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FROM GEOMETRY TO EULER IDENTITIES

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Abstract. We study certain two-dimensional dynamical systems $x_{n+1} = F(x_n, y_n)$; $y_{n+1} = G(x_n, y_n)$.

Contents

1. An exercise

Exercise 7 on page 79 of Luc Moisotte's *1850 Exercices de Mathématiques* [5] begins as follows (see Fig. 1): "Let P and Q be the orthogonal projections of M

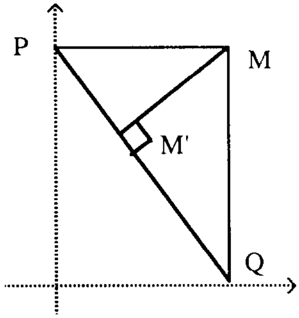

Fig. 1.

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on the axes and let $M' = f(M)$ be the orthogonal projection of M on the line PQ. Suppose that $M(0) = M = M(x, y)$ is given in the first quadrant with $x \neq y$. . ." Thus, at first, we have just an exercise in elementary analytic geometry and in the solution of simultaneous equations. The diagonal PQ is given by $X/x + Y/y = 1$ with slope $-y/x$, so the required orthogonal line through (x, y) is

$$Y - y = \frac{x}{y}(X - x).$$

Solving for the common point $M' = M'(x', y')$ yields

$$x' = \frac{x^3}{x^2 + y^2}, \qquad y' = \frac{y^3}{x^2 + y^2}.$$

Even this is mildly instructive; the initial irrelevance as to which is x and which is y of course entails the symmetry in the result.

But now comes the point (see Fig. 2): "... Show that the sequence of points $M^{(n+1)} = f(M^{(n)})$, $n = 0, 1, 2, \ldots$ always converges on the positive axes and compute its distance from the origin." This is entertaining at a number of levels. One can study the limiting process without actually computing the limit, and, indeed, finding it requires a moment's thought. It turns out to be convenient to set $x > y$ and then $y^{(0)} = 1$ so $0 < x = x^{(0)} < 1$. Plainly, $x^{(\infty)} = 0$ so $M^{(\infty)}$ is on the Y-axis and one finds that

$$y^{(\infty)} = \prod_{n=0}^{\infty} (1 + x^{2.3^n})^{-1}.$$

This result is of itself amusing. Viewing its denominator, $f(x)$ say, as a power series in x, one has $f(x) = \sum_{n \in E} x^n$ with only those powers n of x appearing with n represented in base 3 using only the digits 0 and 2; f is the *Cantor series*, E the set of *Cantor integers*. It happens to be known [1] (actually, already [4]) that $y^{(\infty)}$ is a transcendental number for every algebraic x with $0 < |x| < 1$; say, in particular if

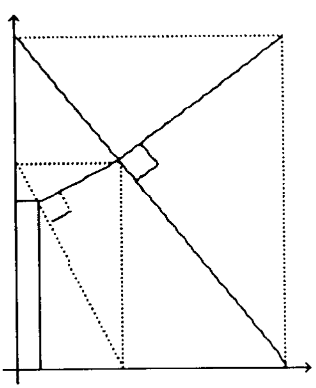

Fig. 2.

$x=1/10$. Of course, it is well-known that almost every real number is transcendental but one also knows well that, mostly, it is extraordinarily difficult to prove any given number to be transcendental. In the survey *FOLDS*! [8] there is mention of and reference to the benefit of viewing a series such as $f(x)$ as a formal series over a finite field. Indeed, in characteristic 3 we have $y^{(\infty)}(x)=(1+x^2)^{1/2}$.

Algebraic functions in positive characteristic have their coefficients generated by finite automata. Series such as f are hypertranscendental—they do not even satisfy an algebraic differential equation. Thus there is no pleasant curve containing all the $M^{(n)}$.

