

Solving PDEs with the aid of two-dimensional Haar wavelets

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ABSTRACT

Two-dimensional Haar wavelets are applied for solution of the partial differential equations (PDEs). The proposed method is mathematically simple and fast. To demonstrate the efficiency of the method, two test problems (solution of the diffusion and Poisson equations) are discussed. Computer simulation showed that the method guarantees the necessary exactness already for a small number of grid points.

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1. Introduction

Wavelet methods have been applied for solving partial differential equations (PDE-s) from beginning of the early 1990s [1,2]. In the last two decades this problem has attracted great attention and numerous papers about this topics have been published. Due to this fact we must confine somewhat our analysis; in the following only PDEs of mathematical physics (elliptic, parabolic and hyperbolic equations) and of elastostatics are considered. From the first field of investigation the papers [3–8] can be cited. As to the elasticity problems we refer to the papers [9–15]. In all these papers different wavelet families have been applied.

In most cases the wavelet coefficients were calculated by the Galerkin or collocation method, by it we have to evaluate integrals of some combinations of the wavelet functions (called also connection coefficients). This is quite a complicated problem, since for most wavelet families an explicit expression is missing. For the Daubechies wavelets formulas for calculation the connection coefficients are derived [16], but only in approximation form and for the first and second order. Connection coefficients for some other families (Shannon and harmonic wavelets) are obtained by Cattani [17] in a finite form and for any power.

Among all the wavelet families the Haar wavelets deserve special attention. They are made up of pairs of piecewise constant functions and are therefore mathematically the simplest of all the wavelet families. A good feature of the Haar wavelets is also the possibility to integrate these wavelets analytically in arbitrary times. A drawback of these wavelets is their discontinuity; since the derivatives do not exist in the breaking points it is not possible to apply these wavelets directly for solving PDEs.

There are two ways for getting out of this situation. One possibility is to regularize the Haar wavelets with interpolating splines (e.g. *B*-splines or Deslauer–Dubuc splines). This approach was realized by Cattano [17] and by some other researchers [1,15]. Another method of solution was proposed by Chen and Hsiao [18] in 1997. They recommended to expand into the Haar series not the function, but its highest derivative appearing in the differential equation; other derivatives and the function itself are obtained through integration. All these ingredients are then integrated into the whole system, which is discretized by the Galerkin's or collocation method. In this way we can get finite formulas for calculating the integrals of

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wavelets and for the connection coefficients. We have applied this technique for solving different 1-D problems [19–22]. The method is fast and with low error.

For calculating the wavelet coefficients we have to solve a linear system of algebraic equations. In the case of a great number of calculation points this system may be nearly singular and difficulties for calculating the inverse matrix appear. To overcome this problem Majak et al. [14] recommended to use a weak formulation based solution, which to some extent improves the stability of the system.

As to the 2-D problems then in some cases it is possible to reduce them to the 1-D problems. This can be done e.g. for the evolution problems, where the governing equation has the form

$$\frac{\partial u}{\partial t} = f \left(t, x, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots \right). \quad (1)$$

For integrating this equation the derivative on the left side is discretized by some finite differences type approximation; integration with regard to the spacial coordinate x is carried out by the wavelet method. This technique is applied for solving different PDEs such as diffusion equation [1,2,4,22], Helmholtz equation [3], Burgers equation [2,14,22], Sine–Gordon equation [22], and Fisher's equation [7] (this list is not complete).

The aim of the present paper is to develop a 2-D Haar wavelet method for solving PDEs, which is fast, mathematically simple and guarantees the necessary accuracy for a relatively small number of grid points. The paper is organized as follows. In Section 2 formulas for calculating the Haar wavelets and their integrals are reported. The method of solution is described in Section 3. In Sections 4 and 5 two test problems – integration of the diffusion and Poisson equations – are solved. Conclusions and possible further directions of research are offered in Section 6.

