COROLLARY. *The following are equivalent.*

(a) $t_0 = 4(m/n) - n$,

(b) $t_1 = 4(m/n) - n$,

(c) *For some r, a divisor of n, G is $T_r(n)$, the complete r-partite graph with the same number, n/r, of vertices in each partite class.*

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