2. A simplification

Suppose our geometrical problem had yielded a simpler, yet similar, sequence defined by

$$x'=\frac{x^2}{x+y},\qquad y'=\frac{y^2}{x+y}.$$

Then we see that it happens (as before, with $y^{(0)}=1$, $0<x=x^{(0)}<1$) that

$$y'-x'=y-x=1-x^{(0)};$$

so, plainly (since $x^{(\infty)}=0$), $y^{(\infty)}=1-x$. Of course, we had $x'/y'=(x/y)^2$ and $y'/y=(1+x/y)^{-1}$ so we have, almost for free as it were,

$$\prod_{n=0}^{\infty}(1+x^{2^n})=(1-x)^{-1}.$$

This is Euler's identity. This is easy enough to see by just multiplying by $1-x$, but it is noticeably more elegant to remark, recalling that $(1-x)^{-1}=\sum_{n=0}^{\infty}x^n$, that it expresses the fact that each nonnegative integer has a unique binary representation. Analogously, recall that

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s}=\prod_{p}(1-p^{-s})^{-1}\quad\text{(product over all primes } p)$$

expresses the fact that each positive integer has a unique factorization into primes.

3. Extensions

Inspired by this we try some other transformations.

3.1. Example. Suppose

$$y'=\frac{y^2}{x+y}\quad\text{and}\quad x'=-\frac{x(x+2y)}{x+y}.$$

Then

$$3y'+x'=\frac{(3y+x)(y-x)}{x+y}, \qquad y'-x'=\frac{y^2+2xy+x^2}{x+y}=y+x;$$

so, congenially,

$$(3y'+x')(y'-x')=(3y+x)(y-x)=(3+x^{(0)})(1-x^{(0)})$$

with, as above, $y^{(0)}=1$ and $x=x^{(0)}$. More generally,

$$(3y^{(n)}+x^{(n)})(y^{(n)}-x^{(n)})=(3+x)(1-x). \tag{1}$$

We have

$$\frac{x'}{y'}=-\frac{x}{y}\left(\frac{x}{y}+2\right):=\varphi\left(\frac{x}{y}\right) \quad \text{and} \quad \frac{y'}{y}=\frac{1}{1+x/y};$$

hence,

$$x^{(n)}=(-1)^n x\prod_{k=0}^{n-1}\left(1+\frac{1}{1+\varphi^k(x)}\right), \qquad y^{(n)}=\prod_{k=0}^{n-1}\frac{1}{1+\varphi^k(x)},$$

where $\varphi^k=\varphi(\varphi^{k-1})$ represents the kth iterate of $\varphi(x)=-x(x+2)$.

Now suppose $x\notin\,]-3,1[$. Then $|\varphi^k(x)|\geq 3$ for all $k\geq 1$, hence $y^{(n)}$ vanishes as n increases to infinity. Equation (1) then implies

$$-(x^{(\infty)})^2=(3+x)(1-x), \qquad (x^{(\infty)})^2=(x-1)(x+3);$$

hence,

$$x^2\prod_{k=0}^{\infty}\left(1+\frac{1}{1+\varphi^k(x)}\right)^2=(x-1)(x+3).$$

Finally, for all $x\notin\,]-3,1[$,

$$\prod_{k=0}^{\infty}\left(1+\frac{1}{1+\varphi^k(x)}\right)=\sqrt{\left(1-\frac{1}{x}\right)\left(1+\frac{3}{x}\right)}.$$

3.2. Example. We now consider the transformation

$$y'=\frac{y^2}{x+y}, \qquad x'=\frac{x^2y^2}{(x+y)(2y^2-x^2)}.$$

With a little care (and foreknowledge) we readily see that

$$y'^2\frac{y'-x'}{y'+x'}=y^2\frac{y-x}{y+x}=\frac{1-x^{(0)}}{1+x^{(0)}}.$$

But

$$\frac{y'}{x'}=2\left(\frac{y}{x}\right)^2-1:=\varphi\left(\frac{y}{x}\right) \quad \text{and} \quad \frac{y}{y'}=1+\frac{1}{y/x};$$

so if $x^{(\infty)} = 0$,

$$\frac{1}{y^{(\infty)}} = \prod_{k=0}^{\infty}\left(1 + \frac{1}{\varphi^k(x^{-1})}\right) = \sqrt{\frac{1+x}{1-x}}$$

with $\varphi(x) = 2x^2 - 1$. More elegantly, replacing x by $1/x$ we obtain for all $|x| > 1$

$$\prod_{k=0}^{\infty}\left(1 + \frac{1}{\varphi^k(x)}\right) = \sqrt{\frac{x+1}{x-1}}\,.$$