2. Haar wavelets

To make the paper self-contained some results of [19] are referred in this Section. Consider the interval $x \in [A, B]$, where A and B are given constants. We shall define the quantity $M = 2^J$, where J is the maximal level of resolution. The interval $[A, B]$ is participated into $2M$ subintervals of equal length; the length of each subinterval is $\Delta x = (B - A)/(2M)$. Next two parameters are introduced: the dilatation parameter $j = 0, 1, \dots, J$ and the translation parameter $k = 0, 1, \dots, m - 1$ (here the notation $m = 2^j$ is introduced). The wavelet number i is identified as $i = m + k + 1$.

The i -th Haar wavelet is defined as

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\xi_1(i), \xi_2(i)], \\ -1 & \text{for } x \in [\xi_2(i), \xi_3(i)], \\ 0 & \text{elsewhere,} \end{cases} \quad (2)$$

where

$$\begin{aligned} \xi_1(i) &= A + 2k\mu\Delta x, & \xi_2(i) &= A + (2k + 1)\mu\Delta x, \\ \xi_3(i) &= A + 2(k + 1)\mu\Delta x, & \mu &= M/m. \end{aligned} \quad (3)$$

The case $i = 1$ corresponds to the scaling function: $h_1(x) = 1$ for $x \in [A, B]$ and $h_1(x) = 0$ elsewhere.

If we want to solve a n -th order PDE we need the integrals

$$p_{\alpha,i}(x) = \underbrace{\int_A^x \int_A^x \dots \int_A^x}_{\alpha\text{-times}} h_i(t) dt^\alpha = \frac{1}{(\alpha - 1)!} \int_A^x (x - t)^{\alpha-1} h_i(t) dt \quad \alpha = 1, 2, \dots, n, \quad i = 1, 2, \dots, 2M. \quad (4)$$

The case $\alpha = 0$ corresponds to function $h_i(t)$.

Taking account of (2) these integrals can be calculated analytically; by doing it we obtain

$$p_{\alpha,i}(x) = \begin{cases} 0 & \text{for } x < \xi_1(i), \\ \frac{1}{\alpha!} [x - \xi_1(i)]^\alpha & \text{for } x \in [\xi_1(i), \xi_2(i)], \\ \frac{1}{\alpha!} \{ [x - \xi_1(i)]^\alpha - 2[x - \xi_2(i)]^\alpha \} & \text{for } x \in [\xi_2(i), \xi_3(i)], \\ \frac{1}{\alpha!} \{ [x - \xi_1(i)]^\alpha - 2[x - \xi_2(i)]^\alpha + [x - \xi_3(i)]^\alpha \} & \text{for } x > \xi_3(i). \end{cases} \quad (5)$$

These formulas hold for $i > 1$. In the case $i = 1$ we have $\xi_1 = A$, $\xi_2 = \xi_3 = B$ and

$$p_{\alpha,1}(x) = \frac{1}{\alpha!} (x - A)^\alpha. \quad (6)$$

For solving boundary value problems we need the values $p_{\alpha,i}(B)$, which can be calculated from (5). In special cases $\alpha = 1$ or $\alpha = 2$ we find

$$q_1(i) = p_{1,i}(B) = \begin{cases} B - A & \text{for } i = 1 \\ 0 & \text{for } i \neq 1 \end{cases} \tag{7}$$

and

$$q_2(i) = p_{2,i}(B) = \begin{cases} 0.5(B - A)^2 & \text{for } i = 1 \\ 0.25 \frac{(B - A)^2}{m^2} & \text{for } i \neq 1. \end{cases} \tag{8}$$

In the present paper the collocation method for solving the PDEs is applied. The collocation points are $x_l = 0.5[\tilde{x}_{l-1} + \tilde{x}_l]$, $l = 1, 2, \dots, 2M$; the symbol \tilde{x}_l denotes the l -th grid point $\tilde{x}_l = A + l\Delta x$, $l = 1, 2, \dots, 2M$.

Eqs. (2) and (5) are discretized by replacing $x \rightarrow x_l$. It is convenient to introduce the Haar matrices $H(i, l) = h_i(x_l)$, $P_v(i, l) = p_{v,i}(x_l)$. In the following Sections computer simulations were carried out with the aid of the *Matlab* programs for which the matrix representation is effective.

3. Problem statement and method of solution

Consider the linear PDE

$$\sum_{\gamma=0}^{\Gamma} \sum_{\lambda=0}^{\Lambda} D_{\gamma\lambda} \frac{\partial^{(\gamma+\lambda)} u}{\partial x^\gamma \partial y^\lambda} = f(x, y), \tag{9}$$

where Γ, Λ are given constants and $D_{\gamma\lambda}, f$ are prescribed functions. The independent variables x, y belong to a domain Σ , which has the boundary σ . We have to calculate the function $u(x, y)$, which satisfies the required boundary conditions. For simplicity sake we confine ourselves to problems, where the domain Σ is a rectangle $x \in [0, L_1], y \in [0, L_2]$ and it divides the intervals $[0, L_1], [0, L_2]$ into $2M_1$ and $2M_2$ parts of equal length, respectively. According to the Haar wavelet method the solution is sought in the form

$$\frac{\partial^{(\Gamma+\Lambda)} u}{\partial x^\Gamma \partial y^\Lambda} = \sum_{i=1}^{2M_1} \sum_{l=1}^{2M_2} a_{il} h_i(x) h_l(y), \tag{10}$$

where a_{il} are the wavelet coefficients and $h_i(x), h_l(y)$ the Haar functions. By multiple integrating (10) the lower order derivatives and the function u itself are calculated. In this procedure unknown functions $\varphi_1(x), \varphi_2(x), \dots, \varphi_{\Gamma-1}(x), \psi_1(y), \psi_2(y), \dots, \psi_{\Lambda-1}(y)$ appear; they are calculated from the boundary conditions. We satisfy (10) in the collocation points x_r, y_s , where $r \in [1, 2M_1], s \in [1, 2M_2]$. By doing this we get the following system of linear equations

$$\sum_{i=1}^{2M_1} \sum_{l=1}^{2M_2} R_{ilrs} = f(x_r, y_s), \tag{11}$$

from which the wavelet coefficients a_{il} can be calculated. Since we do not possess algorithms for dealing with the fourth-order matrices we must transform the system into a form, where only second-order matrices appear. This can be done by introducing new indices

$$\alpha = 2M_1(i - 1) + l, \quad \beta = 2M_2(r - 1) + s. \tag{12}$$

Now (11) obtains the form

$$\sum_{\alpha=1}^{2M_1} \sum_{\beta=1}^{2M_2} b(\alpha) S(\alpha, \beta) = F(\beta). \tag{13}$$

Here b and F are $2M_1 * 2M_2$ dimensional row vectors and S is a $(2M_1)^2 * (2M_2)^2$ dimensional matrix. It is expedient to put (14) into the matrix form

$$bS = F. \tag{14}$$

After evaluating the coefficients $b(\alpha)$ from (14) it is not difficult to restore the original matrix of the wavelet coefficient matrix a_{il} . According to (12) we have $\alpha / (2M_1) = i - 1 + l / (2M_1)$. The integer part of this expression gives $i - 1$, the remainder of this division—the index l .

By integrating (10) Γ -times in regard to x and Λ -times in regard to y , we obtain

$$u(x, y) = \sum_{i=1}^{2M_1} \sum_{l=1}^{2M_2} a_{il} p_\Gamma(x) p_\Lambda(y) + \Psi(x, y). \tag{15}$$

In this formula the integrals $p_\Gamma(x)$ and $p_\Lambda(y)$ are calculated according to (5) the function $\Psi(x, y)$ incorporates the functions $\varphi_1(x), \varphi_2(x), \dots, \psi_1(y), \psi_2(y), \dots$ appearing in the course of integration of (10). Details of this method are explained by solving two problems in Sections 4 and 5.

4. Diffusion equation

Solve the equation

$$\frac{\partial u}{\partial t} = A \frac{\partial^2 u}{\partial x^2}, \quad (x, t) \in [0, 1] \quad (16)$$

for the initial condition $u(x, 0) = g(x)$ and boundary conditions $u(0, t) = u(1, t) = 0$.