3.3. Example. Set

$$y' = \frac{x+y}{x-y}\,y, \qquad x' = x\,\frac{x+y}{x-y}\cdot\frac{x^2-3y^2}{y^2}$$

and verify that

$$y'^2\,\frac{x'+2y'}{x'-2y'} = y^2\,\frac{x+2y}{x-2y} = \frac{x^{(0)}+2}{x^{(0)}-2}\,.$$

Now

$$\frac{x'}{y'} = \left(\frac{x}{y}\right)^3 - 3\left(\frac{x}{y}\right) =: \varphi\left(\frac{x}{y}\right) \quad \text{and} \quad \frac{y'}{y} = \frac{x/y+1}{x/y-1}$$

Thus, if $x^{(\infty)} = \infty$ (e.g., $|x| > 2$),

$$\prod_{k=0}^{\infty}\frac{\varphi^n(x)+1}{\varphi^n(x)-1} = \sqrt{\frac{x+2}{x-2}}, \quad \varphi(x) = x^3 - 3x.$$

3.4. Example. With

$$y' = \frac{x+y}{x}\,y, \qquad x' = \frac{(x+y)(2y^2-x^2)}{xy}$$

we see that

$$y'^2\,\frac{x'^2-4y'^2}{(x'-y')^2} = y^2\,\frac{x^2-4y^2}{(x-y)^2} = \frac{(x^{(0)})^2-4}{(x^{(0)}-1)^2}\,.$$

Hence, if $\varphi(x) = 2 - x^2$,

$$\prod_{k=0}^{\infty}\left(1 + \frac{1}{\varphi^k(x)}\right) = \frac{\sqrt{x^2-4}}{|x-1|} \quad \text{provided } |x| \geqslant 2.$$

3.5. Example. As a final example, take

$$y' = y\,\frac{x+y}{x}, \qquad x' = \frac{(x+y)(x^2+2xy-2y^2)}{xy}\,.$$

Then

$$y'^2 \frac{(x'+2y')^2}{(x'+3y')(x'-y')} = y^2 \frac{(x+2y)^2}{(x+3y)(x-y)} = \frac{(x^{(0)}+2)^2}{(x^{(0)}+3)(x^{(0)}-1)}.$$

We easily conclude that for $\varphi(x)=(x+1)^2-3$ and $x \notin [-3, 1]$

$$\prod_{k=0}^{\infty}\left(1+\frac{1}{\varphi^k(x)}\right) = \frac{|x+2|}{\sqrt{(x+3)(x-1)}}.$$

The principle we have just employed is one of stumbling upon a pleasant curve $f(X, Y)=f(x, 1)$ containing all the points $M^{(n)}=(x^{(n)}, y^{(n)})$, thus making it an easy matter to compute either $x^{(\infty)}$ or $y^{(\infty)}$ by $f(x^{(\infty)}, y^{(\infty)})=f(x, 1)$ once one of them is known. In Example 3.1 we found $y^{(\infty)}=0$, while in Example 3.2, $x^{(\infty)}=0$. In the three last examples, $x^{(\infty)}=\infty$.

The general pattern of computing the above products is described as follows. Consider the transformation

$$y'=yH\left(\frac{x}{y}\right), \qquad x'=xK\left(\frac{x}{y}\right)$$

and define $\varphi(t)=t\,K(t)/H(t)$. Then

$$x^{(\infty)}=x\prod_{k=0}^{\infty} K(\varphi^k(x)), \qquad y^{(\infty)}=\prod_{k=0}^{\infty} H(\varphi^k(x))$$

presuming both infinite products make sense.

4. Ostrowski's work

Of course, our examples were not selected quite at random. They are taken from a work of Ostrowski [7] who studies the question of finding rational functions $\varphi(x)$ so that

$$\prod_{n=0}^{\infty} (1+\varphi^n(x)) = \Psi(x) \tag{2}$$

is an algebraic function of x. In particular, Ostrowski determines all φ so that $\varphi(x)$ or $\varphi(x^{-1})$ is either a polynomial or the reciprocal of a polynomial; we have cited the more interesting examples.