Let us confine to the case $M_1 = M_2 = M$. The wavelet solution is sought in the form

$$\frac{\partial^3 u}{\partial t \partial x^2} = \sum_{i=1}^{2M} \sum_{l=1}^{2M} a_{il} h_i(x) h_l(t). \quad (17)$$

By multiple integration and considering the initial condition we find

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \sum_i \sum_l a_{il} h_i(x) p_{1l}(t) + \frac{\partial^2 g}{\partial x^2} \\ \frac{\partial u}{\partial t} &= \sum_i \sum_l a_{il} p_{2i}(x) h_l(t) + x \frac{\partial^2 u}{\partial t \partial x} \Big|_{x=0} + \frac{\partial u}{\partial t} \Big|_{x=0}. \end{aligned} \quad (18)$$

For simplicity's sake here and in the following the limits of summation are not written out. In view of the boundary conditions we have

$$\frac{\partial u}{\partial t} \Big|_{x=0} = \frac{\partial u}{\partial t} \Big|_{x=1} = 0. \quad (19)$$

It follows for $x = 1$ from (18)

$$\frac{\partial^2 u}{\partial t \partial x} \Big|_{x=0} = - \sum_i \sum_l a_{il} q_2(i) h_l(t). \quad (20)$$

Replacing this result back into (18) we obtain

$$\frac{\partial u}{\partial t} = \sum_i \sum_l a_{il} [p_{2i}(x) - x q_2(i)] h_l(t). \quad (21)$$

These results are substituted into (16); satisfying the obtained equation in the collocation points x_r, y_s we get the system

$$\sum_i \sum_l a_{il} R_{ilrs} = g''(x), \quad (22)$$

where

$$R_{ilrs} = [P_2(i, r) - x_r q_2(i)] H(l, s) - A H(i, l) P_1(r, s). \quad (23)$$

Introducing the indices α, β according to (12) this result obtains the form (13) or (14). The wavelet coefficients a_{ij} are calculated in the way indicated in Section 3. For evaluating the function $u(x, y)$ we integrate (21) with regard to t and obtain

$$u(x, t) = \sum_i \sum_l a_{il} [p_{2i}(x) - x q_2(i)] p_{1l}(t) + g(x). \quad (24)$$

Example 1. Computer simulation was carried out for $A = 0.2$ and $g(x) = x(1 - x)$. For J the values 2, 3, 4 were taken. The solution for $J = 3$ is plotted in Fig. 1.

It is interesting to compare our solution with other results. A solution based on a Fourier transform is [21]:

$$u(x, t) = \sum_n d_n \exp(-\pi^2 n^2 A t) \sin(n\pi x), \quad (25)$$

where $d_n = [2/(n\pi)]^3, n = 1, 3, 5, \dots$

These two solutions were compared in the collocation point $[x(M), y(M)]$. The difference of $u[x(M), y(M)]$ was $5.4E-4$ for $J = 2$, $1.4E-4$ for $J = 3$ and $3.5E-5$ for $J = 4$. So we state that already for $J = 2$ (64 collocation points) the accuracy of the Haar wavelet solution is rather good.

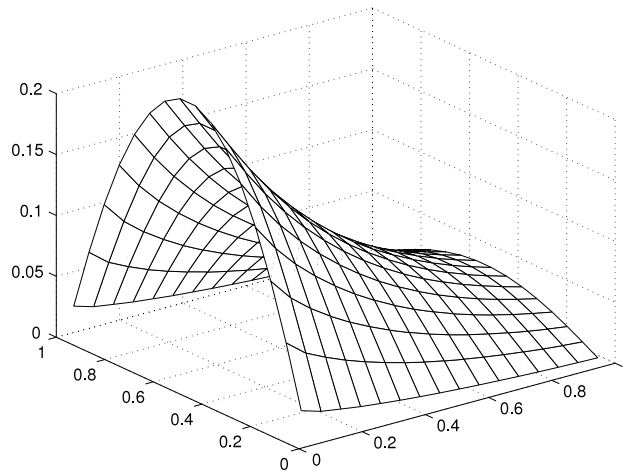


Fig. 1. Solution of the diffusion equation for $J = 3$.

5. Poisson equation

Consider the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \tag{26}$$

on a square $x \in [0, 1], y \in [0, 1]$ with the boundary conditions $u(x, 0) = u(0, y) = u(x, 1) = 0, u(1, y) = g(y)$.

The solution is started by assuming

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} = \sum_i \sum_l a_{il} h_i(x) h_l(y). \tag{27}$$

By integrating this equation twice with regard to x and twice with regard to y we find

$$\frac{\partial^2 u}{\partial x^2} = \sum_i \sum_l a_{il} h_i(x) p_{2l}(y) + y \varphi_1''(x) + \varphi_2''(x) \tag{28}$$

$$\frac{\partial^2 u}{\partial y^2} = \sum_i \sum_l a_{il} p_{2l}(x) h_l(y) + x \psi_1''(y) + \psi_2''(y).$$

By multiple integration we obtain

$$u(x, y) = \sum_i \sum_l a_{il} p_{2i}(x) p_{2l}(y) + y \varphi_1(x) + \varphi_2(x) + x \psi_1(y) + \psi_2(y). \tag{29}$$

Next the boundary conditions are satisfied:

- (i) It follows from $u(x, 0) = 0$ that $\varphi_2(x) = -x \psi_1(0) - \psi_2(0)$.
- (ii) The condition $u(0, y) = 0$ gives $\psi_2(y) = -y \varphi_1(0) - \varphi_2(0)$.
- (iii) Satisfying the condition $u(x, 1) = 0$ we obtain

$$\varphi_1(x) = - \sum_i \sum_l a_{il} p_{2i}(x) q_{2l}(1) + \varphi_1(0) - x[\psi_1(1) - \psi_1(0)].$$

- (iv) In view of $u(1, y) = g(y)$ we find

$$\psi_1(y) = - \sum_i \sum_l a_{il} q_{2l}(i) [-p_{2i}(y) + y q_{2l}(1)] + \psi_1(0) + y[\psi_1(1) - \psi_1(0)] + g(y).$$

Now (28) obtain the form

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \sum_i \sum_l a_{il} h_i(x) [p_{2l}(y) - y q_{2l}(1)] \\ \frac{\partial^2 u}{\partial y^2} &= \sum_i \sum_l a_{il} [p_{2l}(x) - x q_{2l}(i) h_l(y)] + x g''(y). \end{aligned} \tag{30}$$

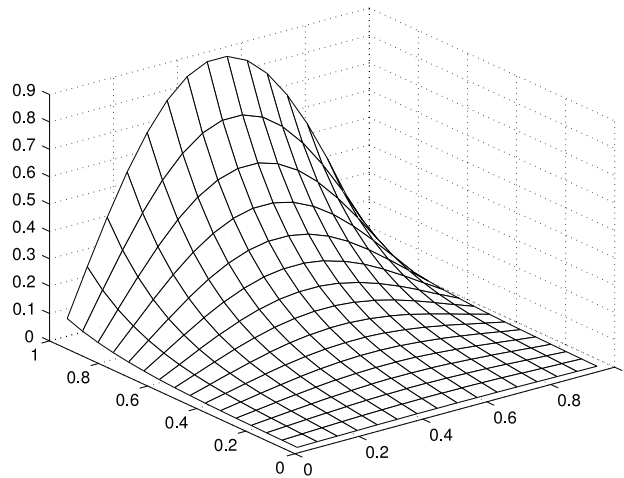


Fig. 2. Solution of the Poisson equation for $J = 3$.