At first it seems surprising that, with φ a rational function, Ψ is necessarily the rth root of a rational function. But this is quite easy to see: Suppose that Ψ is a zero of the irreducible polynomial

$$y^r + a_1(x)y^{r-1} + \cdots + a_r(x) \tag{3}$$

with rational functions $a_i(x)$ as coefficients. From (2) we notice that

$$\Psi(\varphi(x)) = \prod_{n=0}^{\infty} (1+\varphi^{n+1}(x)) = \frac{\Psi(x)}{(1+x)}.$$

On replacing x by $\varphi(x)$ in (3), and multiplying by $(1+x)^r$, we obtain

$$y^r + a_1(\varphi(x))(1+x)y^{r-1} + \cdots + a_r(\varphi(x))(1+x)^r. \tag{4}$$

But (4) is again a polynomial of degree r with $\Psi(x)$ as a zero. Thus it must coincide with (3). In particular, we have

$$\frac{a_r(x)}{a_r(\varphi(x))} = (1+x)^r.$$

Replacing x by $\varphi(x)$, multiplying, and iterating that procedure yields

$$a_r(x) = a_r(\varphi^{(\infty)}(x))(\Psi(x))^r.$$

But $\varphi^{(\infty)}(x) = 0$ by hypothesis. It also follows that $a_r(0) \neq 0$, indeed that $a_r(0) = -1$ since, evidently, $y^r + a_r(x)$ is the polynomial we introduced above.

It turns out that Ostrowski's "interesting" examples are all with $r = 2$. He points out that our Example 3.2 (due to F. Engel [3]) yields a nice approximation

$$\sqrt{2} = \left(1+\frac{1}{3}\right)\left(1+\frac{1}{17}\right)\left(1+\frac{1}{577}\right)\left(1+\frac{1}{665857}\right)\cdots;$$

Example 3.3 provides the very rapidly converging

$$\sqrt{5} = \left(1+\frac{2}{2}\right)\left(1+\frac{2}{17}\right)\left(1+\frac{2}{5777}\right)\left(1+\frac{2}{192900153617}\right)\cdots$$

Evidently, in this case each truncation yields a convergent (in the sense of the theory of regular continued fractions) to $\sqrt{5}$. Example 3.4 provides

$$\frac{1}{2}\sqrt{5} = \left(1+\frac{1}{3}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{47}\right)\left(1-\frac{1}{2207}\right)\left(1-\frac{1}{4870847}\right)\cdots$$

An example we omitted to mention:

$$\prod_{n=0}^{\infty}\left(1+\frac{1}{\varphi^n(x)}\right) = \sqrt{\frac{x+2}{x}} \quad \text{with } \varphi(x) = -2(x+1)^2$$

is easily obtained from Example 3.2 or, of course, directly by our approach. It yields

$$\sqrt{3} = 2\left(1-\frac{1}{8}\right)\left(1-\frac{1}{98}\right)\left(1-\frac{1}{18818}\right)\cdots$$

Ostrowski remarks that Cantor [2] shows that each positive number has a unique representation as a product

$$\prod_{n=0}^{\infty}\left(1+\frac{1}{\varphi^n(x)}\right)$$

in positive integers $x_0 \geqslant 2$ and $\varphi(x) \geqslant x^2$. Cantor noticed representations of certain quadratic irrationals if $\varphi(x) = 2x^2 - 1$ and suggested that all quadratic irrationals might arise in this way. Engel proved (as we have in Example 3.2) that only quadratic irrationals are produced. However, Example 3.5 proves that, contrary to Cantor, certain quadratic irrationals arise from $\varphi(x) = (x+1)^2 - 3$ which, with $\varphi^n(x) > 1$, has $\varphi^{n+1}(x) \geqslant (\varphi^n(x))^2$ and is essentially different.

Acknowledgment

We are indebted to Pierre Terracher for showing one of us the exercise which motivate the present article.

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[8] M. Dekking, M. Mendès France and A.J. van der Poorten, Folds! I, II and III, *Math. Intelligencer* **4** (1982) 130–138, 173–181 and 190–195.

[1] Amusingly (and no doubt by complete coincidence...), "Luc Moisotte" happens to be an anagram of "Louis Comtet", moving us to mention Comtet's excellent book *Combinatorial Analysis* (Reidel, 1974) which is a wonderful compendium of compactly presented facts of counting theory and a minefield of fascinating exercises: see, for example, a footnote in *Math. Intelligencer* **1** (1979) 195–203 concerning the sum $\sum 1/n^4\binom{2n}{n} = \frac{17}{36}\zeta(4)$.