Substituting this result into (26) and satisfying the obtained equation in the collocation points we find

$$u(x, y) = \sum_i \sum_l a_{il} R_{ilrs} = F(r, s) \tag{31}$$

where

$$\begin{aligned} R_{ilrs} &= H(i, r)P_2(l, s) + P_2(i, r)H(l, s) - H(i, r)q_2(l)y_s - q_2(i)x_r H(l, s) \\ F(r, s) &= f(x_r, y_s) - x_r g''(y_s). \end{aligned} \tag{32}$$

For solving this system again the technique of Section 3 is applied. The final result is

$$u(x, y) = \sum_i \sum_l a_{il} \{p_{2i}(x)[p_{2l}(y) - yq_2(l)] + xq_2(i)[-p_{2l}(y) + yq_2(l)]\} + g(y). \tag{33}$$

Example 2. Solve (26) for $g = 0$ and

$$f(x, y) = 6xy[y^2(1 - 2x)(1 - y) - x^2(1 - 2y)(1 - x)]. \tag{34}$$

The exact solution for this case is

$$u_{ex} = x^3 y^3 (1 - x)(1 - y). \tag{35}$$

For estimating the accuracy of the wavelet results the matrix norm from the MATLAB library $\Delta(\eta) = \text{norm}(u - u_{ex}, \eta)$ is used; the case $\eta = 2$ gives the largest column sum, the case $\eta \rightarrow \text{inf}$ —the largest row sum. In the present case due to symmetry both norms are equal and we can estimate the accuracy of our solution by the parameter $\delta = \Delta(2)/(2M)$. Computer simulation gave the following error estimates: $\delta = 9.4E - 5$ for $J = 2$, $\delta = 2.4E - 5$ for $J = 3$, $\delta = 6.0E - 6$ for $J = 4$.

Example 3. Consider the case $f(x) = 3x^2$, $g(y) = A \sin(\pi y)$ The solution for $J = 3$ is plotted in Fig. 2. In the present case we do not know the exact solution, therefore to estimate the accuracy of the solution by the curves $u(x, 0.5)$ and $u(0.5, y)$, which are plotted in Fig. 3 for different values of J ; it turned out that already for $J = 2$ and $J = 3$ these curves visually coincide. In addition to it the value of $V = u(0.5, 0.5)$ was also computed and we found that $V = 0.13494$ for $J = 2$, $V = 0.13542$ for $J = 3$, $V = 0.13560$ for $J = 4$. So we can state that already the solution $J = 2$ (64 grid points) guarantees the necessary exactness in most cases.

6. Conclusions

The main benefits of the proposed method are its simplicity (already a small number of grid points guarantees the necessary accuracy) and universality (the same approach is applicable for a wide class of PDEs). The method is very convenient for solving boundary value problems, since the boundary conditions are taken into account automatically. For numerical calculations useful are the matrix programs of MATLAB. The most time-consuming procedure is to calculate the integrals (5); for this purpose it is suitable to put together an universal subprogram. In this paper only linear problems were

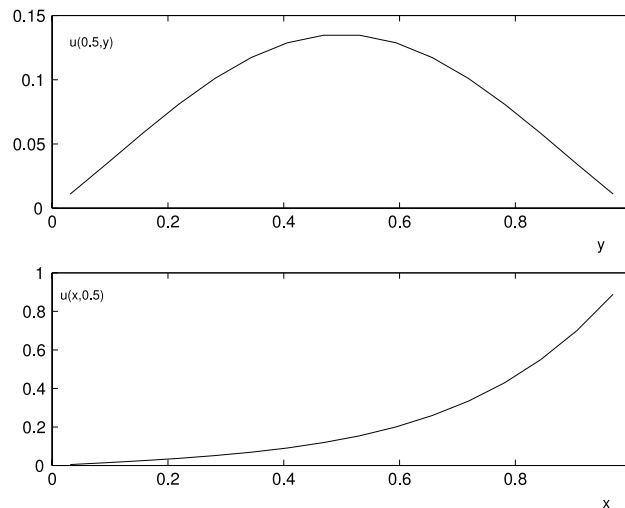


Fig. 3. Poisson equation: sections of the surface $u = u(x, y)$.

considered, but the method is applicable also for nonlinear PDEs: we recommend consulting the paper [22] in which the Burgers' equation was solved.

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